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Servaas van Bilsen, Roger Laeven, Theo Nijman

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Optimal Savings and Portfolio Choice with Risky Labor Income and Reference-Dependent Preferences*

Servaas van Bilsen
Amsterdam School of Economics
University of Amsterdam
and NETSPAR

Roger J. A. Laeven†
Amsterdam School of Economics
University of Amsterdam
EURANDOM and CentER

Theo E. Nijman
Department of Finance
Tilburg University
CentER and NETSPAR

Abstract

This paper explores the joint impact of reference-dependent preferences and non-tradable risky labor income on optimal savings and portfolio decisions. We develop a non-trivial solution procedure to determine the optimal policies. Our results reveal that the impact of permanent labor income shocks on both the optimal savings rate and the optimal portfolio share is more pronounced under reference-dependent preferences than under CRRA preferences. In particular, we find that in a wide range of scenarios, individuals withdraw pension wealth already before retirement. Furthermore, we show that the optimal response of the savings rate and the portfolio share to a fall in labor income exhibits large heterogeneity across the ratio of consumption to the reference level. Finally, we find that the optimal policies are more conservative compared to the case with risk-less labor income and CRRA preferences.

JEL Classification: D81, G11, J24, J26.

OR/MS Classification: Risk, Investment, Portfolio.

Keywords: Non-Tradable Risky Labor Income; Endogenous Reference Level; Optimal Savings Behavior; Optimal Portfolio Behavior.

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†Corresponding author. Mailing Address: PO Box 15867, 1001 NJ Amsterdam, The Netherlands. Phone: +31 (0) 20 525 4219.

1 Introduction

The question of how to optimally save and invest total wealth over the life-cycle has been extensively studied in different contexts since the seminal works of Merton (1969) and Samuelson (1969). Human wealth constitutes the largest part of total wealth. For example, Lustig, van Nieuwerburgh, and Verdelhan (2013) estimate that for the average US household, human wealth is 90% of total wealth; see also Mayers (1972) and Jorgenson and Fraumeni (1989). Furthermore, labor income is not risk-less as has been vividly demonstrated by the recent COVID-19 crisis.[1] Hence, it is of great importance to understand how risky human wealth affects optimal savings and portfolio decisions. This paper extends the literature by analyzing this question for an individual with reference-dependent preferences.

Bodie, Merton, and Samuelson (1992) show that in a setting where labor income is risk-less and CRRA preferences apply, the optimal share of pension wealth invested in the risky stock decreases, on average, with age. Intuitively, bond-like human wealth diversifies stock return risk. Hence, early in life, when human wealth is large, the individual can afford to take more stock return risk. A number of authors (see, e.g., Viceira (2001), Cocco, Gomes, and Maenhout (2005) and Benzoni, Collin-Dufresne, and Goldstein (2007)) explore the role of non-tradable risky labor income for optimal behavior. They find that non-tradable risky labor income may provide an explanation for the hump-shaped equity allocation over the life-cycle as commonly observed in empirical studies (see, e.g., Ameriks and Zeldes (2004)). These authors derive optimal policies assuming standard preferences such as CRRA utility or Epstein-Zin utility. Furthermore, they mainly focus on implications for optimal investment behavior and, to a lesser extent, on implications for optimal savings behavior.

The experimental and empirical literature has shown substantial deviations from standard preferences. One prominent finding is that people tend to evaluate outcomes relative to a reference level; see the classical works of Kahneman and Tversky (1979) and Tversky and Kahneman (1992). Several papers explore the savings and investment implications of reference-dependent preferences.[2] Most of these authors consider a setting where the individual faces stock return risk but does not earn labor income. But

[1]The unemployment rate in the US increased from 3.5% in February 2020 to 14.7% in April 2020. In August 2021, the unemployment rate was 5.2%, which is still considerably higher than in the three years prior to the COVID-19 crisis.

[2]See, e.g., Berkelaar, Kouwenberg, and Post (2004), Gomes (2005), Jin and Zhou (2008), He and Zhou (2011), He and Zhou (2016), Curatola (2017), Guasoni, Huberman, and Ren (2020), and Van Bilsen, Laeven, and Nijman (2020).

how does uncertainty in labor income affect the individual's optimal savings and portfolio behavior? To the best of our knowledge, we are the first to explore the joint impact of reference-dependent preferences and non-tradable risky labor income on optimal savings and portfolio decisions.

We assume that the individual derives utility from the difference between consumption and a reference level. Furthermore, the individual's reference level may depend on own past consumption choices, own past labor income and the past consumption choices of the individual's neighbors. As our model involves market incompleteness and behavioral preferences, we cannot use standard solution methods. Therefore, we develop a non-trivial solution technique to determine the optimal policies and the shadow price of labor income risk. Our solution technique is inspired by the works of He and Pearson (1991), Schroder and Skiadas (2002), and Van Bilsen et al. (2020). We can even apply our solution technique in a setting where the individual is loss-averse.

Our three main findings are as follows. First, we find that the impact of a current permanent labor income shock on both the current optimal savings rate and the current optimal portfolio share is more pronounced under reference-dependent preferences than under CRRA preferences. The excess sensitivity, i.e., over-responsiveness, of the current optimal savings rate is due to a strong preference for consumption smoothing and the endogeneity of the reference level. An implication of the preference for consumption smoothing is that a permanent labor income shock affects consumption not only during the working phase but also during the retirement phase. Hence, after a permanent drop in labor income, the individual decreases the optimal savings rate, so that less pension wealth is available for retirement consumption.[3] An implication of the endogeneity of the reference level is that the individual has a strong preference to protect current consumption. As a result, the individual absorbs a permanent drop in current labor income by reducing the current optimal savings rate.

We even find that in a wide range of economic scenarios, the individual does not save at all and withdraws pension wealth already before retirement. Thus, an institutional setting in which individuals cannot easily unlock pension wealth before retirement can be quite costly in welfare terms. For example, in the US, an individual cannot easily withdraw pension wealth from her 401(k) plan or IRA before age 60, while after losing

[3] We note that a permanent drop in labor income has two counteracting effects on human wealth: a direct effect and an indirect effect. It impacts human wealth directly through a decrease in permanent labor income and indirectly through a decrease in the shadow price of labor income risk. For our parameter values, we find that the net effect is that human wealth decreases following a permanent drop in labor income.

her job, it might be optimal to do so. We find that the welfare loss associated with CRRA preferences in which the savings rate is not excessively sensitive can be as large as 30%.

We can explain the excess sensitivity of the current optimal portfolio share by a time-varying relative risk aversion. After a permanent drop in labor income, the individual reduces the optimal portfolio share not only because her human wealth decreases – which implies less diversification of stock return risk – but also because she becomes more risk averse. Indeed, a drop in income makes her worried whether she will have sufficient pension wealth to finance future reference levels. Hence, her optimal response is to move investments away from risky stocks towards risk-less assets.

Our second main finding is that the optimal response of the current savings rate and the current portfolio share to a permanent labor income shock varies heavily with the ratio of consumption to the reference level. In case the ratio of consumption to the reference level is low, which often applies to low income individuals, the individual drastically reduces the optimal savings rate and the optimal portfolio share following a permanent drop in labor income. Intuitively, as consumption is close to the reference level, the individual has a strong preference to protect current consumption. Hence, the individual limits the impact of a permanent drop in labor income on consumption by reducing the optimal savings rate and by investing less in the risky stock. We note that under CRRA preferences, the optimal response of the current savings rate and the current portfolio share do not directly depend on the ratio between the individual's consumption level and the individual's reference level.

Our third main finding is that the optimal policies are more conservative compared to the case with risk-less labor income and CRRA preferences. Both non-tradable risky labor income and reference-dependent preferences affect the optimal portfolio share. Consistent with existing literature (see, e.g., Viceira (2001)), non-tradable risky labor income causes the optimal share of pension wealth invested in the risky stock to decrease. An endogenous reference level has two additional counteracting effects on the optimal portfolio share. On the one hand, the endogeneity of the reference level leads to a riskier investment strategy. Indeed, since a drop in labor income affects not only consumption levels but also reference levels, the impact of a labor income shock on utility – which depends on the difference between a consumption level and a reference level – is limited. This fact allows the individual to take more investment risk. On the other hand, the individual needs to reserve a substantial part of her pension wealth to make sure that future consumption levels exceed future reference levels with high probability. For a typical range of parameter values, we find that the net effect is that reference dependence leads to a reduction in the optimal share of pension wealth invested in the risky stock. As a result of the conservative optimal portfolio strategy, the optimal consumption level is typically lower as compared to standard CRRA preferences with risk-less labor income. Indeed, the individual benefits less from the expected excess return on the risky stock.

We also test the excess sensitivity of the current optimal savings rate and the heterogeneous response of optimal savings rate using monthly data on total expenditures and incomes.[4] Our dataset provides support for these two main implications. In particular, we find that the optimal savings rate of a low-income individual exhibits a higher degree of excess sensitivity than the optimal savings rate of a high-income individual.

Finally, we explore our main findings for the case in which labor income shocks are less permanent. We develop a continuous-time labor income process with non-permanent labor income shocks. We find that both the optimal savings rate and the optimal portfolio share remain excessively sensitive. What is new is that the optimal savings rate rapidly converges back to the savings rate before the labor income shock. This result is due to the gradual absorption of shocks and the temporary nature of income shocks. Indeed, after some time, income is restored to its old level, while optimal consumption is relatively low. A similar response applies, more or less, to the optimal portfolio share. The portfolio share converges to its old level after some time. Indeed, once the income shock is over, the individual becomes less risk averse and can afford to take more investment risk.

The remainder of the paper is structured as follows. Section 2 introduces the model specification. In Section 3, we present the non-trivial solution technique and the optimal policies. The main implications are discussed in Section 4. This section also relates our findings to empirical analysis. Section 5 explores whether our main findings remain valid in case labor income shocks are less permanent. Finally, Section 6 concludes the paper.

2 Model

This section presents our continuous-time model. Denote by t adult age, which corresponds to the individual's age minus the age at which the individual starts working. For sake of simplicity, we assume that the individual retires at the (non-random) adult age $T_R > 0$ and dies at the (non-random) adult age $T_D > 0$.

[4]It is beyond the scope of this paper to test *all* model implications.

2.1 Preferences

Denote by $c(t)$ and $h(t)$ the individual's consumption choice and the individual's endogenous reference level at adult age t, respectively. The individual derives utility from the difference between consumption and a reference level. Hence, expected lifetime utility is given by

$$U = \mathbb{E}_0 \left[\int_0^{T_D} e^{-\delta t} u\left(c(t) - h(t)\right) \mathrm{d}t \right], \tag{2.1}$$

where $\mathbb{E}_0 [\cdot]$ represents the expectation conditional upon information available at time 0, $\delta \geq 0$ denotes the subjective rate of time preference, and $u(\cdot)$ represents the instantaneous utility function satisfying the usual conditions. We give an example specification of $u(\cdot)$ in Section 4.

We assume that the reference level satisfies the dynamics:

$$\mathrm{d}h(t) = \left(\beta c(t) - \alpha(t) h(t)\right) \mathrm{d}t. \tag{2.2}$$

We allow the depreciation parameter $\alpha(t)$ to be a deterministic function of adult age t, so that, for example, the rate of depreciation during the working phase can differ from the rate of depreciation during the retirement phase. The preference parameter $\beta \geq 0$ models the impact of current consumption $c(t)$ on the individual's current reference level $h(t)$. Our reference level specification (2.2) is in line with the specification considered by, e.g., Constantinides (1990), Detemple and Zapatero (1991), Gomes and Michaelides (2003), Munk (2008), and Van Bilsen et al. (2020).

Finally, in the appendix we consider an even more general reference level specification that depends not only on own past consumption but also on individual past labor income and on past consumption of the individual's neighbors. A reference level specification in which the reference level depends on other people's consumption captures the idea that the individual wants to catch up with the Joneses; see, e.g., Abel (1990), Campbell and Cochrane (1999), and Chen (2017).

2.2 State Variables and Financial Market

We consider an economy with two state variables: non-tradable risky labor income $Y(t)$ and the stock price $S(t)$. We assume that the dynamics of individual labor income are driven by a single standard Brownian motion $Z_Y(t)$. In Section 4, we give an example specification of the dynamics for $Y(t)$.[5]

We assume that the individual can invest her pension (or financial) wealth $F(t)$ into two assets: a risky stock and a risk-less asset. We assume the following dynamics for the stock price $S(t)$ and the price of the risk-less asset $B(t)$:

$$\mathrm{d}S(t) = (r + \lambda_S \sigma_S)\, S(t)\mathrm{d}t + \sigma_S S(t)\mathrm{d}Z_S(t), \tag{2.3}$$

$$\mathrm{d}B(t) = rB(t)\mathrm{d}t. \tag{2.4}$$

Here, $\lambda_S \in \mathbb{R}$ denotes the market price of stock return risk, $\sigma_S > 0$ models the stock return volatility, $Z_S(t)$ is a standard Brownian motion, and $r \in \mathbb{R}$ denotes the risk-less interest rate. We allow the Brownian increments $\mathrm{d}Z_S(t)$ and $\mathrm{d}Z_Y(t)$ to be correlated, and denote the correlation coefficient between $\mathrm{d}Z_S(t)$ and $\mathrm{d}Z_Y(t)$ by $\rho_{SY} \in [-1, +1]$.

2.3 Dynamic Budget Constraint

Let us denote by $\omega(t)$ the share of pension wealth $F(t)$ invested in the risky stock at adult age t. The individual's dynamic budget constraint is now given by

$$\mathrm{d}F(t) = (r + \omega(t)\lambda_S\sigma_S)\, F(t)\mathrm{d}t + \omega(t)\sigma_S F(t)\mathrm{d}Z_S(t) + (Y(t) - c(t))\mathrm{d}t. \tag{2.5}$$

We observe from (2.5) that pension wealth grows because of two reasons: investment results (see the first two terms on the right-hand side of (2.5)) and new savings $Y(t) - c(t)$.

2.4 Dynamic Maximization Problem

The individual faces the following dynamic maximization problem:

$$\begin{aligned} &\max_{c(t),\omega(t)} \quad \mathbb{E}_0\left[\int_0^{T_D} e^{-\delta t} u\left(c(t) - h(t)\right)\mathrm{d}t\right] \\ &\text{s.t.} \quad \begin{aligned} &\mathrm{d}h(t) = \left(\beta c(t) - \alpha(t) h(t)\right)\mathrm{d}t \\ &\mathrm{d}F(t) = (r + \omega(t)\lambda_S\sigma_S)\, F(t)\mathrm{d}t + \omega(t)\sigma_S F(t)\mathrm{d}Z_S(t) + (Y(t) - c(t))\mathrm{d}t. \end{aligned} \end{aligned} \tag{2.6}$$

[5] We note that if $t \geq T_R$, then $Y(t)$ represents the social security payment from the government at adult age t. We assume that the size of the social security payment is determined at the retirement age T_R and is a function of the history of individual labor income $Y(s)$, $s \in [0, T_R]$. In what follows, we refer to $Y(t)$ simply as individual labor income at t.

To solve (2.6), we need to define the stochastic discount factor $m(t)$. We can show that $m(t)$ satisfies the following dynamics (see, e.g., He and Pearson (1991)):

$$\mathrm{d}m(t) = -rm(t)\mathrm{d}t + \phi^{\top}(t)m(t)\mathrm{d}Z(t). \tag{2.7}$$

Here, $\top$ denotes the transpose sign, $Z(t) \equiv (Z_S(t), Z_Y(t))$, and $\phi(t)$ is a vector of factor loadings. We can determine $\phi(t)$ from the vector of prices of risk $\lambda(t) \equiv (\lambda_S, \lambda_Y(t))$ where $\lambda_Y(t)$ denotes the shadow price of labor income risk. We have the following relationship:

$$\phi(t) = -\begin{bmatrix} 1 & \rho_{SY} \\ \rho_{SY} & 1 \end{bmatrix}^{-1} \lambda(t). \tag{2.8}$$

By the principle of no arbitrage, $\lambda_S = (\mu_S - r)/\sigma_S$. However, as labor income risk is non-tradable, every value of $\lambda_Y(t)$ is consistent with the principle of no arbitrage. Section 3 explains how we endogenously determine $\lambda_Y(t)$.

3 Optimal Life-Cycle Policies

The financial market as defined in Section 2.2 is incomplete, because the number of risk sources (i.e., stock return risk and labor income risk) is larger than the number of risky assets (i.e., stock). He and Pearson (1991) show how to determine the optimal consumption and portfolio policies in an incomplete financial market using the martingale method. The idea is as follows. First, given the shadow price of labor income risk $\lambda_Y(t)$, we determine the optimal consumption policy by transforming the individual's dynamic maximization problem (2.6) into a dual maximization problem; see Section 3.1 for more details. Then, we endogenously determine $\omega(t)$ and $\lambda_Y(t)$ such that changes in tradable total wealth match changes in the value of future optimal consumption. Hence, $\lambda_Y(t)$ is chosen such that the individual does not want to hedge non-tradable uncertainty (i.e., the individual's demand for consumption plans that are not marketed is zero). Let us denote the optimal portfolio strategy by $\omega^*(t)$ and the optimal shadow price of labor income risk by $\lambda_Y^*(t)$. Next, we substitute the just found $\lambda_Y^*(t)$ in the optimal consumption policy. For more details, we refer the reader to Appendices A, B and C.

3.1 A Dual Maximization Problem

As indicated at the beginning of this section, we first need to determine the optimal consumption policy under the assumption that the value of the shadow price of labor income risk $\lambda_Y(t)$ is known. However, even if we assume the value of $\lambda_Y(t)$ to be known, the individual's dynamic maximization problem (2.6) is difficult to solve. Appendix A shows that (2.6) is equivalent to a dual maximization problem which is easier to solve. Denote by $\widehat{c}(t) \equiv c(t) - h(t)$ the individual's dual consumption choice at adult age t. The dual problem is now defined as follows:

$$\begin{aligned} \max_{\widehat{c}(t)} \quad & \mathbb{E}_0 \left[\int_0^{T_D} e^{-\delta t} u\left(\widehat{c}(t)\right) \mathrm{d}t \right] \\ \text{s.t.} \quad & \mathbb{E}_0 \left[\int_0^{T_D} \frac{\widehat{m}(t)}{\widehat{m}(0)} \widehat{c}(t) \mathrm{d}t \right] \quad \leq \frac{F(0) - f(0)h(0)}{1 + \beta f(0)} + \mathbb{E}_0 \left[\int_0^{T_D} \frac{\widehat{m}(t)}{\widehat{m}(0)} \widehat{Y}(t) \mathrm{d}t \right], \end{aligned} \tag{3.1}$$

where $\widehat{m}(t) \equiv m(t)\left(1 + \beta f(t)\right)$ and

$$\widehat{Y}(t) \equiv \frac{Y(t)}{1 + \beta f(t)} \tag{3.2}$$

are the dual stochastic discount factor and dual individual labor income at adult age t, respectively. Here, $f(t)$ is defined as follows:

$$f(t) \equiv \mathbb{E}_t \left[\int_t^{T_D} \frac{m(s)}{m(t)} e^{-\int_t^s (\alpha(u) - \beta) \mathrm{d}u} \mathrm{d}s \right]. \tag{3.3}$$

We observe that a solution exists if and only if the right-hand side of the static budget constraint in (3.1) is larger than zero. By substituting the expressions of $\widehat{m}(t)$ and $\widehat{Y}(t)$ into the static budget constraint, we find that a solution exists if and only if

$$F(0) + H(0) \geq f(0)h(0), \tag{3.4}$$

with

$$H(t) \equiv Y(t)\mathbb{E}_t \left[\int_t^{T_D} \frac{m(s)}{m(t)} \frac{Y(s)}{Y(t)} \mathrm{d}s \right] \tag{3.5}$$

denoting human wealth at adult age t. Condition (3.4) states that the sum of initial pension wealth $F(0)$ and initial human wealth $H(0)$ should be larger than the amount of money needed to finance the baseline consumption stream, which is attained when dual consumption $\widehat{c}(t) = 0$ for every adult age t.

Given $\lambda_Y(t)$, we solve the dual problem (3.1). Let us denote the dual optimal consumption choice by $\widehat{c}^{**}(t)$. Then the (primal) optimal consumption choice is given by (see Schroder and Skiadas (2002))

$$c^{**}(t) = \widehat{c}^{**}(t) + \widehat{h}^{**}(t), \tag{3.6}$$

where the optimal reference level is specified as follows:

$$\widehat{h}^{**}(t) = e^{-\int_0^t \alpha(u)\mathrm{d}u} h(0) + \beta \int_0^t e^{-\int_s^t (\alpha(u)-\beta)\mathrm{d}u} \widehat{c}^{**}(s)\mathrm{d}s. \tag{3.7}$$

3.2 Optimal Consumption Choice

We are now ready to present the optimal consumption choice.

Theorem 1 (optimal consumption choice) *Consider an individual who solves the maximization problem (2.6). Assume that condition (3.4) holds true. Let $\lambda_Y^*(t)$ be the optimal (endogenous) shadow price of labor income risk. Then the optimal consumption choice at adult age t is given by*

$$c^*(t) = h^*(t) + (u')^{-1}\left(e^{\delta t} y \frac{\widehat{m}^*(t)}{\widehat{m}^*(0)}\right). \tag{3.8}$$

Here,

$$h^*(t) = e^{-\int_0^t \alpha(u)\mathrm{d}u} h(0) + \beta \int_0^t e^{-\int_s^t \alpha(u)\mathrm{d}u} c^*(s)\mathrm{d}s \tag{3.9}$$

and $\widehat{m}^*(t)$ is the dual stochastic discount factor associated with the optimal shadow price of labor income risk $\lambda_Y^*(t)$.

The Lagrange multiplier y is chosen such that the static budget constraint holds with equality.

Proof. See Appendix B.1.

In Appendix C, we describe how to numerically determine the optimal shadow price of labor income risk $\lambda_Y^*(t)$. Equation (A81) in Appendix C reveals that the optimal shadow price of labor income risk is, under some mild conditions, proportional to the ratio between dual human wealth and dual total wealth (i.e., discounted value of dual future consumption $\widehat{c}(s)$, with $s > t$). This result is consistent with Sangvinatsos and Wachter (2005) and de Jong (2008) who find that under standard CRRA preferences, the shadow price of labor income risk is proportional to the ratio between human wealth and total wealth.

3.3 Optimal Portfolio Choice

The following theorem summarizes the optimal portfolio choice.

Theorem 2 (optimal portfolio choice) *Consider an individual who solves the maximization problem* (2.6). *Assume that condition* (3.4) *holds true. Let*

$$\phi^*(t) = -\begin{bmatrix} 1 & \rho_{SY} \\ \rho_{SY} & 1 \end{bmatrix}^{-1} \times \begin{bmatrix} \lambda_S \\ \lambda_Y^*(t) \end{bmatrix} \tag{3.10}$$

be the vector of optimal factor loadings. And let $F(t)$ be the individual's pension wealth at adult age t. Then the optimal portfolio choice at adult age t is given by

$$\omega^*(t) = (1+\beta f(t))\,\frac{\partial \widehat{F}(t)}{\partial \widehat{m}(t)}\,\frac{\phi_S^*(t)}{\sigma_S}\,\frac{\widehat{m}^*(t)}{F(t)}, \tag{3.11}$$

where

$$\widehat{F}(t) = \frac{F(t) - f(t)h(t)}{1+\beta f(t)} \tag{3.12}$$

and $\widehat{m}^*(t)$ is the dual stochastic discount factor associated with the optimal shadow price of labor income risk $\lambda_Y^*(t)$.

Proof. See Appendix B.2.

3.4 Loss Aversion

We derived the optimal consumption choice (see Theorem 1) under the assumption that the instantaneous utility function $u(c(t) - \theta(t))$ is continuously differentiable. This assumption does not allow for loss aversion. However, in case the individual's instantaneous utility function includes loss aversion, we can still obtain analytical results. More specifically, let us assume that the instantaneous utility function is specified as follows:

$$u\left(c(t)-\theta(t)\right)=\begin{cases}\left(c(t)-\theta(t)\right)^{\gamma_G}, & \text{if } c(t)\geq\theta(t);\\ -\kappa\left(\theta(t)-c(t)\right)^{\gamma_L}, & \text{if } c(t)<\theta(t).\end{cases} \tag{3.13}$$

Here, $\gamma_G \in (0,1)$ and $\gamma_L > 0$ denote the curvature parameters for gains and losses, respectively, and $\kappa \geq 1$ stands for the loss aversion parameter. Van Bilsen et al. (2020) show in a setting without human wealth risk how a loss-averse individual should save and invest her total wealth. We can combine the techniques of the present paper with those developed by Van Bilsen et al. (2020) to obtain the optimal policies in a setting with human wealth risk and where preferences are described by (3.13).[6]

In what follows, we do not consider (3.13) but a utility specification in which the individual cannot consume below the reference level. We note that the main implications as discussed in Section 4 will remain valid in case the individual's preferences are described by (3.13). Indeed, while loss aversion allows the individual to consume below the reference level in severe economic scenarios, our main implications are driven by the inclusion of an endogenous reference level in the utility specification.

4 Main Implications

4.1 Setting

We consider an individual who starts working at age 25, retires at age 65 and passes away at age 85. Each period, the individual decides what share of labor income to consume and how much to invest in the risk-less asset and the risky stock.

We specify individual labor income $Y(t)$ as follows ($t \leq T_R$):

$$Y(t)=Y(0)\exp\left\{\int_0^t \mu_Y(s)\mathrm{d}s+\sigma_Y\int_0^t e^{-\kappa(t-s)}\mathrm{d}Z_Y(s)\right\}. \tag{4.1}$$

Here, $\mu_Y(t)\in\mathbb{R}$ represents the growth rate of median labor income, $\sigma_Y \geq 0$ denotes (instantaneous) labor income volatility, $Z_Y(t)$ is a standard Brownian motion, and $\kappa \geq 0$ models the impact of the labor income shock $\sigma_Y \mathrm{d}Z_Y(t)$ on future median labor income.

[6]The derivations are available upon request. The main idea is to split the individual's dual maximization problem into two separate problems: a gain part problem and a loss part problem. The next step is to solve the gain part problem as well as the loss part problem. In the final step, we determine the global maximum of the dual problem by comparing the two local maxima.

We allow κ to be different from zero to capture the idea that labor income shocks can be permanent or transitory; see, e.g., Caroll (1997). If $\kappa = 0$, then labor income shocks are permanent, while if $\kappa = \infty$, then labor income shocks are transitory. Figure 12 in Section 5 illustrates the impact of a labor income shock on future median labor income for various values of κ. Moreover, we allow the growth rate of median labor income $\mu_Y(t)$ to be time-dependent to capture the idea that the drift of the labor income process typically depends on age; see, e.g., Cocco et al. (2005). Finally, we note that after retirement, $Y(t)$ represents the social security payment from the government at adult age t.

We assume the following parameter values for the financial market and the labor income process. The risk-less interest rate r and the mean stock return μ_S are set at 1% and 5%, respectively. We assume that the stock return volatility σ_S is equal to 20%. These parameter values roughly coincide with the ones reported by Gomes, Kotlikoff, and Viceira (2008). Initial salary $Y(0)$ is equal to \$68,000 per year which corresponds to the average household income in the US in 2019. During retirement, the individual does not receive labor income. In line with Viceira (2001), we set labor income volatility σ_Y equal to 10%.[7] Moreover, we assume that the growth rate of median labor income is 0% (i.e., $\mu_Y(t) = 0$ for every t) and that, in this section, labor income shocks are permanent (i.e., $\kappa = 0$).[8] Initial pension wealth $F(0)$ is zero.

We assume that the instantaneous utility function is given by the familiar form

$$u\big(c(t) - h(t)\big) = \frac{1}{1-\gamma}\big(c(t) - h(t)\big)^{1-\gamma}. \tag{4.2}$$

Here, $\gamma > 0$ is a preference parameter. We report our results for the following values of the individual's preference parameters. The parameter that models the curvature of the utility function, i.e., γ, is equal to 5. We set the parameters that characterize the reference level, i.e., $\alpha(t) = \alpha$ and β, equal to 0.1 and 0.2, respectively. The subjective rate of time preference δ equals 4%. Our main implications remain qualitatively unchanged if we vary the values of the parameters within reasonable limits.

4.2 Optimal Consumption Choice

This section illustrates the individual's optimal consumption choice. We assume that the individual faces non-tradable labor income risk. Furthermore, she derives utility from the difference between consumption and an endogenous reference level.

[7] Viceira's estimate of σ_Y is based on Chamberlain and Hirano (1999) and Caroll and Samwick (1997).
[8] Section 5 explores the impact of different values of κ on the optimal policies.

4.2.1 Excess Sensitivity of Current Optimal Savings Rate

This section analyzes the implications of both a stock return shock and a permanent labor income shock for optimal consumption and the optimal savings rate. But first we need to explore how total wealth responds to a shock. Indeed, total wealth finances optimal consumption. The impact of a shock on total wealth is not immediately clear as a shock also affects total wealth indirectly through its effect on the shadow price of labor income risk; see the last paragraph of Section 3.2 for more details on this subject matter.

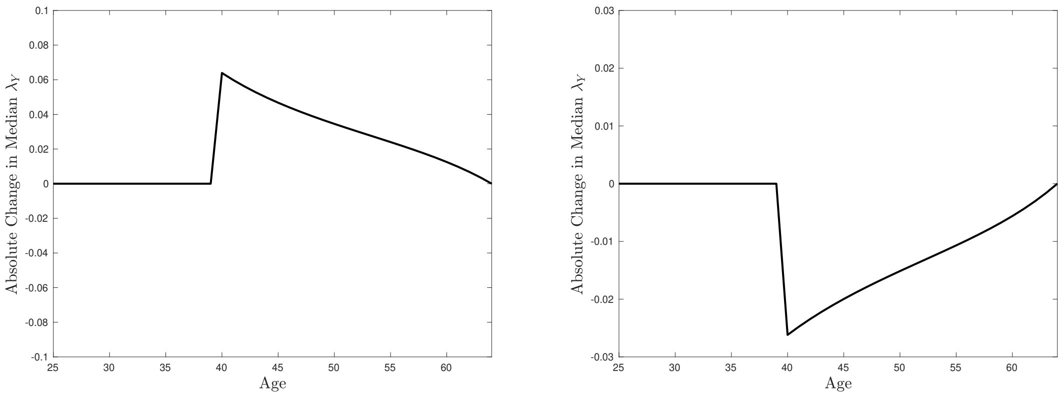
A negative stock return shock obviously leads to a lower amount of pension wealth. Furthermore, we find that a negative stock return shock leads to a decrease in human wealth. Indeed, a lower stock price implies a higher shadow price of labor income risk, which in turn implies a lower discounted value of future labor income; see Figure 1(a).[9] Hence, a stock price decline is extra bad news: both pension wealth and human wealth decrease. We note that even in the absence of correlation between stock return shocks and labor income shocks, human wealth decreases following a stock price decline.

Clearly, a permanent drop in labor income leaves current pension wealth unaffected. Moreover, we find that a permanent drop in labor income has two counteracting effects on human wealth. On the one hand, human wealth decreases since expected labor income is smaller. On the other hand, human wealth increases since the shadow price of labor income risk is lower; see Figure 1(b). We find that for a wide range of parameter values, the first effect dominates the second effect. Hence, human wealth decreases following a negative permanent shock in labor income.

From our analysis above, it becomes clear that both a negative stock return shock and a negative permanent labor income shock lead to a reduction in the individual's total wealth. As a result, the individual needs to adjust future median optimal consumption levels downwards following a negative shock. As illustrated by Figures 2 and 3, we find that optimal consumption responds sluggishly to both shocks. Intuitively, with an endogenous reference level, consumption reductions in the far future are felt less heavily than consumption reductions in the near future. The dashed lines show the case in which the individual has standard CRRA preferences. In that case, a shock affects all future median consumption levels to the same extent.

A number of authors (see, e.g., Van Bilsen et al. (2020)) have already shown that in the presence an endogenous reference level, individuals adjust consumption gradually in

[9]Since the shadow price of labor income risk is – under some mild conditions – proportional to the ratio between dual human wealth and dual total wealth, a reduction in dual pension wealth leads to a higher shadow price of labor income risk.



(a) Impact of Stock Return Shock

(b) Impact of Labor Income Shock

Figure 1. Impact of Shocks on Median Shadow Price of Labor Income Risk The figure shows how the median shadow price of labor income risk $\lambda_Y(t)$ changes following a 50% drop in the stock price at age 40 (left panel) and following a 20% permanent drop in labor income at age 40 (right panel). The parameter values are given in Section 4.1.

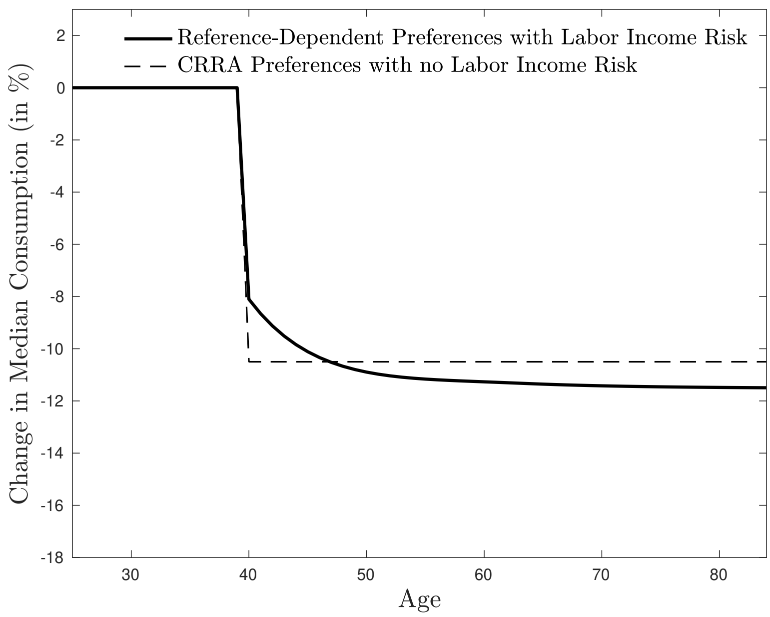


Figure 2. Impact of a Stock Return Shock on Future Median Optimal Consumption The figure shows the impact of a 50% drop in the stock price at age 40 on future median optimal consumption. The dashed line illustrates the case in which the individual has standard CRRA preferences and does not face non-tradable labor income risk. Both cases experience the same relative decline in total wealth. The parameter values are given in Section 4.1.

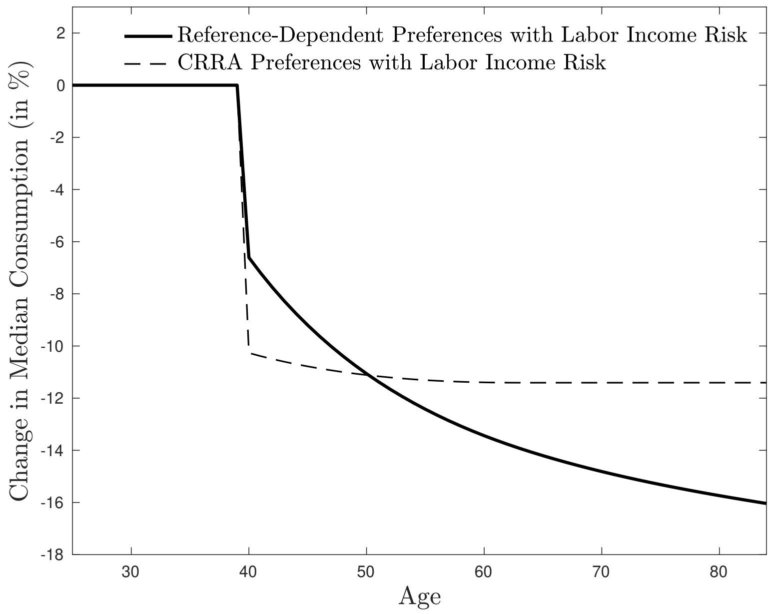
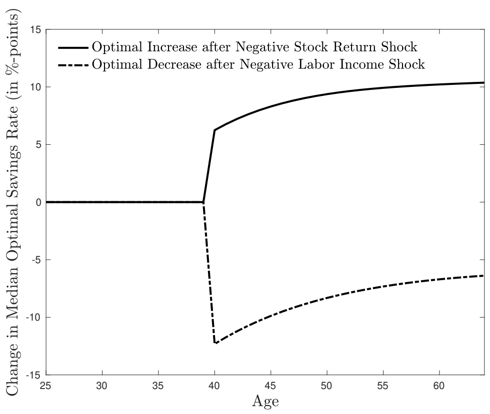


Figure 3. Impact of a Labor Income Shock on Future Median Optimal Consumption The figure shows the impact of a 20% permanent drop in labor income at age 40 on future median optimal consumption. The dashed line illustrates the case in which the individual has standard CRRA preferences and faces non-tradable labor income risk. Both cases experience the same relative decline in total wealth. The parameter values are given in Section 4.1.

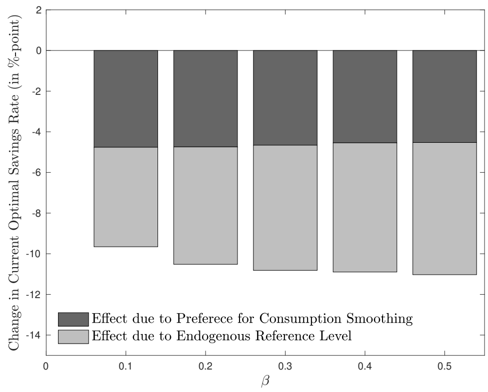
response to a stock return shock. What is new is that with non-tradable labor income risk, individuals adjust consumption gradually following both a stock return shock and a labor income shock. With CRRA utility, we do not observe this gradual adjustment of consumption to shocks; see the dashed line in Figure 2 and the dashed line in Figure 3.

A direct implication of the downward adjustment of median optimal consumption to negative shocks is that the current optimal savings rate, i.e., the optimal share of labor income saved, responds differently to a stock return shock than to a labor income shock; see Figure 4(a). As the individual has a preference for consumption smoothing, she prefers that a stock price decline – which leads to a reduction in pension wealth – affects consumption not only during the retirement phase but also during the working phase. Therefore, she increases the optimal savings rate after a drop in the current stock price; see the solid line in Figure 4(a). We note that the increase in the current optimal savings rate is lower as compared to the increase in the current savings rate under standard CRRA preferences. Indeed, with an endogenous reference level, the individual prefers to protect current consumption.

As illustrated by the dash-dotted line in Figure 4(a), we find that after a permanent drop in current labor income, the current optimal savings rate decreases. We can decompose the optimal response of the current savings rate into two parts. The first part is due to a preference for consumption smoothing, as illustrated by the dark gray area in Figure 4(b). In other words, a labor income shock affects consumption not only during the working phase but also during the retirement phase. To transfer part of the current labor income shock to the retirement phase, the individual reduces the optimal savings rate, so that less pension wealth is available for retirement consumption. The second part is due to an endogenous reference level, as illustrated by the light gray area in Figure 4(b). Indeed, since the individual prefers to protect current consumption, she reduces the optimal savings rate. In our setting, we find that the impact due to an endogenous reference level is larger than the impact due to a preference for consumption smoothing. We conclude that the current optimal savings rate is excessively sensitive, i.e., over-responsive, to a permanent labor income shock. This is our first main finding.



(a) Change in Optimal Savings Rate



(b) Decomposition of Current Optimal Response

Figure 4. Impact of Shocks on Optimal Savings Rate The left panel of this figure shows how the median optimal savings rate responds following a negative stock return shock (solid line) and following a negative labor income shock (dash-dotted line) at age 40. The right panel decomposes, for various values of β, the optimal response of the current savings rate into two parts. The first part is due to a preference for consumption smoothing (dark gray area), while the second part is due to an endogenous reference level (light gray area). In the right panel, we assume that $\alpha(t) = \alpha = \beta$. The remaining parameter values are given in Section 4.1.

Figure 5 shows the new level of the optimal savings rate for a wide rage of labor income shocks. We assume that the old level of the optimal savings rate is equal to 18%. We find that in a wide range of economic scenarios, i.e., scenarios in which labor income shocks are larger than 29%, the individual does not save at all and withdraws pension wealth already before retirement; this is indicated by the gray area in Figure 5. As illustrated by Figure 4(b), the preference for dissaving is larger under reference-dependent preferences than under CRRA preferences.

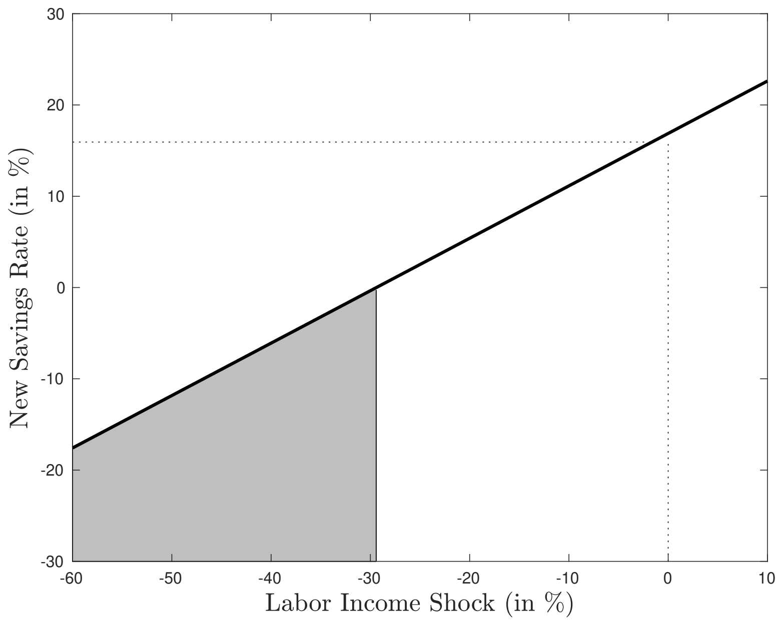


Figure 5. New Savings Rate for a Wide Range of Labor Income Shocks The figure shows the new optimal savings rate for a wide range of labor income shocks. We assume that the old optimal savings rate is equal to 18%. Furthermore, the ratio of consumption to the reference level and the ratio of pension wealth to human wealth are assumed to be equal to 2 and 1, respectively. The gray area indicates the economic scenarios in which the optimal savings is negative, i.e., the individual withdraws pension wealth already before retirement. The parameter values are given in Section 4.1.

4.2.2 Heterogeneity in Optimal Response of Savings Rate

The optimal response of the current savings rate heavily varies with the ratio of consumption to the reference level. This is our second main finding. We illustrate this finding by Figure 6. Note that the ratio of consumption to the reference level can be seen as a proxy for income. Indeed, the closer current consumption is to the reference level, the lower the individual's income level typically will be. We find that in case the ratio of consumption to the reference level is small, the optimal savings rate is heavily reduced following a permanent drop in current labor income; see Figure 6. Intuitively, as consumption is very close to the reference level, the individual has a strong need to protect current consumption. Hence, after a permanent labor income shock, she prefers to maintain, more or less, the same optimal savings rate. Also note that under CRRA preferences, the optimal response of the savings rate is independent of the ratio between the individual's consumption level and the individual's reference level, as this ratio does not appear in a CRRA preference specification.

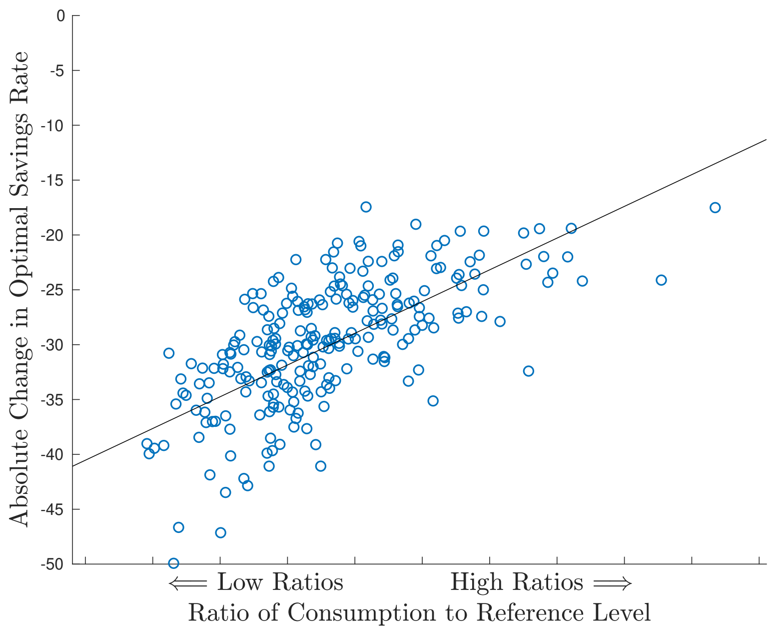


Figure 6. Heterogeneity in Optimal Savings Rate The figure shows the optimal response (in percentage points) of the current savings rate to a negative permanent labor income shock as a function of the ratio of consumption to the reference level. The parameter values are given in Section 4.1.

The optimal response of the savings rate varies not only with the ratio of consumption to the reference level but also with the ratio of human wealth to pension wealth. We find that in case human wealth is small compared to pension wealth, the individual heavily adjusts the optimal savings rate. Intuitively, if a worker's human wealth is low compared to pension wealth, then a permanent drop in current labor income affects total wealth only to a limited extent, so that there is no need to adjust current consumption. Such a worker fully absorbs the labor income shock by reducing the optimal savings rate. We note that the ratio of human wealth to pension wealth can be seen a proxy for age. Indeed, old workers typically have a low ratio of human wealth to pension wealth, while this ratio is much larger for young workers.

4.2.3 Conservative Optimal Consumption Choice

The presence of both non-tradable labor income risk and reference-dependent preferences lead to a conservative optimal portfolio strategy; see Section 4.3.3. As a result of the conservative optimal portfolio strategy, the optimal consumption level is typically lower as compared to standard CRRA preferences with risk-less labor income. Indeed, the individual benefits less from the expected excess return on the risky stock. Figure 7 illustrates the median optimal consumption choice for the case with reference-dependent preferences and non-tradable labor income risk and for the case with standard CRRA preferences and risk-less labor income. We observe that up to age 55, optimal consumption is (substantially) lower. The underlying conservative optimal portfolio strategy explains this observation.

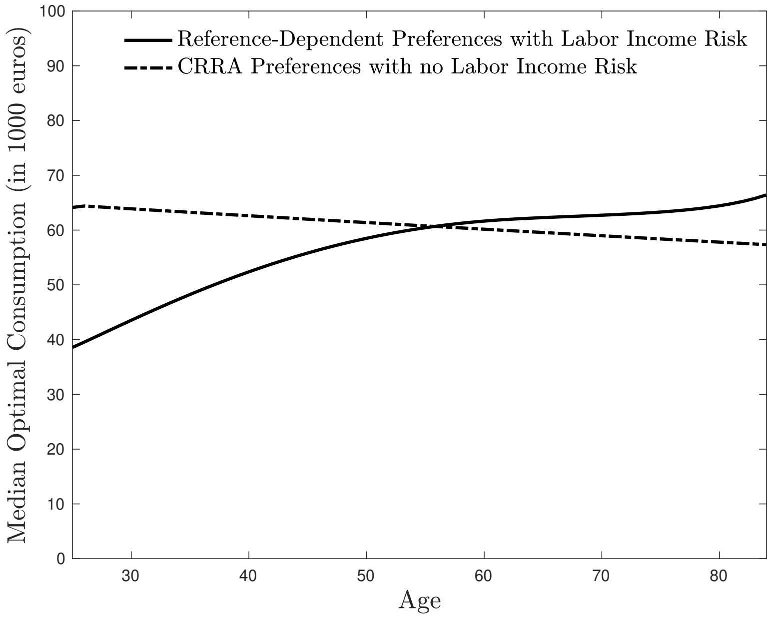


Figure 7. Conservative Median Optimal Consumption Choice The figure illustrates the median optimal consumption choice for the case with reference-dependent preferences and non-tradable labor income risk (solid line) and for the case with standard CRRA preferences and risk-less labor income (dash-dotted line). Both cases have the same total initial wealth. The parameter values are given in Section 4.1.

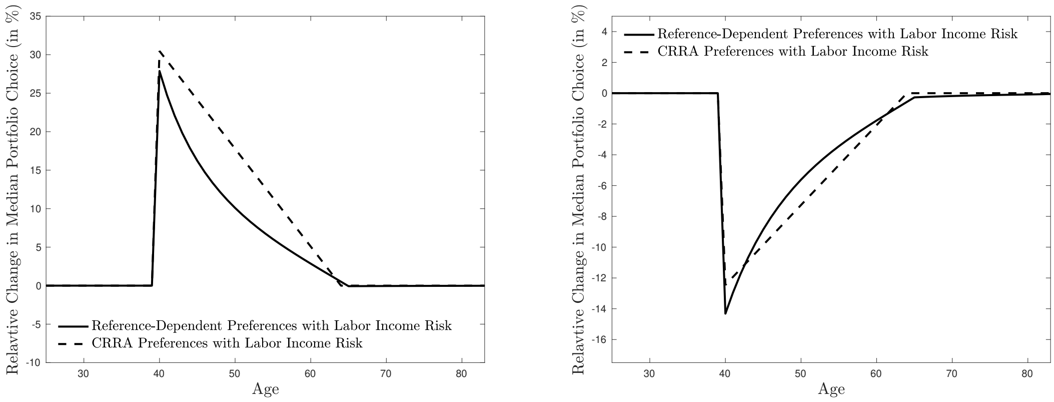
4.3 Optimal Portfolio Choice

This section illustrates the individual's optimal portfolio choice. As in Section 3.2, the individual faces non-tradable labor income risk and derives utility from the difference between consumption and an endogenous reference level.

4.3.1 Excess Sensitivity of the Current Optimal Portfolio Share

This section analyzes the implications of both a stock return shock and a permanent labor income shock for the optimal share of pension wealth invested in the risky stock (i.e., portfolio share). In case of non-tradable risky labor income and reference-dependent preferences, the impact of a stock return shock and a labor income shock on the optimal portfolio share is not immediately clear. It is obvious that a negative stock return shock leads to a lower amount of pension wealth. However, human wealth also decreases following a drop in the stock price. Indeed, as illustrated by Figure 1(a), the shadow price of labor income risk goes up after a negative stock return shock. As both pension wealth and human wealth decrease, it is not directly clear how the individual should adjust the optimal share of pension wealth invested in the stock. We find that the impact of a stock return shock is typically much larger on pension wealth than on human wealth. Hence, in our benchmark case – in which the individual aims for a constant exposure of total wealth to stock return risk – the individual increases the optimal share of pension wealth invested in the stock following a decline in the stock price; see the dashed line in Figure 8(a). Under reference-dependent preferences, a negative stock return shock leads to an increase in the individual's relative risk aversion. Indeed, a relatively larger part of total wealth needs to be invested in the risk-less asset to guarantee that future consumption levels exceed future reference levels. This reason dampens the impact of a negative stock return shock on the optimal portfolio share. Therefore, we find that the impact of a stock return shock on the optimal portfolio share is more pronounced under CRRA preferences than under reference-dependent preferences; see Figure 8(a).

As mentioned in Section 3.2, a negative permanent labor income shock has two counteracting effects on human wealth: a direct effect and an indirect effect. It impacts human wealth directly through its effect on future labor income and indirectly through its effect on the shadow price of labor income risk; see Figure 1(b). We find that the net effect is that human wealth decreases following a permanent decline in labor income. As a result, in our benchmark case – in which the individual aims for a constant exposure of total wealth to stock return risk – the individual decreases the optimal share of pension wealth invested in the stock following a negative labor income shock; see the dashed line in Figure 8(b). Under reference-dependent preferences, relative risk aversion increases following a negative permanent labor income shock. Indeed, the individual's current consumption gets closer to the individual's reference level. This is the second reason, which is not present under CRRA preferences, why the individual reduces the optimal portfolio share. We conclude that not only the optimal savings rate but also the optimal portfolio share is excessively sensitive to labor income shocks; see Figure 8(b).



(a) Impact of Stock Return Shock

(b) Impact of Labor Income Shock

Figure 8. Impact of Shocks on Optimal Portfolio Share The figure shows how the optimal share of pension wealth invested in the risky stock changes as a result of a shock at age 40. The left panel shows the optimal response to a 50% decline in the stock price while the right panel illustrates the optimal response to a 20% permanent decline in labor income. The dashed lines illustrate the case in which the individual has standard CRRA preferences and faces non-tradable labor income risk. The parameter values are given in Section 4.1.

4.3.2 Heterogeneity in Optimal Response of Portfolio Share

Both the optimal response of the savings rate and the optimal response of the portfolio share vary with the ratio of consumption to the reference level. We illustrate this finding in Figure 9. However, a major difference between Figure 6 – which illustrates the heterogeneity in the optimal savings rate – and Figure 9 is that the optimal portfolio share will almost never drop below zero, while this is not the case for the optimal savings rate.

Figure 9 illustrates how the optimal response of the portfolio share varies with the ratio of consumption to the reference level. We find that in case the ratio of consumption to the reference level is high, the optimal portfolio share is relatively insensitive to a permanent drop in current labor income. Intuitively, as the individual's consumption is high compared to the reference level, a permanent drop in labor income affects her relative risk aversion only to a limited extent. Hence, there is less need to de-risk. Under CRRA preferences, we do not observe this behavior, as relative risk aversion is constant.

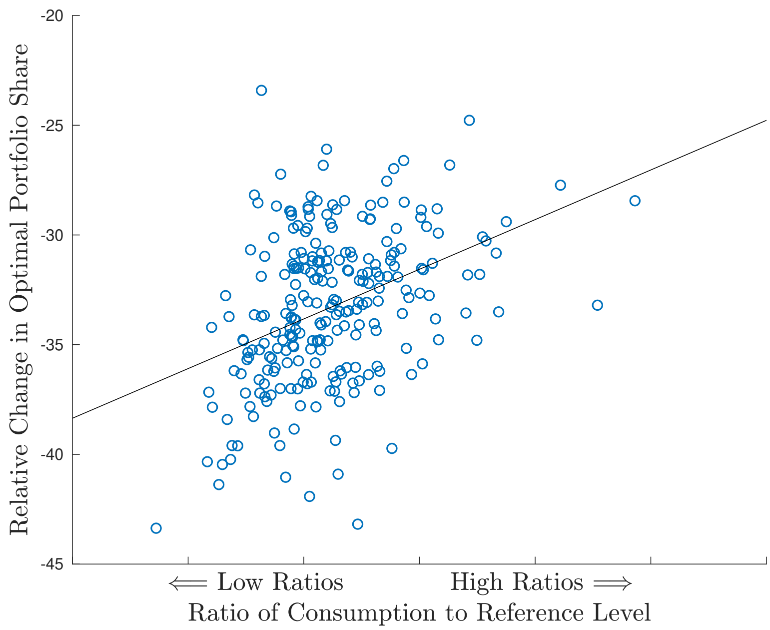


Figure 9. Heterogeneity in Optimal Portfolio Share The figure shows the optimal response (in percentage points) of the current optimal portfolio share to a negative permanent labor income shock as a function of the ratio of consumption to the reference level. The parameter values are given in Section 4.1.

We also find that the optimal response of the portfolio share varies with the ratio of human wealth to pension wealth. In particular, in case the ratio of human wealth to pension wealth is relatively small, the individual barely adjusts the optimal portfolio share. Intuitively, if a worker's human wealth is small compared to a worker's pension wealth, then a drop in current labor income affects total wealth only to a limited extent, so that there is no reason to significantly adjust the optimal portfolio share.

4.3.3 Conservative Optimal Portfolio Choice

In a setting with non-tradable labor income risk and reference-dependent preferences, the individual still applies a life-cycle portfolio strategy: the optimal share of pension wealth invested in the stock decreases, on average, with age. This is consistent with conventional wisdom (see, e.g., Bodie et al. (1992)). However, for our parameter values, the optimal portfolio strategy is more conservative compared to the case with risk-less labor income and CRRA preferences; see Figure 10. This finding holds true even if there is no correlation between stock return shocks and labor income return shocks.[10]

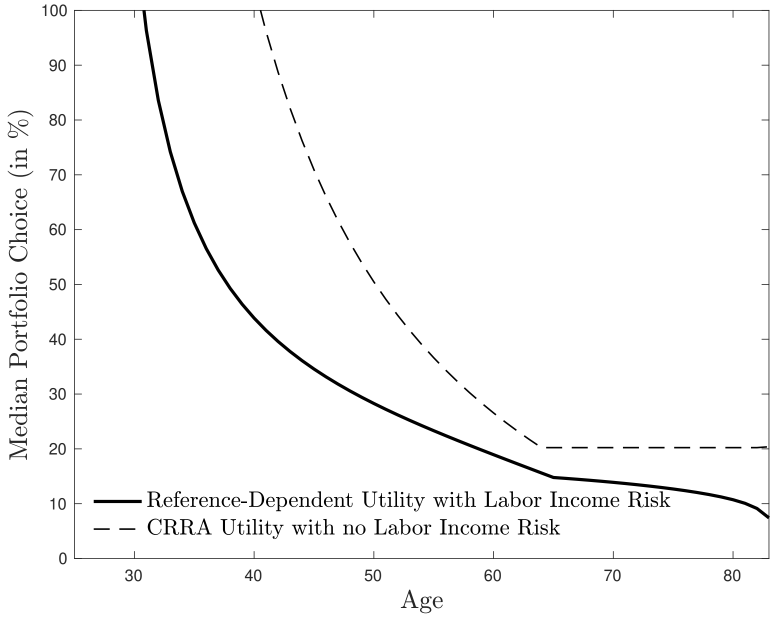


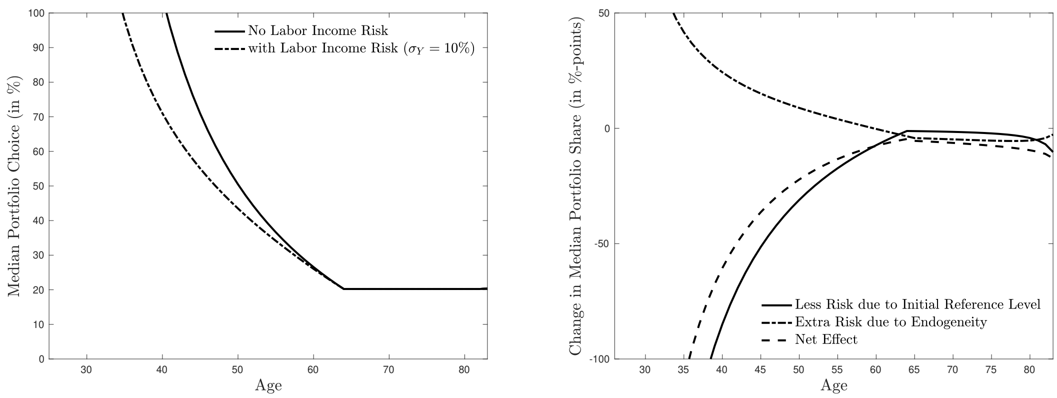
Figure 10. Conservative Optimal Portfolio Share The figure shows the median optimal share of pension wealth invested in the risky stock as a function of age. The dashed line illustrates the case in which the individual has standard CRRA preferences and does not face non-tradable labor income risk. The parameter values are given in Section 4.1.

Both non-tradable risky labor income and reference-dependent preferences affect the optimal portfolio strategy. Figure 11(a) illustrates the impact of non-tradable risky labor income on the optimal portfolio strategy. We find that non-tradable risky labor income causes the optimal portfolio share to decrease, especially at young ages. Indeed, due to a positive shadow price of labor income risk, human wealth, which is equal to the discounted value of future labor income, becomes smaller. The individual has thus a lower implicit portfolio holding of the non-tradable asset human wealth. As a result,

[10]The impact of the correlation coefficient on the optimal portfolio strategy is twofold. First, a higher ρ_{SY} means that human wealth carries more stock return risk, so that the individual's willingness to invest in the risky stock decreases. Second, a higher ρ_{SY} leads, in our setting, to a higher shadow price of labor income risk, which causes human wealth to decrease. This in turn leads to a reduction of the optimal portfolio share.

the individual should invest less in risky stocks to achieve the optimal exposure of total wealth to stock return risk. We find that in our setting, the median portfolio strategy at age 40 decreases by 25 percent.

The presence of a reference level has two additional counteracting effects on the optimal portfolio choice as illustrated by Figure 11(b). The first effect, which is illustrated by the solid line in Figure 11(b), is due to the presence of a reference level. This fact implies that the individual needs to reserve part of her pension wealth to guarantee that future consumption levels exceed future reference levels. As a result, the optimal portfolio share decreases. The second effect, which is illustrated by the dash-dotted line in Figure 11(b), is due to the endogeneity of the reference level. This fact allows the individual to take more investment risk. Indeed, a negative stock return shock leads to lower future consumption levels as well as lower future reference levels. As the impact of a given stock return shock on utility is limited compared to CRRA preferences, the individual can afford to take more investment risk. We observe that, in our parameter setting, the first effect is much stronger than the second effect: reference-dependent preferences lead to a conservative investment strategy.



(a) Impact of Non-Tradable Labor Income

(b) Impact of Reference-Dependence

Figure 11. Impact of Non-Tradable Risky Labor Income and Reference-Dependent Preferences The left panel shows the impact of non-tradable risky labor income on the optimal portfolio share, while the right panel illustrates the impact of reference-dependent preferences on the optimal share. The parameter values are given in Section 4.1.

4.4 Welfare Costs

This section computes the welfare costs associated with standard CRRA preferences. Our setting is as follows. An individual with references-dependent preferences who faces non-tradable labor income risk delegates her consumption and portfolio decisions to a professional asset manager. However, this asset manager offers strategies that are exclusively based on standard CRRA preferences. Each strategy offered by the asset manager corresponds to a value of the relative risk aversion coefficient. The individual is smart and chooses the value of the relative risk aversion coefficient such that the difference between the individual's optimal utility (i.e., the utility associated with the optimal policies) and the sub-optimal utility (i.e., the utility associated with standard CRRA preferences) is minimized. Table 1 reports our results. We observe that a strategy in which the savings rate does not respond excessively sensitive to a labor income shock can be quite costly in welfare terms. For our benchmark setting (i.e., $\alpha(t) = \alpha = 0.2$ and $\beta = 0.1$), we find a minimum welfare loss of 35%.

Table 1. **Welfare Costs** The table reports the minimum welfare losses associated with standard CRRA preferences in which the savings rate does not respond excessively sensitive to a labor income shock. We compute the minimum welfare loss for different optimal values of $\alpha(t) = \alpha$ and β. We measure welfare losses in terms of the relative decline in certainty equivalent consumption. The remaining parameter values are given in Section 4.1.

true parameters α and β		
α	β	minimum welfare loss (in %)
0.1	0.2	35.08
0.05	0.1	38.04
0.2	0.3	30.13
0.3	0.4	26.04
0.4	0.5	23.52

4.5 Relating Our Findings to Empirical Analysis

We can test the main implications of our model with reference-dependent preferences and non-tradable risky labor income using data on savings behavior and portfolio holdings. This section shows that we find support for some of our main implications in the data. It is beyond the scope of this paper to carry out a full empirical analysis.

4.5.1 Testable Implications

Our first main implication is that both the optimal savings rate and the optimal portfolio choice are excessively sensitive with respect to permanent labor income shocks; see Sections 4.2.1 and 4.3.1. The excess sensitivity of the optimal savings rate, i.e., the current optimal savings rate *over*-responds to a current income shock, implies that optimal consumption is excessively smooth, i.e., current optimal consumption *under*-responds to a current income shock; see Figure 3. We can test the excess smoothness of consumption by regressing changes in current (log) income on changes in current (log) consumption:

$$\Delta \log c(t) = \beta \Delta \log Y(t-1) + \epsilon(t), \tag{4.3}$$

where $\Delta \log c(t)$ denotes the change in the log total consumption between adult age $t-1$ and adult age t, $\Delta \log Y(t-1)$ represents the change in the log income level between adult age $t-2$ and adult age $t-1$, and $\epsilon(t)$ is the error term. If $0 \leq \beta < 1$, then consumption *under*-responds to a current income shock.[11] In a similar fashion, we can test the excess sensitivity of the current optimal portfolio share by relating income shocks to changes in investment behavior.

Our second main implication is that the optimal response of the savings rate and the portfolio share to a current income shock heavily varies with the ratio of consumption to the reference level; see Sections 4.2.2 and 4.3.2. We can test this model implication using panel data on savings decisions, portfolio holdings, income levels, current consumption levels and current reference levels. Our hypothesis is that in case current consumption is close to the current reference level, the savings rate and the portfolio share will heavily respond to a permanent income shock, while in case consumption is far away from the current reference level, individuals barely adjust savings rates and portfolio shares following a permanent income shock. As we do not have data on reference levels, in the empirical analysis that follows, we use the income level as a proxy for the ratio between consumption and the reference level. Indeed, a high income typically corresponds to a large ratio of consumption to reference level, while a low income is typically associated with a low ratio of consumption to reference level.

Our third main implication is that an individual with reference-dependent preferences who faces non-tradable labor income risk implements a conservative consumption strategy as well as a conservative investment strategy; see Sections 4.2.3 and 4.3.3. We can test this

[11] We have verified in simulations that $0 \leq \beta < 1$.

model finding using data on consumption decisions and portfolio holdings and compare the empirical findings with what the standard theory (Bodie et al. (1992)) predicts. Existing empirical evidence already suggests that people are reluctant to invest in risky stocks (see, e.g., Haliassos and Bertaut (1995)).

In the remainder of this section, we test whether consumption is excessively smooth and whether the excess smoothness of consumption varies with ratio between consumption and the reference level.

4.5.2 Data Description

We obtain data from the US Bureau of Labor Statistics. It collects data on expenditures, incomes and demographic characteristics of consumers in the United States. We construct a panel data set to provide support for some of our main implications, using monthly data on total consumption and income levels (before taxes) for 15,381 unique individuals. Our dataset runs from January 2020 to August 2021 (20 periods). We use this dataset to explore whether we find support for two of our main implications: the excess smoothness of the optimal consumption choice and the heterogeneity of the optimal response of the consumption choice.

4.5.3 Results

We run regression model (4.4) to explore whether consumption is excessively smooth. To explore whether we find evidence for the heterogeneity of the optimal response of the consumption choice, we divide the data set into three categories:

1. low incomes (monthly income is lower 2053 dollars);
2. middle incomes (monthly income is between 2053 dollars and 3902 dollars);
3. and high incomes (monthly income is higher than 3902 dollars).

We run the regression model (4.4) for each category. The model predicts that the β-coefficient in (4.4) varies with the income level. Table 2 reports our results.
Table 2 provides support for the finding that consumption is excessively smooth (all β-estimates are between 0 and 1) and for the finding that the optimal response of the consumption level depends on income. Indeed, we observe that the optimal consumption of a low-income individual responds less strong to an income shock compared to the optimal consumption of a high-income individual. Thus, the optimal savings rate of

Table 2. **Regression Results** The table reports the estimate of the β-coefficient. We compute the β-estimate for three categories: low incomes, middle incomes and high incomes.

category	estimate of β	95% confidence interval β-estimate
low incomes	0.0270	[0.0211;0.0333]
middle incomes	0.1200	[0.0259;0.2141]
high incomes	0.2159	[0.1368;0.2951]

a low-income individual exhibits a higher degree of excess sensitivity than the optimal savings rate of a high-income individual. This is consistent with our finding; see also Figure 6.

As indicated by Table 2, current consumption of a low-income individual heavily *under*-responds to a current income shock. Our model predicts that consumption in the far future is more sensitive to a current income shock than consumption close to today; see Figure 3. We can test this additional implication by regressing changes in current income on changes in future consumption. More specifically, we run the following regression model:

$$\sum_{i=0}^{n} \Delta \log c(t+i) = \beta_n \Delta \log Y(t-1) + \epsilon(t). \tag{4.4}$$

If β_n increases with n, a labor income shock will have a bigger impact on total expenditures in the far future compared to the near future. Table 3 provides our results.

Table 3. **Regression Results** The table reports the estimate of the β_n-coefficient. We compute the β_n-estimate for low incomes.

Periods ahead	estimate of β_n
Immediate ($n = 0$)	0.0270
Beginning of second quarter ($n = 3$)	0.0621
Beginning of third quarter ($n = 6$)	0.2043

We clearly see that the estimate of β_n increases with n: low-income individuals postpone reductions in total consumption following a current labor income shock.

5 Different Types of Labor Income Shocks

5.1 Labor Income Process

So far we have assumed that labor income shocks are permanent, i.e., $\kappa = 0$. This section explores our main findings for the case in which labor income shocks are less permanent.[12] More specifically, we consider the case $\kappa = 0.25$. Figure 12 shows how median income behaves following a 20% decline in labor income at age 40 for different types of labor income shocks. In case $\kappa = 0.25$, we observe that at age 60, the individual's labor income is roughly equal to labor income before the shock.

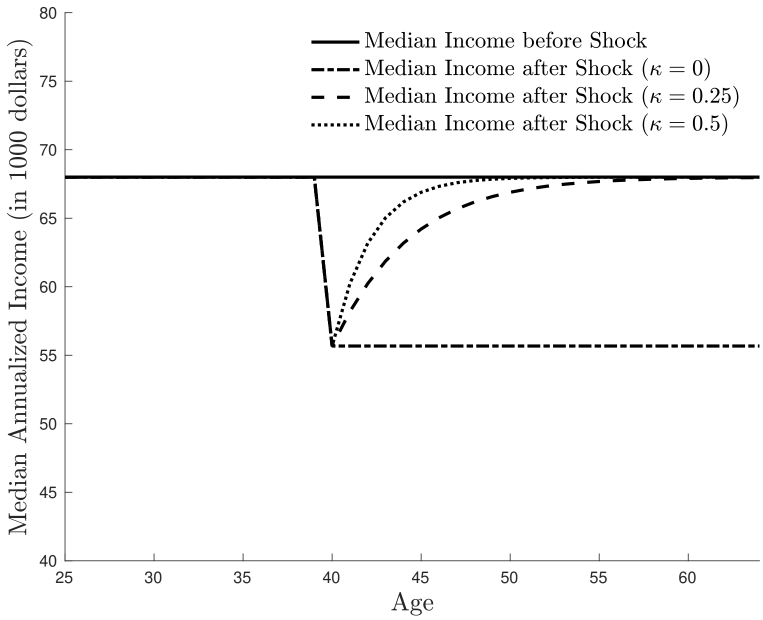


Figure 12. Impact of Different Types of Labor Income Shocks on Future Median Consumption The figure shows the impact of a 20% drop in labor income at age 40 on future median income for various values of κ. The remaining parameter values are given in Section 4.1.

[12]We can easily extend our numerical method to the case $\kappa > 0$. While the value of κ affects how labor income evolves over time, it essentially does not affect the numerical determination of the optimal policies and the shadow price of labor income risk. Indeed, the optimal strategy is a function of the current state variables (dual) labor income and the (dual) stochastic discount factor.

5.2 Optimal Consumption Choice

Figure 13 illustrates the relative change in median consumption as a result of a 20% decline in labor income at age 40. If we compare the case $\kappa = 0.25$ (less permanent labor income shock) with the case $\kappa = 0$ (permanent labor income shock), we observe that the relative change in median consumption is smaller. Indeed, in case labor income shocks are less permanent, the impact of a labor income shock on the individual's total wealth will be smaller. Furthermore, in both cases, we observe that the individual gradually absorbs a labor income shock. Indeed, with an endogenous reference level, it is optimal to postpone reductions in consumption.

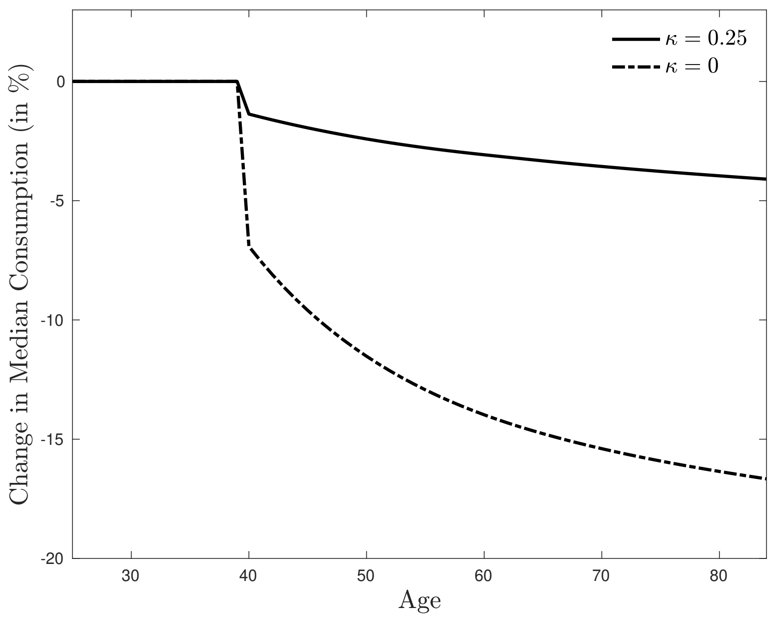


Figure 13. Impact of a Labor Income Shock on Future Median Optimal Consumption The figure shows the impact of a 20% drop in labor income at age 40 on future median optimal consumption. We assume that $\kappa = 0.25$ (less permanent labor income shock) and $\kappa = 0$ (permanent labor income shock). The remaining parameter values are given in Section 4.1.

Next, we analyze how the optimal savings rate responds to labor income shocks. Figure 14 illustrates the optimal response of the median savings rate following a 20% drop in labor income at age 40. We observe the after the labor income shock, the individual immediately decreases the optimal savings rate. Indeed, the individual wants to protect current consumption. Furthermore, she wants to transfer part of the labor income shock to the retirement phase. We also observe this behavior in case $\kappa = 0$

(permanent labor income shock). What is new is that the optimal savings rate rapidly converges back to the savings rate before the shock. After some time, the individual even saves more. This result is due to the gradual absorption of shocks and the temporary nature of income shocks. Indeed, after some time, income is restored to its old level, while optimal consumption is relatively low; see Figure 12 and the solid line in Figure 13. We conclude that less permanent income shocks still have a substantial impact on the *current* optimal savings rate.

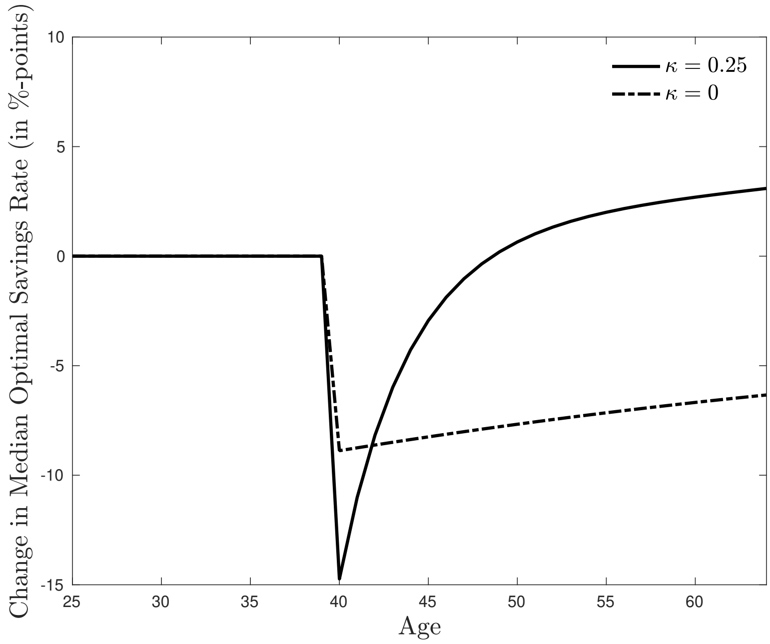


Figure 14. Impact of Labor Income Shock on Optimal Savings Rate The figure shows how the median optimal savings rate responds following a 20% drop in labor income at age 40. We assume that $\kappa = 0.25$ (less permanent labor income shock) and $\kappa = 0$ (permanent labor income shock). The remaining parameter values are given in Section 4.1.

5.3 Optimal Portfolio Choice

This section explores the impact of a labor income shock on the optimal portfolio share. Figure 15 illustrates the optimal response of the median portfolio share following a 20% drop in labor income at age 40. Comparing the case $\kappa = 0.25$ (less permanent labor income shock) with the case $\kappa = 0$ (permanent labor income shock), we observe that the change in the optimal portfolio share is smaller. Indeed, when labor income shocks are less permanent, the impact of a shock on total wealth is smaller, so that the individual has a weaker preference to de-risk. Furthermore, we observe that in the median scenario, the optimal portfolio share even goes up after some years. As income is restored to its old level and optimal consumption is relatively small, the individual can afford to take more investment risk.

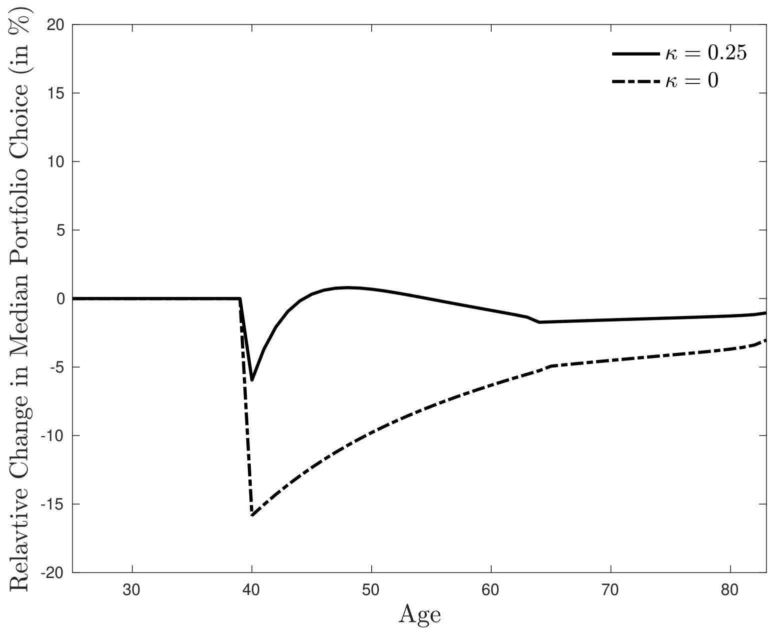


Figure 15. Impact of Labor Income Shock on Optimal Portfolio Share The figure shows how the optimal share of pension wealth invested in the risky stock changes as a result of a 20% decline in labor income at age 40. We assume that $\kappa = 0.25$ (less permanent labor income shock) and $\kappa = 0$ (permanent labor income shock). The remaining parameter values are given in Section 4.1.

6 Conclusion

We have explored the joint impact of reference-dependent preferences and non-tradable risky labor income on optimal savings and portfolio decisions. To analyze the optimal policies and to determine the shadow price of labor income risk, we have developed a non-trivial solution procedure. We have shown the following results. First, we have shown that both the current optimal savings rate and the current optimal portfolio share are excessively sensitive to labor income shocks. We find that in a wide rage of economic scenarios, the individual does not save at all and withdraws pension wealth already before retirement. Second, the optimal response of the savings rate and the portfolio share to a permanent labor income shock vary heavily with the ratio of consumption to the reference level. Third, we find that the optimal policies are more conservative compared to the case with risk-less labor income and CRRA preferences. Both non-tradable risky labor income and reference-dependent preferences contribute to this finding.

A Dual Formulation

In what follows, we assume a rich specification of the reference level. More specifically, the reference level depends not only on own consumption but also on individual labor income $Y(t)$ and consumption of the individual's neighbor $\bar{C}(t)$ which is exogenously given. We assume that the reference level can de decomposed as follows:

$$h(t) = h_c(t) + h_{\bar{C}}(t) + h_Y(t). \tag{A1}$$

Here, $h_c(t)$, $h_{\bar{C}}(t)$ and $h_Y(t)$ satisfies the dynamics:

$$\mathrm{d}h_c(t) = (\beta_c c(t) - \alpha_c(t)h(t))\,\mathrm{d}t, \tag{A2}$$

$$\mathrm{d}h_{\bar{C}}(t) = \big(\beta_{\bar{C}}\bar{C}(t) - \alpha_{\bar{C}}(t)h_{\bar{C}}(t)\big)\,\mathrm{d}t, \tag{A3}$$

$$\mathrm{d}h_Y(t) = (\beta_Y Y(t) - \alpha_Y(t)h_Y(t))\,\mathrm{d}t. \tag{A4}$$

The results in the main text arise as a special case when we assume $\beta = \beta_c$ and $\alpha(t) = \alpha_c(t)$.

By Itô's Lemma, we find that the reference level $h(t)$ satisfies

$$\mathrm{d}h(t) = \big(\beta_c c(t) + \beta_{\bar{C}}\bar{C}(t) + \beta_Y Y(t) - \alpha(t)h(t)\big)\,\mathrm{d}t, \tag{A5}$$

where

$$\alpha(t) \equiv \alpha_c(t)\frac{h_c(t)}{h(t)} + \alpha_{\bar{C}}(t)\frac{h_{\bar{C}}(t)}{h(t)} + \alpha_Y(t)\frac{h_Y(t)}{h(t)} \tag{A6}$$

models the rate at which the individual's reference level $h(t)$ depreciates.

In line with the permanent income hypothesis (see Hall (1978)), we assume that changes in consumption of the individual's neighbor are unpredictable. More specifically, we assume that $\bar{C}(t)$ is given by

$$\bar{C}(t) = \bar{C}(0)\exp\left\{\int_0^t \mu_{\bar{C}}\mathrm{d}s + \sigma_{\bar{C}}\int_0^t \mathrm{d}Z_{\bar{C}}(s)\right\}. \tag{A7}$$

Here, $\mu_{\bar{C}} \in \mathbb{R}$ models the expected growth of $\bar{C}(t)$, $\sigma_{\bar{C}} \geq 0$ represents volatility, and $Z_{\bar{C}}(t)$ is a standard Brownian motion.

We now can, by virtue of the martingale approach (Pliska (1986), Karatzas, Lehoczky, and Shreve (1987), and Cox and Huang (1989, 1991)), transform the individual's dynamic maximization problem into the following equivalent problem:

$$\begin{aligned} \max_{c(t)} \quad & \mathbb{E}_0\left[\int_0^{T_D} e^{-\delta t} u\left(c(t)-h(t)\right) \mathrm{d}t\right] \\ \text{s.t.} \quad & \mathbb{E}_0\left[\int_0^{T_D} m(t)\left(c(t)-Y(t)\right) \mathrm{d}t\right] \leq F(0), \\ & \mathrm{d}h(t) = \big(\beta_c c(t) + \beta_{\bar{C}} \bar{C}(t) + \beta_Y Y(t) - \alpha(t) h(t)\big)\, \mathrm{d}t, \end{aligned} \tag{A8}$$

with $\alpha(t)$, $\bar{C}(t)$, $Y(t)$ and $m(t)$ defined in (A6), (A7), (4.1) and (2.7).

Inspired by Schroder and Skiadas (2002) and Van Bilsen et al. (2020), we apply the following transformation. Let us denote by $\widehat{c}(t) = c(t) - h(t)$ the individual's dual consumption choice at adult age t. By substituting $c(t) = h(t) + \widehat{c}(t)$ into (A8), we find that (A8) is equivalent to:

$$\begin{aligned} \max_{\widehat{c}(t)} \quad & \mathbb{E}_0\left[\int_0^{T_D} e^{-\delta t} u\left(\widehat{c}(t)\right) \mathrm{d}t\right] \\ \text{s.t.} \quad & \mathbb{E}_0\left[\int_0^{T_D} m(t)\left(h(t) + \widehat{c}(t) - Y(t)\right) \mathrm{d}t\right] \leq F(0), \\ & \mathrm{d}h(t) = \big(\beta_c \widehat{c}(t) + \beta_{\bar{C}} \bar{C}(t) + \beta_Y Y(t) - (\alpha(t) - \beta_c) h(t)\big)\, \mathrm{d}t. \end{aligned} \tag{A9}$$

By repeated substitution, we are able to derive the analytical form of the reference level. More specifically, we find

$$\begin{aligned} h(t) = {} & e^{-\int_0^t (\alpha_c(u)-\beta_c)\mathrm{d}u} h_c(0) + \beta_c \int_0^t e^{-\int_s^t (\alpha_c(u)-\beta_c)\mathrm{d}u} \widehat{c}(s) \mathrm{d}s \\ & + \left[\beta_c \int_0^t e^{-\int_0^s \alpha_{\bar{C}}(u)\mathrm{d}u - \int_s^t (\alpha_c(u)-\beta_c)\mathrm{d}u} \mathrm{d}s + e^{-\int_0^t \alpha_{\bar{C}}(u)\mathrm{d}u}\right] h_{\bar{C}}(0) \\ & + \beta_{\bar{C}} \int_0^t \left[\beta_c \int_s^t e^{-\int_s^v \alpha_{\bar{C}}(u)\mathrm{d}u - \int_v^t (\alpha_c(u)-\beta_c)\mathrm{d}u} \mathrm{d}v + e^{-\int_s^t \alpha_{\bar{C}}(u)\mathrm{d}u}\right] \bar{C}(s) \mathrm{d}s \\ & + \left[\beta_c \int_0^t e^{-\int_0^s \alpha_Y(u)\mathrm{d}u - \int_s^t (\alpha_c(u)-\beta_c)\mathrm{d}u} \mathrm{d}s + e^{-\int_0^t \alpha_Y(u)\mathrm{d}u}\right] h_Y(0) \\ & + \beta_Y \int_0^t \left[\beta_c \int_s^t e^{-\int_s^v \alpha_Y(u)\mathrm{d}u - \int_v^t (\alpha_c(u)-\beta_c)\mathrm{d}u} \mathrm{d}v + e^{-\int_s^t \alpha_Y(u)\mathrm{d}u}\right] Y(s) \mathrm{d}s. \end{aligned} \tag{A10}$$

Substitution of (A10) into the static budget constraint in (A9) results in the following new static budget constraint:

$$F(0) \geq \mathbb{E}_0\Bigg[\int_0^{T_D} \{f_c(0)h_c(0) + f_{\bar{C}}(0)h_{\bar{C}}(0) + f_Y(0)h_Y(0) + m(t)\left(1+\beta_c f_c(t)\right)\widehat{c}(t) + \beta_{\bar{C}} f_{\bar{C}}(t)\bar{C}(t)m(t) - Y(t)m(t)\left(1-\beta_Y f_Y(t)\right)\}\,\mathrm{d}t\Bigg]. \tag{A11}$$

Here,

$$f_c(t) \equiv \mathbb{E}_t\left[\int_t^{T_D} \frac{m(s)}{m(t)} e^{-\int_t^s(\alpha_c(u)-\beta_c)\mathrm{d}u}\mathrm{d}s\right], \tag{A12}$$

$$f_{\bar{C}}(t) \equiv \mathbb{E}_t\left[\int_t^{T_D} \frac{m(s)}{m(t)}\left\{\beta_c \int_t^s e^{-\int_t^v \alpha_{\bar{C}}(u)\mathrm{d}u - \int_v^s(\alpha_c(u)-\beta_c)\mathrm{d}u}\mathrm{d}v + e^{-\int_t^s \alpha_{\bar{C}}(u)\mathrm{d}u}\right\}\mathrm{d}s\right], \tag{A13}$$

$$f_Y(t) \equiv \mathbb{E}_t\left[\int_t^{T_D} \frac{m(s)}{m(t)}\left\{\beta_c \int_t^s e^{-\int_t^v \alpha_Y(u)\mathrm{d}u - \int_v^s(\alpha_c(u)-\beta_c)\mathrm{d}u}\mathrm{d}v + e^{-\int_t^s \alpha_Y(u)\mathrm{d}u}\right\}\mathrm{d}s\right]. \tag{A14}$$

We denote by $\widehat{m}(t)$, $\widehat{Y}(t)$ and $\widehat{F}(t)$ the dual counterparts of the pricing kernel, individual labor income and pension wealth, respectively. These variables are defined as follows:

$$\widehat{m}(t) \equiv m(t)\left(1+\beta_c f_c(t)\right), \tag{A15}$$

$$\widehat{Y}(t) \equiv \frac{Y(t)\left(1-\beta_Y f_Y(t)\right) - \beta_{\bar{C}} f_{\bar{C}}(t)\bar{C}(t)}{1+\beta_c f_c(t)}, \tag{A16}$$

$$\widehat{F}(t) \equiv \frac{F(t) - f_c(t)h_c(t) - f_{\bar{C}}(t)h_{\bar{C}}(t) - f_Y(t)h_Y(t)}{1+\beta_c f_c(t)}. \tag{A17}$$

We can now write the new static budget constraint (A11) in familiar form as:

$$\mathbb{E}_0\left[\int_0^{T_D} \frac{\widehat{m}(t)}{\widehat{m}(0)}\left(\widehat{c}(t) - \widehat{Y}(t)\right)\mathrm{d}t\right] \leq \widehat{F}(0). \tag{A18}$$

Hence, the individual maximizes the following static dual maximization problem:

$$\begin{aligned} \max_{\widehat{c}(t)} \quad & \mathbb{E}_0\left[\int_0^{T_D} e^{-\delta t}u\left(\widehat{c}(t)\right)\mathrm{d}t\right] \\ \text{s.t.} \quad & \mathbb{E}_0\left[\int_0^{T_D} \frac{\widehat{m}(t)}{\widehat{m}(0)}\left(\widehat{c}(t) - \widehat{Y}(t)\right)\mathrm{d}t\right] \leq \widehat{F}(0). \end{aligned} \tag{A19}$$

Problem (A19) is typically easier to solve than the original individual's dynamic maximization problem (2.6). We note that the dual pricing kernel $\widehat{m}(t) \equiv m(t)\left(1 + \beta_c f_c(t)\right)$ satisfies the following dynamics:

$$\frac{\mathrm{d}\widehat{m}(t)}{\widehat{m}(t)} = -\widehat{r}(t)\mathrm{d}t + \widehat{\phi}(t)^\top \mathrm{d}Z(t), \tag{A20}$$

with $\widehat{\phi}(t) = \phi(t)$ and

$$\widehat{r}(t) \equiv \beta_c + \frac{r - \alpha_c(t)\beta_c f_c(t)}{1 + \beta_c f_c(t)}. \tag{A21}$$

B Derivation of Optimal Policies

B.1 Derivation of Optimal Consumption Choice

Let us first assume that the financial market is complete (i.e., aggregate consumption risk and individual labor income risk are tradable), so that $\phi(t)$ is uniquely determined. We now determine the individual's optimal consumption choice given this assumption. The Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = \mathbb{E}_0\left[\int_0^{T_D} e^{-\delta t} u\left(\widehat{c}(t)\right)\mathrm{d}t\right] + y\left(\widehat{F}(0) - \mathbb{E}_0\left[\int_0^{T_D} \frac{\widehat{m}(t)}{\widehat{m}(0)}\left(\widehat{c}(t) - \widehat{Y}(t)\right)\mathrm{d}t\right]\right). \tag{A1}$$

Here, $y \geq 0$ denotes the Lagrange multiplier associated with the static budget constraint. For every t, the individual maximizes $e^{-\delta t}u\left(\widehat{c}(t)\right) - y\widehat{m}(t)\widehat{c}(t)/\widehat{m}(0)$. We find that the optimal consumption in the *complete* market case is given by

$$\widehat{c}^{\,+}(t) = (u')^{-1}\left(e^{\delta t} y \frac{\widehat{m}(t)}{\widehat{m}(0)}\right). \tag{A2}$$

Let us now assume that the market is incomplete (i.e., aggregate consumption risk and individual labor income risk are *not* tradable), so that $\phi(t)$ is not uniquely determined. In line with, e.g., He and Pearson (1991), we determine the vector of factor loadings such that changes in dual total wealth match changes in the value of future dual consumption; see Appendices B.2 and C for more details. We denote the vector of factor loadings that satisfies this condition by $\phi^*(t)$. The pricing kernel implied by $\phi^*(t)$ is denoted by $m^*(t)$.

It now follows that

$$\widehat{c}^*(t) = (u')^{-1}\left(e^{\delta t} y \frac{\widehat{m}^*(t)}{\widehat{m}^*(0)}\right) \tag{A3}$$

not only maximizes the individual's static maximization problem (A19) but also can be replicated in our financial market as defined in Section 2.2.

The optimal (primal) consumption choice is given by

$$c^*(t) = h^*(t) + \widehat{c}^*(t), \tag{A4}$$

with the optimal reference level $h^*(t)$ defined as follows:

$$\begin{aligned}
h^*(t) &= e^{-\int_0^t (\alpha_c(u)-\beta_c)\mathrm{d}u} h_c(0) + \beta_c \int_0^t e^{-\int_s^t (\alpha_c(u)-\beta_c)\mathrm{d}u} \widehat{c}^*(s)\mathrm{d}s \\
&\quad + \left[\beta_c \int_0^t e^{-\int_0^s \alpha_{\bar{C}}(u)\mathrm{d}u - \int_s^t (\alpha_c(u)-\beta_c)\mathrm{d}u}\mathrm{d}s + e^{-\int_0^t \alpha_{\bar{C}}(u)\mathrm{d}u}\right] h_{\bar{C}}(0) \\
&\quad + \beta_{\bar{C}} \int_0^t \left[\beta_c \int_s^t e^{-\int_s^v \alpha_{\bar{C}}(u)\mathrm{d}u - \int_v^t (\alpha_c(u)-\beta_c)\mathrm{d}u}\mathrm{d}v + e^{-\int_s^t \alpha_{\bar{C}}(u)\mathrm{d}u}\right] \bar{C}(s)\mathrm{d}s \\
&\quad + \left[\beta_c \int_0^t e^{-\int_0^s \alpha_Y(u)\mathrm{d}u - \int_s^t (\alpha_c(u)-\beta_c)\mathrm{d}u}\mathrm{d}s + e^{-\int_0^t \alpha_Y(u)\mathrm{d}u}\right] h_Y(0) \\
&\quad + \beta_Y \int_0^t \left[\beta_c \int_s^t e^{-\int_s^v \alpha_Y(u)\mathrm{d}u - \int_v^t (\alpha_c(u)-\beta_c)\mathrm{d}u}\mathrm{d}v + e^{-\int_s^t \alpha_Y(u)\mathrm{d}u}\right] Y(s)\mathrm{d}s.
\end{aligned} \tag{A5}$$

B.2 Derivation of Optimal Portfolio Choice and Optimal Vector of Factor Loadings

We determine the dual portfolio strategy $\widehat{\omega}(t)$ and the vector of dual factor loadings $\widehat{\phi}(t) = -\rho^{-1}\widehat{\lambda}(t)$ such that changes in dual total wealth $\mathrm{d}\widehat{W}(t)$ match changes in the value of future optimal dual consumption $\mathrm{d}\widehat{V}(t)$ where the value of future optimal dual consumption is defined as follows:

$$\widehat{V}(t) \equiv \mathbb{E}_t\left[\int_t^{T_D} \frac{\widehat{m}(s)}{\widehat{m}(t)}(u')^{-1}\left(e^{\delta s} y \frac{\widehat{m}(s)}{\widehat{m}(0)}\right)\mathrm{d}s\right] = f_{\widehat{V}}(t, \widehat{Y}(t), \widehat{m}(t)). \tag{A6}$$

We note that the individual's dual total wealth $\widehat{W}(t)$ is the sum of the individual's dual pension wealth $\widehat{F}(t)$ and the individual's dual human wealth $\widehat{H}(t)$, where $\widehat{H}(t)$ is defined as follows:

$$\widehat{H}(t) \equiv \mathbb{E}_t \left[\int_t^{T_D} \frac{\widehat{m}(s)}{\widehat{m}(t)} \widehat{Y}(s) \mathrm{d}s \right] = f_{\widehat{H}} \left(t, \widehat{Y}(t), \widehat{m}(t) \right). \tag{A7}$$

Hence, to derive the dynamics of $\widehat{W}(t)$, we first need to derive the dynamics of $\widehat{F}(t)$ and $\widehat{H}(t)$. By Itô's Lemma, we find

$$\begin{aligned} \mathrm{d}\widehat{H}(t) = (\dots)\,\mathrm{d}t &+ \left(\frac{\partial \widehat{H}(t)}{\partial \widehat{Y}(t)} \frac{\partial \widehat{Y}(t)}{\partial \bar{C}(t)} \sigma_{\bar{C}} \bar{C}(t) + \frac{\partial \widehat{H}(t)}{\partial \widehat{m}(t)} \widehat{\phi}_{\bar{C}}(t) \widehat{m}(t) \right) \mathrm{d}Z_{\bar{C}}(t) \\ &+ \left(\frac{\partial \widehat{H}(t)}{\partial \widehat{Y}(t)} \frac{\partial \widehat{Y}(t)}{\partial Y(t)} \sigma_Y Y(t) + \frac{\partial \widehat{H}(t)}{\partial \widehat{m}(t)} \widehat{\phi}_Y(t) \widehat{m}(t) \right) \mathrm{d}Z_Y(t) \\ &+ \frac{\partial \widehat{H}(t)}{\partial \widehat{m}(t)} \widehat{\phi}_S(t) \mathrm{d}Z_S(t). \end{aligned} \tag{A8}$$

Here,

$$\widehat{\phi}_S(t) = -\widehat{\lambda}_S - \rho_{2,1}^{-1} \widehat{\lambda}_{\bar{C}}(t) - \rho_{3,1}^{-1} \widehat{\lambda}_Y(t), \tag{A9}$$

$$\widehat{\phi}_{\bar{C}}(t) = -\rho_{1,2}^{-1} \widehat{\lambda}_S - \widehat{\lambda}_{\bar{C}}(t) - \rho_{3,2}^{-1} \widehat{\lambda}_Y(t), \tag{A10}$$

$$\widehat{\phi}_Y(t) = -\rho_{1,3}^{-1} \widehat{\lambda}_S - \rho_{2,3}^{-1} \widehat{\lambda}_{\bar{C}}(t) - \widehat{\lambda}_Y(t), \tag{A11}$$

with $\rho_{i,j}^{-1}$ the (i,j)th element of ρ^{-1}. Note that $\widehat{\lambda}_S = \lambda_S = (\mu_S - r)/\sigma_S$.

The individual's dual pension wealth $\widehat{F}(t)$ evolves as follows:

$$\mathrm{d}\widehat{F}(t) = (\dots)\,\mathrm{d}t + \widehat{\omega}(t) \sigma_S \widehat{F}(t) \mathrm{d}Z_S(t), \tag{A12}$$

where $\widehat{\omega}(t)$ represents the share of dual pension wealth invested in the risky stock.

Thus,

$$\begin{aligned} \mathrm{d}\widehat{W}(t) &= \mathrm{d}\widehat{F}(t) + \mathrm{d}\widehat{H}(t) \\ &= (\dots)\,\mathrm{d}t + \left(\frac{\partial \widehat{H}(t)}{\partial \widehat{Y}(t)} \frac{\partial \widehat{Y}(t)}{\partial \bar{C}(t)} \sigma_{\bar{C}} \bar{C}(t) + \frac{\partial \widehat{H}(t)}{\partial \widehat{m}(t)} \widehat{\phi}_{\bar{C}}(t) \widehat{m}(t) \right) \mathrm{d}Z_{\bar{C}}(t) \\ &\quad + \left(\frac{\partial \widehat{H}(t)}{\partial \widehat{Y}(t)} \frac{\partial \widehat{Y}(t)}{\partial Y(t)} \sigma_Y Y(t) + \frac{\partial \widehat{H}(t)}{\partial \widehat{m}(t)} \widehat{\phi}_Y(t) \widehat{m}(t) \right) \mathrm{d}Z_Y(t) \\ &\quad + \left(\widehat{\omega}(t) \sigma_S \widehat{F}(t) + \frac{\partial \widehat{H}(t)}{\partial \widehat{m}(t)} \widehat{\phi}_S(t) \right) \mathrm{d}Z_S(t). \end{aligned} \tag{A13}$$

By Itô's Lemma, we also find

$$\begin{aligned}\mathrm{d}\widehat{V}(t) &= (\dots)\,\mathrm{d}t + \left(\frac{\partial \widehat{V}(t)}{\partial \widehat{Y}(t)}\frac{\partial \widehat{Y}(t)}{\partial \bar{C}(t)}\sigma_{\bar{C}}\bar{C}(t) + \frac{\partial \widehat{V}(t)}{\partial \widehat{m}(t)}\widehat{\phi}_{\bar{C}}(t)\widehat{m}(t)\right)\mathrm{d}Z_{\bar{C}}(t) \\ &\quad + \left(\frac{\partial \widehat{V}(t)}{\partial \widehat{Y}(t)}\frac{\partial \widehat{Y}(t)}{\partial Y(t)}\sigma_{Y}Y(t) + \frac{\partial \widehat{V}(t)}{\partial \widehat{m}(t)}\widehat{\phi}_{Y}(t)\widehat{m}(t)\right)\mathrm{d}Z_{Y}(t) \\ &\quad + \frac{\partial \widehat{V}(t)}{\partial \widehat{m}(t)}\widehat{\phi}_{S}(t)\widehat{m}(t)\mathrm{d}Z_{S}(t).\end{aligned} \tag{A14}$$

Solving $\mathrm{d}\widehat{V}(t) = \mathrm{d}\widehat{W}(t)$, we find that the optimal dual portfolio strategy and the optimal dual factor loadings should satisfy

$$\widehat{\omega}^*(t)\sigma_S\widehat{F}(t) = \frac{\partial \widehat{F}(t)}{\partial \widehat{m}(t)}\widehat{\phi}_S^*(t)\widehat{m}(t), \tag{A15}$$

$$\widehat{\phi}_{\bar{C}}^*(t) = \left(\frac{\partial \widehat{H}(t)}{\partial \bar{C}(t)} - \frac{\partial \widehat{V}(t)}{\partial \bar{C}(t)}\right)\left(\frac{\partial \widehat{V}(t)}{\partial \widehat{m}(t)} - \frac{\partial \widehat{H}(t)}{\partial \widehat{m}(t)}\right)^{-1}\frac{\sigma_{\bar{C}}\bar{C}(t)}{\widehat{m}(t)}, \tag{A16}$$

$$\widehat{\phi}_{Y}^*(t) = \left(\frac{\partial \widehat{H}(t)}{\partial Y(t)} - \frac{\partial \widehat{V}(t)}{\partial Y(t)}\right)\left(\frac{\partial \widehat{V}(t)}{\partial \widehat{m}(t)} - \frac{\partial \widehat{H}(t)}{\partial \widehat{m}(t)}\right)^{-1}\frac{\sigma_{Y}Y(t)}{\widehat{m}(t)}. \tag{A17}$$

Appendix C describes how to numerically determine $\widehat{\omega}^*(t)$, $\widehat{\lambda}_{\bar{C}}^*(t)$ and $\widehat{\lambda}_{Y}^*(t)$.

Note that $F(t) = f_c(t)h_c(t) + f_{\bar{C}}(t)h_{\bar{C}}(t) + f_Y(t)h_Y(t) + \widehat{F}(t)\left(1 + \beta_c f_c(t)\right)$. Hence, by Itô's Lemma,

$$\mathrm{d}F(t) = (\dots)\,\mathrm{d}t + \left(1 + \beta_c f_c(t)\right)\widehat{\omega}(t)\sigma_S\widehat{F}(t)\mathrm{d}Z_S(t). \tag{A18}$$

Comparing (A18) with the dynamic budget constraint

$$\mathrm{d}F(t) = (\dots)\,\mathrm{d}t + \omega(t)\sigma_S F(t)\mathrm{d}Z_S(t), \tag{A19}$$

we find that the optimal portfolio strategy $\omega^*(t)$ is given by

$$\omega^*(t) = \widehat{\omega}^*(t)\left(1 + \beta_c f_c(t)\right)\frac{\widehat{F}(t)}{F(t)}. \tag{A20}$$

C Numerical Method

Appendix B.2 shows that the dual portfolio strategy and the dual factor loadings are determined such that the changes in the value of future dual consumption match the changes in dual total wealth. This appendix illustrates how we implement this numerically. We assume that the instantaneous utility function is given by

$$u(\widehat{c}(t)) = \frac{1}{1-\gamma}\widehat{c}(t)^{1-\gamma}. \tag{A1}$$

We need to determine the dual portfolio strategy $\widehat{\omega}(t)$, the dual market price of aggregate consumption risk $\widehat{\lambda}_{\bar{C}}(t)$ and the dual market price of labor income risk $\widehat{\lambda}_Y(t)$. Note that the dual market price of stock return risk $\widehat{\lambda}_S$ is exogenously given (i.e., $\widehat{\lambda}_S = \lambda_S = (\mu_S - r)/\sigma_S$).

Let us denote by T_R the number of working periods (e.g., $T_R = 40$) and by T_D the total number of periods (e.g., $T_D = 60$). We assume that the individual receives dual labor income at the beginning of a period. Dual consumption also takes places at the beginning of a period. When we refer to a variable $X(t)$, its value is always known at the beginning of period t.

C.1 Algorithm

We now describe the algorithm to compute $\widehat{\omega}^*(t)$, $\widehat{\lambda}^*_{\bar{C}}(t)$ and $\widehat{\lambda}^*_Y(t)$ for every $t \in \{1,\dots,T_D-1\}$. First, we construct a grid of $\widehat{Y}_1(t) \equiv Y(t)\left(1-\beta_Y f_Y(t)\right)/(1+\beta_c f_c(t))$, $\widehat{Y}_2(t) \equiv -\beta_{\bar{C}} f_{\bar{C}}(t)/\bar{C}(t)(1+\beta_c f_c(t))$ and $\widehat{m}(t)$.[13] Although $\widehat{m}(t)$ is determined endogenously, we can still construct a grid of $\widehat{m}(t)$ (we should define the smallest and largest grid point such that most realizations of $\widehat{m}(t)$ lie between these lower and upper bounds). We denote by $\widehat{Y}_1^i(t)$, $\widehat{Y}_2^j(t)$ and $\widehat{m}^k(t)$ the ith grid point of $\widehat{Y}_1(t)$, the jth grid point of $\widehat{Y}_2(t)$ and the kth grid point of $\widehat{m}(t)$, respectively. The corresponding optimal dual portfolio strategy and optimal dual market prices of risk are denoted by $\widehat{\omega}^{*i,j,k}(t)$, $\widehat{\lambda}^{*i,j,k}_{\bar{C}}(t)$ and $\widehat{\lambda}^{*i,j,k}_Y(t)$. In the remainder, we use bold face to indicate that a value of a variable is *not* known given current information $\{\widehat{Y}_1^i(t), \widehat{Y}_2^j(t), \widehat{m}^k(t)\}$.

[13] Note that $\widehat{Y}(t) = \widehat{Y}_1(t) + \widehat{Y}_2(t)$.

C.1.1 Last Period of the Retirement Phase

During retirement, labor income is zero. Hence, $\widehat{Y}_1(t) = 0$ for all $t \in \{T_R + 1, \dots, T_D - 1\}$. In the retirement phase, current information is thus represented by $\left\{\widehat{Y}_2^j(t), \widehat{m}^k(t)\right\}$.

For every combination $\left\{\widehat{Y}_2^j(T_D-1), \widehat{m}^k(T_D-1)\right\}$, we determine $\widehat{\omega}^{*j,k}(T_D-1)$, $\widehat{\lambda}_{\bar{C}}^{*j,k}(T_D-1)$ and $\widehat{\lambda}_Y^{*j,k}(T_D-1)$. To do so, we need to derive the change in dual human wealth, the change in the value of future dual consumption and the change in dual pension wealth. We start by deriving the change in dual human wealth.

We find that $\widehat{H}^{j,k}(T_D-1) = \widehat{Y}_2^j(T_D-1) + \mathbb{E}_{T_D-1}\left[\frac{\widehat{\boldsymbol{m}}(T_D)}{\widehat{m}^k(T_D-1)}\widehat{\boldsymbol{Y}}_2(T_D)\right]$ and $\widehat{\boldsymbol{H}}(T_D) = \widehat{\boldsymbol{Y}}_2(T_D)$. Hence,

$$\begin{aligned}\widehat{H}^{j,k}(T_D-1) &= \widehat{Y}_2^j(T_D-1) + \widehat{Y}_2^j(T_D-1)\mathbb{E}_{T_D-1}\left[\frac{\widehat{\boldsymbol{m}}(T_D)}{\widehat{m}^k(T_D-1)}\frac{\widehat{\boldsymbol{Y}}_{\boldsymbol{2}}(\boldsymbol{T_D})}{\widehat{Y}_2^j(T_D-1)}\right] \\ &= \widehat{Y}_2^j(T_D-1) + \widehat{Y}_2^j(T_D-1)e^{-\widehat{r}(T_D-1)+\mu_{\widehat{Y}_2}(T_D-1)+\frac{1}{2}\sigma_{\bar{C}}^2-\sigma_{\bar{C}}\widehat{\lambda}_{\bar{C}}^{j,k}(T_D-1)},\end{aligned} \tag{A2}$$

with $\mu_{\widehat{Y}_2}(t) \equiv \mu_{\bar{C}} + \mathrm{d}\log f_{\bar{C}}(t)/\mathrm{d}t - \mathrm{d}\log\left(1+\beta_c f_c(t)\right)/\mathrm{d}t$. Furthermore, we find that

$$\widehat{\boldsymbol{H}}(T_D) = \widehat{Y}_2^j(T_D-1)e^{\mu_{\widehat{Y}_2}(T_D-1)+\sigma_{\bar{C}}\epsilon_{\bar{C}}(T_D)}. \tag{A3}$$

Here, $\epsilon_{\bar{C}}(T_D)$ is the unexpected aggregate consumption shock between the beginning of period T_D-1 and the beginning of period T_D. Hence, dual human wealth changes as follows:

$$\begin{aligned}\Delta\widehat{\boldsymbol{H}}(T_D) &= \widehat{\boldsymbol{H}}(T_D) - \widehat{H}^{j,k}(T_D-1) \\ &= \widehat{Y}_2^j(T_D-1)\Big\{e^{\mu_{\widehat{Y}_2}(T_D-1)+\sigma_{\bar{C}}\epsilon_{\bar{C}}(T_D)} \\ &\qquad -e^{-\widehat{r}(T_D-1)+\mu_{\widehat{Y}_2}(T_D-1)+\frac{1}{2}\sigma_{\bar{C}}^2-\sigma_{\bar{C}}\widehat{\lambda}_{\bar{C}}^{j,k}(T_D-1)} - 1\Big\} \\ &\approx \widehat{Y}_2^j(T_D-1)\Big\{\widehat{r}(T_D-1) + \sigma_{\bar{C}}\widehat{\lambda}_{\bar{C}}^{j,k}(T_D-1) + \sigma_{\bar{C}}\epsilon_{\bar{C}}(T_D) - 1\Big\}.\end{aligned} \tag{A4}$$

Here, we have used the approximations $e^x \approx 1+x$ and $e^{-\frac{1}{2}\sigma^2+\sigma\epsilon} \approx 1+\sigma\epsilon$ with $\epsilon \sim N(0,1)$.

We now derive the change in the value of future dual consumption. We find

$$\widehat{\boldsymbol{V}}(T_D) = \widehat{\boldsymbol{c}}(T_D), \tag{A5}$$

$$\widehat{V}^{j,k}(T_D-1) = \widehat{c}^{\,k}(T_D-1) + \mathbb{E}_{T_D-1}\left[\frac{\widehat{\boldsymbol{m}}(T_D)}{\widehat{m}^k(T_D-1)}\widehat{\boldsymbol{c}}(T_D)\right]. \tag{A6}$$

Here,

$$\widehat{c}^{\,k}(T_D-1) = \left(e^{\delta(T_D-2)} y \frac{\widehat{m}^k(T_D-1)}{\widehat{m}(1)}\right)^{-\frac{1}{\gamma}}. \tag{A7}$$

Hence,

$$\begin{aligned}\widehat{\boldsymbol{V}}(T_D) &= \widehat{c}^{\,k}(T_D-1) e^{-\frac{\delta}{\gamma}+\frac{1}{\gamma}\left(\widehat{r}(T_D-1)+\frac{1}{2}\widehat{\phi}^{\top j,k}(T_D-1)\rho\widehat{\phi}^{j,k}(T_D-1)\right)} \\ &\quad \times e^{-\frac{1}{\gamma}\left(\widehat{\phi}_S^{j,k}(T_D-1)\epsilon_S(T_D)+\widehat{\phi}_{\bar{C}}^{j,k}(T_D-1)\epsilon_{\bar{C}}(T_D)+\widehat{\phi}_Y^{j,k}(T_D-1)\epsilon_Y(T_D)\right)},\end{aligned} \tag{A8}$$

and

$$\begin{aligned}\widehat{V}^{j,k}(T_D-1) &= \widehat{c}^{\,k}(T_D-1)\left(1+\mathbb{E}_{T_D-1}\left[e^{-\frac{\delta}{\gamma}}\left(\frac{\widehat{\boldsymbol{m}}(T_D)}{\widehat{m}^k(T_D-1)}\right)^{1-\frac{1}{\gamma}}\right]\right) \\ &= \widehat{c}^{\,k}(T_D-1)\left\{e^{-\frac{\delta}{\gamma}-\frac{\gamma-1}{\gamma}\left(\widehat{r}(T_D-1)+\frac{1}{2}\widehat{\phi}^{\top j,k}(T_D-1)\rho\widehat{\phi}^{j,k}(T_D-1)\right)}\right. \\ &\quad \left.\times\, e^{\frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)^2\widehat{\phi}^{\top j,k}(T_D-1)\rho\widehat{\phi}^{j,k}(T_D-1)}\right\} + \widehat{c}^{\,k}(T_D-1).\end{aligned} \tag{A9}$$

The change in the value of future dual consumption is thus given by

$$\begin{aligned}\Delta\widehat{\boldsymbol{V}}(T_D) &= \widehat{\boldsymbol{V}}(T_D) - \widehat{V}^{j,k}(T_D-1) \\ &\approx \widehat{c}^{\,k}(T_D-1)\left(\widehat{r}(T_D-1)+\frac{1}{\gamma}\widehat{\phi}^{\top j,k}(T_D-1)\rho\widehat{\phi}^{j,k}(T_D-1)\right. \\ &\quad -\frac{\widehat{\phi}_S^{j,k}(T_D-1)}{\gamma}\epsilon_S(T_D)-\frac{\widehat{\phi}_{\bar{C}}^{j,k}(T_D-1)}{\gamma}\epsilon_{\bar{C}}(T_D) \\ &\quad \left.-\frac{\widehat{\phi}_Y^{j,k}(T_D-1)}{\gamma}\epsilon_Y(T_D)\right) - \widehat{c}^{\,k}(T_D-1).\end{aligned} \tag{A10}$$

We finally derive the change in dual pension wealth. We find

$$\begin{aligned}\widehat{\boldsymbol{F}}(T_D) &= \widehat{Y}_2^j(T_D-1) - \widehat{c}^{\,k}(T_D-1) \\ &\quad + \widehat{F}^{j,k}(T_D-1)e^{\widehat{r}(T_D-1)+\widehat{\omega}^{j,k}(T_D-1)\lambda_S\sigma_S-\frac{1}{2}\widehat{\omega}^{j,k}(T_D-1)^2\sigma_S^2+\widehat{\omega}^{j,k}(T_D-1)\sigma_S\epsilon_S(T_D)},\end{aligned} \tag{A11}$$

with

$$\widehat{F}^{j,k}(T_D-1) = \widehat{c}^{\,k}(T_D-1) - \widehat{Y}_2^j(T_D-1). \tag{A12}$$

Hence,

$$
\begin{aligned}
\Delta\widehat{\boldsymbol{F}}(T_D) &= \widehat{\boldsymbol{F}}(T_D) - \widehat{F}^{j,k}(T_D-1) \\
&\approx \widehat{Y}_2^j(T_D-1) - \widehat{c}^{\,k}(T_D-1) \\
&\quad + \widehat{F}^{j,k}(T_D-1)\left(\widehat{r}(T_D-1) + \widehat{\omega}^{j,k}(T_D-1)\lambda_S\sigma_S \right. \\
&\qquad\qquad \left. + \widehat{\omega}^{j,k}(T_D-1)\sigma_S\epsilon_S(T_D)\right).
\end{aligned}
\tag{A13}
$$

Using the condition $\Delta\widehat{\boldsymbol{V}}(T_D) = \Delta\widehat{\boldsymbol{H}}(T_D) + \Delta\widehat{\boldsymbol{F}}(T_D)$, we arrive at

$$
\widehat{\omega}^{*j,k}(T_D-1) = -\frac{\widehat{c}^{\,k}(T_D-1)}{\widehat{F}^{j,k}(T_D-1)}\frac{\widehat{\phi}_S^{j,k}(T_D-1)}{\gamma\sigma_S}, \tag{A14}
$$

$$
\widehat{\phi}_{\bar{C}}^{*j,k}(T_D-1) = -\frac{\sigma_{\bar{C}}\gamma\widehat{Y}^j(T_D-1)}{\widehat{c}^{\,k}(T_D-1)}, \tag{A15}
$$

$$
\widehat{\phi}_Y^{*j,k}(T_D-1) = 0. \tag{A16}
$$

Note that for all t

$$
\widehat{\phi}_S^{j,k}(t) = -\widehat{\lambda}_S - \rho_{1,2}^{-1}\widehat{\lambda}_{\bar{C}}^{j,k}(t) - \rho_{1,3}^{-1}\widehat{\lambda}_Y^{j,k}(t), \tag{A17}
$$

$$
\widehat{\lambda}_{\bar{C}}^{j,k}(t) = -\rho_{2,1}\widehat{\phi}_S^{j,k}(t) - \widehat{\phi}_{\bar{C}}^{j,k}(t) - \rho_{2,3}\widehat{\phi}_Y^{j,k}(t) \tag{A18}
$$

$$
\widehat{\lambda}_Y^{j,k}(t) = -\rho_{3,1}\widehat{\phi}_S^{j,k}(t) - \rho_{3,2}\widehat{\phi}_{\bar{C}}^{j,k}(t) - \widehat{\phi}_Y^{j,k}(t). \tag{A19}
$$

Hence,

$$
\begin{aligned}
\widehat{\phi}_S^{j,k}(T_D-1) &= \frac{-\widehat{\lambda}_S + \rho_{1,2}^{-1}\left(\widehat{\phi}_{\bar{C}}^{j,k}(T_D-1) + \rho_{2,3}\widehat{\phi}_Y^{j,k}(T_D-1)\right)}{1-\rho_{1,2}^{-1}\rho_{2,1} - \rho_{1,3}^{-1}\rho_{3,1}} \\
&\quad + \frac{\rho_{1,3}^{-1}\left(\widehat{\phi}_Y^{j,k}(T_D-1) + \rho_{3,2}\widehat{\phi}_{\bar{C}}^{j,k}(T_D-1)\right)}{1-\rho_{1,2}^{-1}\rho_{2,1} - \rho_{1,3}^{-1}\rho_{3,1}}.
\end{aligned}
\tag{A20}
$$

C.1.2 Remaining Periods of the Retirement Phase

We now determine $\widehat{\omega}^{*j,k}(t)$, $\widehat{\lambda}^{*j,k}_{\bar{C}}(t)$ and $\widehat{\lambda}^{*j,k}_{Y}(t)$ for all $t \in \{T_R, \dots, T_D - 2\}$. We start by deriving the change in dual human wealth. We find

$$\widehat{\boldsymbol{H}}(t+1) = \mathbb{E}_{t+1}\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{\boldsymbol{m}}(t+1)} \widehat{\boldsymbol{Y}}_2(t+1+l)\right], \tag{A21}$$

$$\begin{aligned} \widehat{H}^{j,k}(t) &= \widehat{Y}^j_2(t) + \mathbb{E}_t\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+l+1)}{\widehat{m}^k(t)} \widehat{\boldsymbol{Y}}_2(t+l+1)\right] \\ &= \widehat{Y}^j_2(t) + \mathbb{E}_t\left\{\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)} \mathbb{E}_{t+1}\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+l+1)}{\widehat{\boldsymbol{m}}(t+1)} \widehat{\boldsymbol{Y}}_2(t+l+1)\right]\right\}, \\ &= \widehat{Y}^j_2(t) + \mathbb{E}_t\left[\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)} \widehat{\boldsymbol{H}}(t+1)\right]. \end{aligned} \tag{A22}$$

In the previous step, we have determined $\widehat{\boldsymbol{H}}(t+1)$. Using OLS regression, we now determine the coefficients $\eta^H_0(t+1)$, $\eta^H_1(t+1)$ and $\eta^H_2(t+1)$ such that

$$\log\left(-\widehat{\boldsymbol{H}}(t+1)\right) \approx \eta^H_0(t+1) + \eta^H_1(t+1)\log\left(-\widehat{\boldsymbol{Y}}_2(t+1)\right) + \eta^H_2(t+1)\log \widehat{\boldsymbol{m}}(t+1). \tag{A23}$$

Hence,

$$\begin{aligned} \widehat{H}^{j,k}(t) = \widehat{Y}^j_2(t) - e^{\eta^H_0(t+1)}&\left(-\widehat{Y}^j_2(t)\right)^{\eta^H_1(t+1)} \left(\widehat{m}^k(t)\right)^{\eta^H_2(t+1)} \\ &\times \mathbb{E}_t\left[\left(\frac{\widehat{\boldsymbol{Y}}_2(t+1)}{\widehat{Y}^j_2(t)}\right)^{\eta^H_1(t+1)} \left(\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\right)^{\eta^H_2(t+1)+1}\right]. \end{aligned} \tag{A24}$$

We now find that

$$\begin{aligned} \Delta\widehat{\boldsymbol{H}}(t+1) \approx -e^{\eta^H_0(t+1)}&\left(-\widehat{Y}^j_2(t)\right)^{\eta^H_1(t+1)} \left(\widehat{m}^k(t)\right)^{\eta^H_2(t+1)} \\ &\times\left\{\dots + \left(\eta^H_1(t+1)\sigma_{\bar{C}} + \eta^H_2(t+1)\widehat{\phi}^{j,k}_{\bar{C}}(t)\right)\epsilon_{\bar{C}}(t+1)\right. \\ &\qquad \left. + \eta^H_2(t+1)\widehat{\phi}^{j,k}_S(t)\epsilon_S(t+1)\right\} - \widehat{Y}^j_2(t). \end{aligned} \tag{A25}$$

We now derive the change in the value of future dual consumption. We find

$$\widehat{\boldsymbol{V}}(t+1) = \mathbb{E}_{t+1}\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{\boldsymbol{m}}(t+1)}\widehat{\boldsymbol{c}}(t+1+l)\right], \tag{A26}$$

$$\begin{aligned}
\widehat{V}^{j,k}(t) &= \widehat{c}^{\,k}(t) + \mathbb{E}_t\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{m}^k(t)}\widehat{\boldsymbol{c}}(t+1+l)\right]. \\
&= \widehat{c}^{\,k}(t) + \mathbb{E}_t\left\{\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\mathbb{E}_{t+1}\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{\boldsymbol{m}}(t+1)}\widehat{\boldsymbol{c}}(t+l+1)\right]\right\}, \qquad \text{(A27)} \\
&= \widehat{c}^{\,k}(t) + \mathbb{E}_t\left[\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\widehat{\boldsymbol{V}}(t+1)\right],
\end{aligned}$$

with

$$\widehat{c}^{\,k}(t) = \left(e^{\delta(t-1)} y \frac{\widehat{m}^k(t)}{\widehat{m}(1)}\right)^{-\frac{1}{\gamma}}. \tag{A28}$$

In the previous step, we have determined $\widehat{\boldsymbol{V}}(t+1)$. Using OLS regression, we now determine the coefficients $\eta_0^V(t+1)$, $\eta_1^V(t+1)$ and $\eta_2^V(t+1)$ such that

$$\log\widehat{\boldsymbol{V}}(t+1) \approx \eta_0^V(t+1) + \eta_1^V(t+1)\log\left(-\widehat{\boldsymbol{Y}}_2(t+1)\right) + \eta_2^V(t+1)\log\widehat{\boldsymbol{m}}(t+1). \tag{A29}$$

Hence,

$$\begin{aligned}
\widehat{V}^{j,k}(t) &= \widehat{c}^{\,k}(t) + e^{\eta_0^V(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_1^V(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_2^V(t+1)} \\
&\quad \times \mathbb{E}_t\left[\left(\frac{\widehat{\boldsymbol{Y}}_2(t+1)}{\widehat{Y}_2^j(t)}\right)^{\eta_1^V(t+1)}\left(\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\right)^{\eta_2^V(t+1)+1}\right].
\end{aligned} \tag{A30}$$

We now find that

$$\begin{aligned}
\Delta\widehat{\boldsymbol{V}}(t+1) &\approx e^{\eta_0^V(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_1^V(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_2^V(t+1)} \\
&\quad \times\left\{\ldots + \left(\eta_1^V(t+1)\sigma_{\bar{C}} + \eta_2^V(t+1)\widehat{\phi}_{\bar{C}}^{j,k}(t)\right)\epsilon_{\bar{C}}(t+1)\right. \\
&\quad\quad \left. + \eta_2^V(t+1)\widehat{\phi}_S^{j,k}(t)\epsilon_S(t+1)\right\} - \widehat{c}^{\,k}(t).
\end{aligned} \tag{A31}$$

We finally derive the change in dual pension wealth. We find

$$\Delta\widehat{\boldsymbol{F}}(t+1) \approx \widehat{Y}_2^j(t) - \widehat{c}^{\,k}(t) + \widehat{F}^{j,k}(t)\left(\widehat{r}(t) + \widehat{\omega}^{j,k}(t)\lambda_S\sigma_S + \widehat{\omega}^{j,k}(t)\sigma_S\epsilon_S(t+1)\right). \tag{A32}$$

Here,

$$\begin{aligned}\widehat{F}^{j,k}(t) = {} & e^{\eta_0^V(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_1^V(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_2^V(t+1)} \\ & + e^{\eta_0^H(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_1^H(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_2^H(t+1)}.\end{aligned} \tag{A33}$$

Using the condition $\Delta\widehat{\boldsymbol{V}}(t+1) = \Delta\widehat{\boldsymbol{H}}(t+1) + \Delta\widehat{\boldsymbol{F}}(t+1)$, we arrive at

$$\widehat{\omega}^{*j,k}(t) = \eta_2^V(t+1)\widehat{\phi}_S^{j,k}(t)\frac{\widetilde{V}^{j,k}(t)}{\sigma_S\widehat{F}^{j,k}(t)} - \eta_2^H(t+1)\widehat{\phi}_S^{j,k}(t)\frac{\widetilde{H}^{j,k}(t)}{\sigma_S\widehat{F}^{j,k}(t)}, \tag{A34}$$

$$\widehat{\phi}_{\bar{C}}^{*j,k}(t) = \sigma_{\bar{C}}\frac{\eta_1^V(t+1)\widetilde{V}^{j,k}(t) - \eta_1^H(t+1)\widetilde{H}^{j,k}(t)}{\eta_2^H(t+1)\widetilde{H}^{j,k}(t) - \eta_2^V(t+1)\widetilde{V}^{j,k}(t)}, \tag{A35}$$

$$\widehat{\phi}_Y^{*j,k}(t) = 0, \tag{A36}$$

with

$$\widetilde{H}^{j,k}(t) \equiv -e^{\eta_0^H(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_1^H(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_2^H(t+1)}, \tag{A37}$$

$$\widetilde{V}^{j,k}(t) \equiv e^{\eta_0^V(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_1^V(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_2^V(t+1)}. \tag{A38}$$

C.1.3 Last Period of the Working Phase

Now we determine for every combination $\left\{\widehat{Y}_1^i(T_R-1), \widehat{Y}_2^j(T_R-1), \widehat{m}^k(T_R-1)\right\}$, $\widehat{\omega}^{*i,j,k}(T_R-1)$, $\widehat{\lambda}_{\bar{C}}^{*i,j,k}(T_R-1)$ and $\widehat{\lambda}_Y^{*i,j,k}(T_R-1)$. To do so, we need to derive the change in dual human wealth, the change in the value of future dual consumption and the change in dual pension wealth. We start by deriving the change in dual human wealth.

We find that $\widehat{H}_1^{i,j,k}(T_R-1) = \widehat{Y}_1^i(T_R-1) + \mathbb{E}_{T_R-1}\left[\frac{\widehat{\boldsymbol{m}}(T_R)}{\widehat{m}^k(T_R-1)}\widehat{\boldsymbol{Y}}_1(T_R)\right]$ and $\widehat{\boldsymbol{H}}_1(T_R) = \widehat{\boldsymbol{Y}}_1(T_R)$. Hence,

$$\begin{aligned}\widehat{H}_1^{i,j,k}(T_R-1) &= \widehat{Y}_1^i(T_R-1) + \widehat{Y}_1^i(T_R-1)\mathbb{E}_{T_R-1}\left[\frac{\widehat{\boldsymbol{m}}(T_R)}{\widehat{m}^k(T_R-1)}\frac{\widehat{\boldsymbol{Y}}_{\boldsymbol{1}}(\boldsymbol{T_R})}{\widehat{Y}_1^i(T_R-1)}\right] \\ &= \widehat{Y}_1^i(T_R-1) + \widehat{Y}_1^i(T_R-1)e^{-\widehat{r}(T_R-1)+\mu_{\widehat{Y}_1}(T_R-1)+\frac{1}{2}\sigma_Y^2-\sigma_Y\widehat{\lambda}_Y^{i,j,k}(T_R-1)},\end{aligned} \tag{A39}$$

with $\mu_{\widehat{Y}_1}(t) \equiv \mu_Y + \mathrm{d}\log\left(1-\beta_Y f_Y(t)\right)/\mathrm{d}t - \mathrm{d}\log\left(1+\beta_c f_c(t)\right)/\mathrm{d}t$. Furthermore, we find that

$$\widehat{\boldsymbol{H}}_1(T_R) = \widehat{Y}_1^i(T_R-1)e^{\mu_{\widehat{Y}_1}(T_R-1)+\sigma_Y\epsilon_Y(T_R)}. \tag{A40}$$

Here, $\epsilon_Y(T_R)$ is the unexpected non-tradable labor income shock between the beginning of period T_R-1 and the beginning of period T_R. The first part of dual human wealth changes as follows:

$$\begin{aligned}\Delta\widehat{\boldsymbol{H}}_1(T_R) &= \widehat{\boldsymbol{H}}_1(T_R) - \widehat{H}_1^{i,j,k}(T_R-1)\\ &= \widehat{Y}_1^i(T_R-1)\Big\{e^{\mu_{\widehat{Y}_1}(T_R-1)+\sigma_Y\epsilon_Y(T_R)}\\ &\qquad -e^{-\widehat{r}(T_R-1)+\mu_{\widehat{Y}_1}(T_R-1)+\frac{1}{2}\sigma_Y^2-\sigma_Y\widehat{\lambda}_Y^{i,j,k}(T_R-1)}-1\Big\}\\ &\approx \widehat{Y}_1^i(T_R-1)\Big\{\widehat{r}(T_R-1)+\sigma_Y\widehat{\lambda}_Y^{i,j,k}(T_R-1)+\sigma_Y\epsilon_Y(T_R)-1\Big\}.\end{aligned} \tag{A41}$$

Here, we have used the approximations $e^x \approx 1+x$ and $e^{-\frac{1}{2}\sigma^2+\sigma\epsilon} \approx 1+\sigma\epsilon$ with $\epsilon \sim N(0,1)$.

We now derive the second part of dual human wealth. We find

$$\widehat{\boldsymbol{H}}_2(T_R) = \mathbb{E}_{T_R}\left[\sum_{l=0}^{T_D-T_R}\frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{\boldsymbol{m}}(T_R)}\widehat{\boldsymbol{Y}}_2(T_R+l)\right], \tag{A42}$$

$$\begin{aligned}\widehat{H}_2^{i,j,k}(T_R-1) &= \widehat{Y}_2^j(T_R-1) + \mathbb{E}_{T_R-1}\left[\sum_{l=0}^{T_D-T_R}\frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{m}^k(T_R-1)}\widehat{\boldsymbol{Y}}_2(T_R+l)\right]\\ &= \widehat{Y}_2^j(T_R-1) + \mathbb{E}_{T_R-1}\Bigg\{\frac{\widehat{\boldsymbol{m}}(T_R)}{\widehat{m}^k(T_R-1)}\\ &\qquad \mathbb{E}_{T_R}\left[\sum_{l=0}^{T_D-T_R}\frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{\boldsymbol{m}}(T_R)}\widehat{\boldsymbol{Y}}_2(T_R+l)\right]\Bigg\},\\ &= \widehat{Y}_2^j(T_R-1) + \mathbb{E}_{T_R-1}\left[\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(T_R-1)}\widehat{\boldsymbol{H}}_2(T_R)\right].\end{aligned} \tag{A43}$$

In the previous step, we have determined $\widehat{\boldsymbol{H}}_2(T_R) = \widehat{\boldsymbol{H}}(T_R)$. Using OLS regression, we now determine the coefficients $\eta_0^{H_2}(T_R)$, $\eta_1^{H_2}(T_R)$ and $\eta_2^{H_2}(T_R)$ such that

$$\log\left(-\widehat{\boldsymbol{H}}_2(T_R)\right) \approx \eta_0^{H_2}(T_R) + \eta_1^{H_2}(T_R)\log\left(-\widehat{\boldsymbol{Y}}_2(T_R)\right) + \eta_2^{H_2}(T_R)\log\widehat{\boldsymbol{m}}(T_R). \tag{A44}$$

Hence,

$$
\begin{aligned}
\widehat{H}_2^{i,j,k}(T_R-1) = {}& \widehat{Y}_2^j(T_R-1) \\
& - e^{\eta_0^{H_2}(T_R)} \left(-\widehat{Y}_2^j(T_R-1)\right)^{\eta_1^{H_2}(T_R)} \left(\widehat{m}^k(T_R-1)\right)^{\eta_2^{H_2}(T_R)} \\
& \times \mathbb{E}_{T_R-1}\left[\left(\frac{\widehat{\boldsymbol{Y}}_2(T_R)}{\widehat{Y}_2^j(T_R-1)}\right)^{\eta_1^{H_2}(T_R)} \left(\frac{\widehat{\boldsymbol{m}}(T_R)}{\widehat{m}^k(T_R-1)}\right)^{\eta_2^{H_2}(T_R)+1}\right].
\end{aligned} \tag{A45}
$$

We now find that

$$
\begin{aligned}
\Delta\widehat{\boldsymbol{H}}_2(T_R) \approx {}& -e^{\eta_0^{H_2}(T_R)} \left(-\widehat{Y}_2^j(T_R-1)\right)^{\eta_1^{H_2}(T_R)} \left(\widehat{m}^k(T_R-1)\right)^{\eta_2^{H_2}(T_R)} \\
& \times \Big\{ \ldots + \left(\eta_1^{H_2}(T_R)\sigma_{\bar{C}} + \eta_2^{H_2}(T_R)\widehat{\phi}_{\bar{C}}^{i,j,k}(T_R-1)\right)\epsilon_{\bar{C}}(T_R) \\
& \quad + \eta_2^{H_2}(T_R)\widehat{\phi}_Y^{i,j,k}(T_R-1)\epsilon_Y(T_R) + \eta_2^{H_2}(T_R)\widehat{\phi}_S^{i,j,k}(T_R-1)\epsilon_S(T_R)\Big\} \\
& - \widehat{Y}_2^j(T_R-1).
\end{aligned} \tag{A46}
$$

We now derive the change in the value of future dual consumption. We find

$$
\widehat{\boldsymbol{V}}(T_R) = \mathbb{E}_{T_R}\left[\sum_{l=0}^{T_D-T_R} \frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{\boldsymbol{m}}(T_R)}\widehat{\boldsymbol{c}}(T_R+l)\right], \tag{A47}
$$

$$
\begin{aligned}
\widehat{V}^{i,j,k}(T_R-1) &= \widehat{c}^{\,k}(T_R-1) + \mathbb{E}_{T_R-1}\left[\sum_{l=0}^{T_D-T_R} \frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{m}^k(T_R-1)}\widehat{\boldsymbol{c}}(T_R+l)\right]. \\
&= \widehat{c}^{\,k}(T_R-1) + \mathbb{E}_{T_R-1}\left\{\frac{\widehat{\boldsymbol{m}}(T_R)}{\widehat{m}^k(T_R-1)}\right. \\
&\qquad \left. \mathbb{E}_{T_R}\left[\sum_{l=0}^{T_D-T_R} \frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{\boldsymbol{m}}(T_R)}\widehat{\boldsymbol{c}}(T_R+l)\right]\right\}, \\
&= \widehat{c}^{\,k}(T_R-1) + \mathbb{E}_{T_R-1}\left[\frac{\widehat{\boldsymbol{m}}(T_R)}{\widehat{m}^k(T_R-1)}\widehat{\boldsymbol{V}}(T_R)\right],
\end{aligned} \tag{A48}
$$

with

$$
\widehat{c}^{\,k}(T_R-1) = \left(e^{\delta(T_R-2)} y \frac{\widehat{m}^k(T_R-1)}{\widehat{m}(1)}\right)^{-\frac{1}{\gamma}}. \tag{A49}
$$

In the previous step, we have determined $\widehat{\boldsymbol{V}}(T_R)$. Using OLS regression, we now determine the coefficients $\eta_0^V(T_R)$, $\eta_1^V(T_R)$ and $\eta_2^V(T_R)$ such that

$$\log \widehat{\boldsymbol{V}}(T_R) \approx \eta_0^V(T_R) + \eta_1^V(T_R) \log\left(-\widehat{\boldsymbol{Y}}_2(T_R)\right) + \eta_2^V(T_R) \log \widehat{\boldsymbol{m}}(T_R). \tag{A50}$$

Hence,

$$\begin{aligned}\widehat{V}^{i,j,k}(T_R-1) = \widehat{c}^{\,k}(T_R-1) + e^{\eta_0^V(T_R)} \left(-\widehat{Y}_2^j(T_R-1)\right)^{\eta_1^V(T_R)} \left(\widehat{m}^k(T_R-1)\right)^{\eta_2^V(T_R)} \\ \times \mathbb{E}_{T_R-1}\left[\left(\frac{\widehat{\boldsymbol{Y}}_2(T_R)}{\widehat{Y}_2^j(T_R-1)}\right)^{\eta_1^V(T_R)} \left(\frac{\widehat{\boldsymbol{m}}(T_R)}{\widehat{m}^k(T_R-1)}\right)^{\eta_2^V(T_R)+1}\right].\end{aligned} \tag{A51}$$

We now find that

$$\begin{aligned}\Delta\widehat{\boldsymbol{V}}(T_R) \approx{}& e^{\eta_0^V(T_R)} \left(-\widehat{Y}_2^j(T_R-1)\right)^{\eta_1^V(T_R)} \left(\widehat{m}^k(T_R-1)\right)^{\eta_2^V(T_R)} \\ &\times \Big\{ \ldots + \left(\eta_1^V(T_R)\sigma_{\bar{C}} + \eta_2^V(T_R)\widehat{\phi}_{\bar{C}}^{i,j,k}(T_R-1)\right)\epsilon_{\bar{C}}(T_R) \\ &\quad + \eta_2^V(T_R)\widehat{\phi}_Y^{i,j,k}(T_R-1)\epsilon_Y(T_R) + \eta_2^V(T_R)\widehat{\phi}_S^{i,j,k}(T_R-1)\epsilon_S(T_R)\Big\} \\ &- \widehat{c}^{\,k}(T_R-1).\end{aligned} \tag{A52}$$

We finally derive the change in optimal dual pension wealth. We find

$$\begin{aligned}\Delta\widehat{\boldsymbol{F}}(T_R) \approx{}& \widehat{Y}_1^i(T_R-1) + \widehat{Y}_2^j(T_R-1) - \widehat{c}^{\,k}(T_R-1) \\ &+ \widehat{F}^{i,j,k}(T_R-1)\big(\widehat{r}(T_R-1) + \widehat{\omega}^{i,j,k}(T_R-1)\lambda_S\sigma_S \\ &\qquad + \widehat{\omega}^{i,j,k}(T_R-1)\sigma_S\epsilon_S(T_R)\big).\end{aligned} \tag{A53}$$

Here,

$$\begin{aligned}\widehat{F}^{i,j,k}(T_R-1) ={}& e^{\eta_0^V(T_R)} \left(-\widehat{Y}_2^j(T_R-1)\right)^{\eta_1^V(T_R)} \left(\widehat{m}^k(T_R-1)\right)^{\eta_2^V(T_R)} \\ &+ e^{\eta_0^{H_2}(T_R)} \left(-\widehat{Y}_2^j(T_R-1)\right)^{\eta_1^{H_2}(T_R)} \left(\widehat{m}^k(T_R-1)\right)^{\eta_2^{H_2}(T_R)} \\ &- \widehat{Y}_1^i(T_R-1).\end{aligned} \tag{A54}$$

Using the condition $\Delta\widehat{\boldsymbol{V}}(T_R) = \Delta\widehat{\boldsymbol{H}}_1(T_R) + \Delta\widehat{\boldsymbol{H}}_2(T_R) + \Delta\widehat{\boldsymbol{F}}(T_R)$, we arrive at

$$\begin{aligned}\widehat{\omega}^{*i,j,k}(T_R-1) = \eta_2^V(T_R)\widehat{\phi}_S^{i,j,k}(T_R-1)\frac{\widetilde{V}^{i,j,k}(T_R-1)}{\sigma_S\widehat{F}^{i,j,k}(T_R-1)} \\ - \eta_2^{H_2}(T_R)\widehat{\phi}_S^{i,j,k}(T_R-1)\frac{\widetilde{H}_2^{i,j,k}(T_R-1)}{\sigma_S\widehat{F}^{i,j,k}(T_R-1)},\end{aligned} \tag{A55}$$

$$\widehat{\phi}_{\bar{C}}^{*i,j,k}(T_R-1) = \sigma_{\bar{C}}\frac{\eta_1^V(T_R)\widetilde{V}^{i,j,k}(T_R-1) - \eta_1^{H_2}(T_R)\widetilde{H}_2^{i,j,k}(T_R-1)}{\eta_2^{H_2}(T_R)\widetilde{H}_2^{i,j,k}(T_R-1) - \eta_2^V(T_R)\widetilde{V}^{i,j,k}(T_R-1)}, \tag{A56}$$

$$\widehat{\phi}_Y^{*i,j,k}(T_R-1) = \frac{\sigma_Y\widehat{Y}_1^i(T_R-1)}{\widetilde{V}^{i,j,k}(T_R-1)\eta_2^V(T_R) - \widetilde{H}_2^{i,j,k}(T_R-1)\eta_2^{H_2}(T_R)}, \tag{A57}$$

with

$$\widetilde{H}_2^{i,j,k}(T_R-1) \equiv -e^{\eta_0^{H_2}(T_R)}\left(-\widehat{Y}_2^j(T_R-1)\right)^{\eta_1^{H_2}(T_R)}\left(\widehat{m}^k(T_R-1)\right)^{\eta_2^{H_2}(T_R)}, \tag{A58}$$

$$\widetilde{V}^{i,j,k}(T_R-1) \equiv e^{\eta_0^V(T_R)}\left(-\widehat{Y}_2^j(T_R-1)\right)^{\eta_1^V(T_R)}\left(\widehat{m}^k(T_R-1)\right)^{\eta_2^V(T_R)}. \tag{A59}$$

C.1.4 Remaining Periods of the Working Phase

We now determine $\widehat{\omega}^{*i,j,k}(t)$, $\widehat{\lambda}_{\bar{C}}^{*i,j,k}(t)$ and $\widehat{\lambda}_Y^{*i,j,k}(t)$ for all $t \in \{1,\ldots,T_R-2\}$. We start by deriving the change in the first and second part of dual human wealth. We find

$$\widehat{\boldsymbol{H}}_1(t+1) = \mathbb{E}_{t+1}\left[\sum_{l=0}^{T_R-t-1}\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{\boldsymbol{m}}(t+1)}\widehat{\boldsymbol{Y}}_1(t+1+l)\right], \tag{A60}$$

$$\begin{aligned}\widehat{H}_1^{i,j,k}(t) &= \widehat{Y}_1^i(t) + \mathbb{E}_t\left[\sum_{l=0}^{T_R-t-1}\frac{\widehat{\boldsymbol{m}}(t+l+1)}{\widehat{m}^k(t)}\widehat{\boldsymbol{Y}}_1(t+l+1)\right] \\ &= \widehat{Y}_1^i(t) + \mathbb{E}_t\left\{\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\mathbb{E}_{t+1}\left[\sum_{l=0}^{T_R-t-1}\frac{\widehat{\boldsymbol{m}}(t+l+1)}{\widehat{\boldsymbol{m}}(t+1)}\widehat{\boldsymbol{Y}}_1(t+l+1)\right]\right\}, \\ &= \widehat{Y}_1^i(t) + \mathbb{E}_t\left[\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\widehat{\boldsymbol{H}}_1(t+1)\right],\end{aligned} \tag{A61}$$

and

$$\widehat{\boldsymbol{H}}_2(t+1) = \mathbb{E}_{t+1}\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{\boldsymbol{m}}(t+1)} \widehat{\boldsymbol{Y}}_2(t+1+l)\right], \tag{A62}$$

$$\begin{aligned}
\widehat{H}_2^{i,j,k}(t) &= \widehat{Y}_2^j(t) + \mathbb{E}_t\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+l+1)}{\widehat{m}^k(t)} \widehat{\boldsymbol{Y}}_2(t+l+1)\right] \\
&= \widehat{Y}_2^j(t) + \mathbb{E}_t\left\{\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)} \mathbb{E}_{t+1}\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+l+1)}{\widehat{\boldsymbol{m}}(t+1)} \widehat{\boldsymbol{Y}}_2(t+l+1)\right]\right\} \\
&= \widehat{Y}_2^j(t) + \mathbb{E}_t\left[\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)} \widehat{\boldsymbol{H}}_2(t+1)\right].
\end{aligned} \tag{A63}$$

In the previous step, we have determined $\widehat{\boldsymbol{H}}_1(t+1)$ and $\widehat{\boldsymbol{H}}_2(t+1)$. Using OLS regression, we now determine the coefficients $\eta_0^{H_1}(t+1)$, $\eta_0^{H_2}(t+1)$, $\eta_1^{H_1}(t+1)$, $\eta_1^{H_2}(t+1)$, $\eta_2^{H_1}(t+1)$, $\eta_2^{H_2}(t+1)$, $\eta_3^{H_1}(t+1)$ and $\eta_3^{H_2}(t+1)$ such that

$$\begin{aligned}
\log\left(\widehat{\boldsymbol{H}}_1(t+1)\right) \approx& \eta_0^{H_1}(t+1) + \eta_1^{H_1}(t+1)\log\left(-\widehat{\boldsymbol{Y}}_2(t+1)\right) \\
&+ \eta_2^{H_1}(t+1)\log\widehat{\boldsymbol{m}}(t+1) + \eta_3^{H_1}(t+1)\log\widehat{\boldsymbol{Y}}_1(t+1),
\end{aligned} \tag{A64}$$

$$\begin{aligned}
\log\left(-\widehat{\boldsymbol{H}}_2(t+1)\right) \approx& \eta_0^{H_2}(t+1) + \eta_1^{H_2}(t+1)\log\left(-\widehat{\boldsymbol{Y}}_2(t+1)\right) \\
&+ \eta_2^{H_2}(t+1)\log\widehat{\boldsymbol{m}}(t+1) + \eta_3^{H_2}(t+1)\log\widehat{\boldsymbol{Y}}_1(t+1).
\end{aligned} \tag{A65}$$

Hence,

$$\begin{aligned}
\widehat{H}_1^{i,j,k}(t) =& \widehat{Y}_1^i(t) + e^{\eta_0^{H_1}(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_1^{H_1}(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_2^{H_1}(t+1)}\left(\widehat{Y}_1^i(t)\right)^{\eta_3^{H_1}(t+1)} \\
&\times \mathbb{E}_t\left[\left(\frac{\widehat{\boldsymbol{Y}}_2(t+1)}{\widehat{Y}_2^j(t)}\right)^{\eta_1^{H_1}(t+1)}\left(\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\right)^{\eta_2^{H_1}(t+1)+1}\left(\frac{\widehat{\boldsymbol{Y}}_1(t+1)}{\widehat{Y}_1^i(t)}\right)^{\eta_3^{H_1}(t+1)}\right], \\
\widehat{H}_2^{i,j,k}(t) =& \widehat{Y}_2^j(t) - e^{\eta_0^{H_2}(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_1^{H_2}(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_2^{H_2}(t+1)}\left(\widehat{Y}_1^i(t)\right)^{\eta_3^{H_2}(t+1)} \\
&\times \mathbb{E}_t\left[\left(\frac{\widehat{\boldsymbol{Y}}_2(t+1)}{\widehat{Y}_2^j(t)}\right)^{\eta_1^{H_2}(t+1)}\left(\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\right)^{\eta_2^{H_2}(t+1)+1}\left(\frac{\widehat{\boldsymbol{Y}}_1(t+1)}{\widehat{Y}_1^i(t)}\right)^{\eta_3^{H_2}(t+1)}\right].
\end{aligned}$$

We now find that

$$
\begin{aligned}
\Delta\widehat{\boldsymbol{H}}_1(t+1) \approx\, & e^{\eta_0^{H_1}(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_1^{H_1}(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_2^{H_1}(t+1)}\left(\widehat{Y}_1^i(t)\right)^{\eta_3^{H_1}(t+1)} \\
& \times\Big\{\ldots+\left(\eta_1^{H_1}(t+1)\sigma_{\bar{C}}+\eta_2^{H_1}(t+1)\widehat{\phi}_{\bar{C}}^{i,j,k}(t)\right)\epsilon_{\bar{C}}(t+1) \\
& \quad+\left(\eta_3^{H_1}(t+1)\sigma_Y+\eta_2^{H_1}(t+1)\widehat{\phi}_Y^{i,j,k}(t)\right)\epsilon_Y(t+1) \\
& \quad+\eta_2^{H_1}(t+1)\widehat{\phi}_S^{i,j,k}(t)\epsilon_S(t+1)\Big\}-\widehat{Y}_1^i(t),
\end{aligned}
\tag{A66}
$$

$$
\begin{aligned}
\Delta\widehat{\boldsymbol{H}}_2(t+1) \approx\, & -e^{\eta_0^{H_2}(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_1^{H_2}(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_2^{H_2}(t+1)}\left(\widehat{Y}_1^i(t)\right)^{\eta_3^{H_2}(t+1)} \\
& \times\Big\{\ldots+\left(\eta_1^{H_2}(t+1)\sigma_{\bar{C}}+\eta_2^{H_2}(t+1)\widehat{\phi}_{\bar{C}}^{i,j,k}(t)\right)\epsilon_{\bar{C}}(t+1) \\
& \quad+\left(\eta_3^{H_2}(t+1)\sigma_Y+\eta_2^{H_2}(t+1)\widehat{\phi}_Y^{i,j,k}(t)\right)\epsilon_Y(t+1) \\
& \quad+\eta_2^{H_2}(t+1)\widehat{\phi}_S^{i,j,k}(t)\epsilon_S(t+1)\Big\}-\widehat{Y}_2^j(t).
\end{aligned}
\tag{A67}
$$

We now derive the change in the value of future dual consumption. We find

$$
\widehat{\boldsymbol{V}}(t+1)=\mathbb{E}_{t+1}\left[\sum_{l=0}^{T_D-t-1}\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{\boldsymbol{m}}(t+1)}\widehat{\boldsymbol{c}}(t+1+l)\right], \tag{A68}
$$

$$
\begin{aligned}
\widehat{V}^{i,j,k}(t) &= \widehat{c}^{\,k}(t)+\mathbb{E}_t\left[\sum_{l=0}^{T_D-t-1}\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{m}^k(t)}\widehat{\boldsymbol{c}}(t+1+l)\right]. \\
&= \widehat{c}^{\,k}(t)+\mathbb{E}_t\left\{\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\mathbb{E}_{t+1}\left[\sum_{l=0}^{T_D-t-1}\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{\boldsymbol{m}}(t+1)}\widehat{\boldsymbol{c}}(t+l+1)\right]\right\}, \\
&= \widehat{c}^{\,k}(t)+\mathbb{E}_t\left[\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\widehat{\boldsymbol{V}}(t+1)\right],
\end{aligned}
\tag{A69}
$$

with

$$
\widehat{c}^{\,k}(t)=\left(e^{\delta(t-1)}y\frac{\widehat{m}^k(t)}{\widehat{m}(1)}\right)^{-\frac{1}{\gamma}}. \tag{A70}
$$

In the previous step, we have determined $\widehat{\boldsymbol{V}}(t+1)$. Using OLS regression, we now determine the coefficients $\eta_0^V(t+1)$, $\eta_1^V(t+1)$, $\eta_2^V(t+1)$ and $\eta_3^V(t+1)$ such that

$$
\begin{aligned}
\log\left(\widehat{\boldsymbol{V}}(t+1)\right) \approx\, & \eta_0^V(t+1)+\eta_1^V(t+1)\log\left(-\widehat{\boldsymbol{Y}}_2(t+1)\right) \\
& +\eta_2^V(t+1)\log\widehat{\boldsymbol{m}}(t+1)+\eta_3^V(t+1)\log\widehat{\boldsymbol{Y}}_1(t+1).
\end{aligned}
\tag{A71}
$$

Hence,

$$\widehat{V}^{i,j,k}(t) = \widehat{c}^{\,k}(t) + e^{\eta_0^V(t+1)} \left(-\widehat{Y}_2^j(t)\right)^{\eta_1^V(t+1)} \left(\widehat{m}^k(t)\right)^{\eta_2^V(t+1)} \left(\widehat{Y}_1^i(t)\right)^{\eta_3^V(t+1)} \times \mathbb{E}_t \left[\left(\frac{\widehat{\boldsymbol{Y}}_2(t+1)}{\widehat{Y}_2^j(t)}\right)^{\eta_1^V(t+1)} \left(\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\right)^{\eta_2^V(t+1)+1} \left(\frac{\widehat{\boldsymbol{Y}}_1(t+1)}{\widehat{Y}_1^i(t)}\right)^{\eta_3^V(t+1)} \right].$$

We now find that

$$\begin{aligned} \Delta\widehat{\boldsymbol{V}}(t+1) \approx{} & e^{\eta_0^V(t+1)} \left(-\widehat{Y}_2^j(t)\right)^{\eta_1^V(t+1)} \left(\widehat{m}^k(t)\right)^{\eta_2^V(t+1)} \left(\widehat{Y}_1^i(t)\right)^{\eta_3^V(t+1)} \\ & \times \Big\{ \ldots + \left(\eta_1^V(t+1)\sigma_{\bar{C}} + \eta_2^V(t+1)\widehat{\phi}_{\bar{C}}^{i,j,k}(t)\right)\epsilon_{\bar{C}}(t+1) \\ & \times \Big\{ \ldots + \left(\eta_3^V(t+1)\sigma_Y + \eta_2^V(t+1)\widehat{\phi}_Y^{i,j,k}(t)\right)\epsilon_Y(t+1) \\ & + \eta_2^V(t+1)\widehat{\phi}_S^{i,j,k}(t)\epsilon_S(t+1)\Big\} - \widehat{c}^{\,k}(t). \end{aligned} \tag{A72}$$

We finally derive the change in dual pension wealth. We find

$$\begin{aligned} \Delta\widehat{\boldsymbol{F}}(t+1) \approx{} & \widehat{Y}_1^i(t) + \widehat{Y}_2^j(t) - \widehat{c}^{\,k}(t) \\ & + \widehat{F}^{i,j,k}(t)\left(\widehat{r}(t) + \widehat{\omega}^{i,j,k}(t)\lambda_S\sigma_S + \widehat{\omega}^{i,j,k}(t)\sigma_S\epsilon_S(t+1)\right). \end{aligned} \tag{A73}$$

Here,

$$\begin{aligned} \widehat{F}^{i,j,k}(t) ={} & e^{\eta_0^V(t+1)} \left(-\widehat{Y}_2^j(t)\right)^{\eta_1^V(t+1)} \left(\widehat{m}^k(t)\right)^{\eta_2^V(t+1)} \left(\widehat{Y}_1^i(t)\right)^{\eta_3^V(t+1)} \\ & + e^{\eta_0^{H_2}(t+1)} \left(-\widehat{Y}_2^j(t)\right)^{\eta_1^{H_2}(t+1)} \left(\widehat{m}^k(t)\right)^{\eta_2^{H_2}(t+1)} \left(\widehat{Y}_1^i(t)\right)^{\eta_3^{H_2}(t+1)} \\ & - e^{\eta_0^{H_1}(t+1)} \left(-\widehat{Y}_2^j(t)\right)^{\eta_1^{H_1}(t+1)} \left(\widehat{m}^k(t)\right)^{\eta_2^{H_1}(t+1)} \left(\widehat{Y}_1^i(t)\right)^{\eta_3^{H_1}(t+1)}. \end{aligned} \tag{A74}$$

Using the condition $\Delta\widehat{\boldsymbol{V}}(t+1) = \Delta\widehat{\boldsymbol{H}}_1(t+1) + \Delta\widehat{\boldsymbol{H}}_2(t+1) + \Delta\widehat{\boldsymbol{F}}(t+1)$, we arrive at

$$\begin{aligned}\widehat{\omega}^{*i,j,k}(t) &= \eta_2^V(t+1)\widehat{\phi}_S^{i,j,k}(t)\frac{\widetilde{V}^{i,j,k}(t)}{\sigma_S\widehat{F}^{i,j,k}(t)} - \eta_2^{H_1}(t+1)\widehat{\phi}_S^{i,j,k}(t)\frac{\widetilde{H}_1^{i,j,k}(t)}{\sigma_S\widehat{F}^{i,j,k}(t)} \\ &\quad - \eta_2^{H_2}(t+1)\widehat{\phi}_S^{i,j,k}(t)\frac{\widetilde{H}_2^{i,j,k}(t)}{\sigma_S\widehat{F}^{i,j,k}(t)},\end{aligned} \tag{A75}$$

$$\widehat{\phi}_{\bar{C}}^{*i,j,k}(t) = \sigma_{\bar{C}}\frac{\eta_1^V(t+1)\widetilde{V}^{i,j,k}(t) - \eta_1^{H_1}(t+1)\widetilde{H}_1^{i,j,k}(t) - \eta_1^{H_2}(t+1)\widetilde{H}_2^{i,j,k}(t)}{\eta_2^{H_1}(t+1)\widetilde{H}_1^{i,j,k}(t) + \eta_2^{H_2}(t+1)\widetilde{H}_2^{i,j,k}(t) - \eta_2^V(t+1)\widetilde{V}^{i,j,k}(t)}, \tag{A76}$$

$$\widehat{\phi}_Y^{*i,j,k}(t) = \sigma_Y\frac{\eta_3^V(t+1)\widetilde{V}^{i,j,k}(t) - \eta_3^{H_1}(t+1)\widetilde{H}_1^{i,j,k}(t) - \eta_3^{H_2}(t+1)\widetilde{H}_2^{i,j,k}(t)}{\eta_2^{H_1}(t+1)\widetilde{H}_1^{i,j,k}(t) + \eta_2^{H_2}(t+1)\widetilde{H}_2^{i,j,k}(t) - \eta_2^V(t+1)\widetilde{V}^{i,j,k}(t)}, \tag{A77}$$

with

$$\widetilde{H}_1^{i,j,k}(t) \equiv e^{\eta_0^{H_1}(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_1^{H_1}(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_2^{H_1}(t+1)}\left(\widehat{Y}_1^i(t)\right)^{\eta_3^{H_1}(t+1)}, \tag{A78}$$

$$\widetilde{H}_2^{i,j,k}(t) \equiv -e^{\eta_0^{H_2}(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_1^{H_2}(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_2^{H_2}(t+1)}\left(\widehat{Y}_1^i(t)\right)^{\eta_3^{H_2}(t+1)}, \tag{A79}$$

$$\widetilde{V}^{i,j,k}(t) \equiv e^{\eta_0^{V}(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_1^{V}(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_2^{V}(t+1)}\left(\widehat{Y}_1^i(t)\right)^{\eta_3^{V}(t+1)}. \tag{A80}$$

We note that in applications $\eta_3^V(t+1)$, $\eta_3^{H_2}(t+1)$, $\eta_2^{H_1}(t+1)$ and $\eta_2^{H_2}(t+1)$ are approximately equal to zero. Under these conditions, we thus find that

$$\widehat{\phi}_Y^{*i,j,k}(t) = \sigma_Y\frac{\eta_3^{H_1}(t+1)\widetilde{H}_1^{i,j,k}(t)}{\eta_2^V(t+1)\widetilde{V}^{i,j,k}(t)}, \tag{A81}$$

which reduces to $-\widehat{\lambda}_Y^{*i,j,k}(t)$ if $\rho_{3,1} = \rho_{3,2} = 0$.

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