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Abstract

We investigate the impact of stochastic volatility on target-based investment strategies. We compare the results under CRRA risk preference and SAHARA risk preference. By setting a positive reference level under SAHARA risk preference, investors tend to be highly risk averse around the reference level, which leads investors towards target-based investment strategies, and therefore leads to a distribution that is more centred around the reference level. We show that the target-based investment strategies are less sensitive to the value of the diffusive volatility risk premium than CRRA-based investment strategies. This is especially useful when it is difficult to estimate the value of diffusive volatility risk premium.

Keywords: Dynamic asset allocation, reference-dependent utility, stochastic volatility, optimal portfolio, target-based investment

JEL Classification: C61, D15, D53, G2, G11

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1 Introduction

Empirical work on index returns and options find strong evidence for stochastic volatility, for instance, Andersen, Benzoni, and Lund (2002) using the dataset S&P 500 returns for the period from 1953 to 1996, Chernov, Gallant, Ghysels, and Tauchen (2003) using the dataset Dow Jones 30 returns for the period from 1953 to 1999, and Bakshi, Cao, and Chen (1997) using a cross section of S&P 500 options for the period from 1988 to 1991. Stochastic volatility can impact the optimal allocation of assets in a long-term investment portfolio which typically has a horizon of several years or even decades. As volatility fluctuates, long-term investors who focus on the fundamental value of their investments rather than short-term price movements, may need to adjust their asset allocation to achieve their risk and return objectives. By accounting for the fluctuating nature of volatility, long-term investors can make more informed decisions about asset location within their portfolios.

One popular stochastic volatility model is the Heston model (Heston, 1993). The Heson model consists of two stochastic differential equations, one for the underlying asset's price and the other for its variance. The variance process follows a square root mean reverting process, namely the CIR process. The model helps capture the observed phenomena in financial markets such as volatility clustering, heavy tails and the smile of implied volatilities, see Tankov (2003). It also gives closed-form option pricing formula, which has a considerable computational advantage. Under the Heston model, perfect hedging by only trading the risky asset and the risk-free asset is not possible. Liu and Pan (2003) introduce derivatives to hedge the additional random source and assume that the risk premia are determined exogenously, then they are able to derive closed-form solutions. In their study, the portfolio improvement from participating in the derivatives market is considerable, in terms of the annualized, continuously compounded return in certainty-equivalent wealth for an investor with relative risk aversion of three (CRRA). It becomes higher when the market becomes more volatile. Moreover, this portfolio improvement is very sensitive to how diffusive volatility risk is priced, stated by Liu and Pan (2003).

Regarding the diffusive volatility risk premium, Broadie, Chernov, and Johannes (2007) point out that the evidence is inconclusive. First, theory provides no guidance regarding the sign of the diffusive volatility risk premium. Additionally, the studies that formally estimate diffusive volatility risk premia obtain conflicting results, because the estimates depend on the data set and the model specification used, see for instance Chernov and Ghysels (2000), Pan (2002), Jones (2003), Eraker (2004). Particularly, Jones (2003) show that the value as well as the sign of the estimates could differ when different data samples are used.

Regarding the utility functions, CRRA utility functions are broadly used within the academic literature. CRRA utility is defined over the level of the outcome being evaluated, hence it may be appropriate for an investor who is concerned with comparing outcomes on the basis of their level and spread but whose risk aversion does not change with the overall level. For investors who evaluate outcomes relative to some reference level, reference-dependent utility functions may be more suitable, stated in Warren (2019). Investing with a reference level typically involves setting a specific financial goal or objective before the investor starts investing. Within the framework of prospect theory, final and interim targets are usually considered as reference levels, see for instance Blake, Wright, and Zhang (2013) and Donnelly, Khemka, and Lim (2022). This helps create a focused investment strategy. Alternatively, the starting level can be considered as a reference level. Touré-Tillery and Fishbach (2012) suggest that people seem to be more motivated using the starting point as a reference level in the initial states of goal pursuit. Koop and Johnson (2012) argue that minimum requirement could be established as a reference level and that multiple reference levels could be included within the value function; Higgins and Liberman (2018) argue that people may use many kinds of reference levels at different times. In addition, Larrick, Heath, and Wu (2009) point out that goals induce risk taking and that specific, challenging goals lead to riskier strategies.

One reference-dependent utility function is SAHARA utility function. SAHARA stands for Symmetric Asymptotic Hyperbolic Absolute Risk Aversion. This class of utility functions was introduced by Chen, Pelsser, and Vellekoop (2011). It deviates from CRRA utility function in the functional form and consists of three parameters, i.e. parameter of risk aversion, scale and threshold. We interpret the threshold in the SAHARA utility function as the reference level. Nevertheless, CRRA utility function is a special case of SAHARA utility function, namely, given positive wealth and the threshold being set at zero wealth, the SAHARA utility function converges to the CRRA utility function (apart from an overall change of scale), when the value of the scale parameter converges to zero. By allowing the scale parameter to take small values, the SAHARA utility function can have potentially important implications for investment behaviour[1], such as locking-in certain gains, becoming overconfident when they have made gains and taking extra risks when they have made losses, which CRRA utility function fails to describe.

This paper contributes to the literature on optimal investment with reference-dependent preferences in a number of ways. We investigate the optimal dynamic investment strategies for long-term investors under the Heston model when the investors have SAHARA type of risk preference. We find that, when the reference level is imposed at zero wealth and the value of the scale parameter is small, the distribution under SAHARA risk preference is similar to that under CRRA risk preference with the same risk aversion. This enables us to analyse the impact of a positive reference level on the distribution of optimal terminal wealth. In addition, the reference level is time-varying, and our interim target fund can be modelled as a proportion of the future value of the initial fund under the risk-neutral measure throughout time. Compared

[1]See Mitchell and Utkus (2004) for a review of investment decision making in pension plans in a behavioural context.

to CRRA risk preference, the distribution of the optimal terminal wealth under SAHARA risk preference is more concentrated around the reference level. Namely, under SAHARA risk preference, investors become highly risk averse around the reference level, which introduces opportunity cost by limiting their ability to invest in risky assets, especially when the financial market is favourable, but also prevent them from making losses when the market is unfavourable. More importantly, this investment strategy leads to a certainty-equivalent wealth which is less sensitive to the value of the diffusive volatility risk premium.

Another important issue we address is model mis-specification, as the true model is in general unknown and model mis-specification can drastically affect the portfolio performance, see Branger, Schlag, and Schneider (2008). We consider a particular model mis-specification where investors wrongly use a model without stochastic volatility, namely the Black-Scholes model, while the true data are generated by the Heston model. We explore the impact of model mis-specification on the outcomes and examine whether it is likely to reduce the loss in the certainty-equivalent wealth of a utility-maximizing investor in case of model mis-specification. We follow Harrison and Kreps (1979) and assume a no-arbitrage financial market in both cases, which implies the existence of a pricing kernel in both financial markets. We utilize the method of moments to match the variance of the pricing kernel relatives, and make use of SAHARA utility function as well as the time-varying reference level. We find that the feature of being highly risk averse around the reference level helps reduce the loss in the certainty-equivalent wealth when the market returns deviate from the assumptions under which the investment strategies are derived.

The rest of the paper is organized as follows. Section 2 briefly introduces the models, i.e. the Black-Scholes model and the Heston model. Section 3 introduces two classes of utility functions, i.e. the CRRA utility function and the SAHARA utility function, and provides the problem setting and the solution to the expected utility-maximization problem under these utility functions. Section 4 deals with model mis-specification. Section 5 provides numerical implementations comparing results under CRRA and SAHARA risk preferences, as well as the role of SAHARA risk preference in reducing the loss in the certainty-equivalent wealth when investors use the Black-Scholes model with deterministic volatility instead of the true model, i.e. the Heston model. Section 6 draws conclusions.

2 Models

2.1 Black-Scholes model

We assume a financial market in continuous time without transaction costs. The first model we consider is the Black-Scholes model. The asset price dynamics for the bank account and the risky stock S are given by

$$
\begin{aligned}
dB_t &= rB_t dt,\ B_0 = 1, \\
dS_t &= \mu_t S_t dt + \sigma_t S_t dW_t,\ S_0 > 0,
\end{aligned}
\tag{2.1}
$$

where the parameters r, μ_t and σ_t represent the constant risk-free interest rate, the instantaneous rate of return and the volatility of the stock, respectively. σ_t is a deterministic function of time. λ_t is the diffusive volatility risk premium. Moreover, $\mu_t = r + \lambda_t\sigma_t$. Following Heston (1993) and Chen, Nguyen, and Stadje (2018), the diffusive volatility risk premium is assumed to be proportional to the volatility of the risky asset, i.e. $\lambda_t = \lambda\sigma_t$ and λ is a constant.

The pricing kernel is

$$
M_t = M_0 e^{-rt} e^{-\int_0^t \lambda_u dW_u - \frac{1}{2}\int_0^t \lambda_u^2 du}, \tag{2.2}
$$

where $M_0 = 1$ and M_t is log-normally distributed.

Moreover,

$$
\mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^m \,\Big|\, \mathscr{F}_t\right] = e^{-rm(T-t)} e^{\frac{1}{2}m(m-1)\int_t^T \lambda_u^2 du}. \tag{2.3}
$$

The investor invests a proportion π_t of the wealth, X_t, in the risky stock S and the rest in the bank account B. The wealth process satisfies

$$
dX_t = X_t(r + \pi_t\lambda_t\sigma_t)dt + \pi_t X_t \sigma_t dW_t. \tag{2.4}
$$

2.2 Heston stochastic volatility model

The second model we consider is the Heston model. Let the asset price for the stock S be given by

$$
\begin{aligned}
dS_t &= \mu_t S_t\, dt + \sqrt{v_t} S_t\, dW_{S,t},\ S_0 > 0, \\
dv_t &= \kappa(\bar{v} - v_t)dt + \delta\sqrt{v_t}\left(\rho\, dW_{S,t} + \sqrt{1-\rho^2}\, dW_{S,t}^{\perp}\right),
\end{aligned}
\tag{2.5}
$$

where W_S and $W_S^{\perp}$ are two independent standard Brownian motions under measure $\mathbb{P}$ and they are called the fundamental risk factors in the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. ρ is between -1 and 1. $\kappa > 0$ determines the speed of adjustment of the variance towards its theoretical mean $\bar{v} > 0$, and $\delta > 0$ is the volatility of the variance. Furthermore, Feller condition $v_0 > 0$ and $2\kappa\bar{v} \geq \delta^2$ guarantees that the value of v_t is always positive.

The diffusive-risk premium is assumed to be proportional to the standard deviation of the risky asset variance and it is specified as

$$
\frac{\mu_t - r}{\sqrt{v_t}} = \lambda_1\sqrt{v_t}. \tag{2.6}
$$

The volatility risk premium is defined by $\lambda_2\sqrt{v_t}$. λ_1 and λ_2 are constants. We set

$$d\tilde{W}_{S,t} = dW_{S,t} + \frac{\mu_t - r}{\sqrt{v_t}}dt = dW_{S,t} + \lambda_1\sqrt{v_t}dt, \tag{2.7}$$

and

$$d\tilde{W}^{\perp}_{S,t} = dW^{\perp}_{S,t} + \lambda_2\sqrt{v_t}dt, \tag{2.8}$$

then the covariance between $\tilde{W}_{S,t}$ and $\tilde{W}^{\perp}_{S,t}$ is zero, and they are standard independent Brownian motions under a new measure $\tilde{\mathbb{P}}$ defined by

$$\zeta_T := \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = e^{-\lambda_1\int_0^T\sqrt{v_u}dW_{S,u} - \lambda_2\int_0^T\sqrt{v_u}dW^{\perp}_{S,u} - \frac{\lambda_1^2+\lambda_2^2}{2}\int_0^T v_u du}. \tag{2.9}$$

We introduce

$$\begin{aligned} dW_{v,t} &:= \rho\, dW_{S,t} + \sqrt{1-\rho^2}\, dW^{\perp}_{S,t}, \\ dW^{\perp}_{v,t} &:= \sqrt{1-\rho^2}dW_{S,t} - \rho dW^{\perp}_{S,t}, \end{aligned} \tag{2.10}$$

$W_{v,t}$ and $W^{\perp}_{v,t}$ are independent as they are normally distributed and their covariance is zero. Furthermore, following Chen et al. (2018) we introduce these two quantities:

$$\lambda_3 := \rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2,\ \lambda_4 := \sqrt{1-\rho^2}\lambda_1 - \rho\lambda_2, \tag{2.11}$$

Then $\zeta_T = \zeta_{1,T}\zeta_{2,T}$, where

$$\begin{aligned} \zeta_{1,T} &: = e^{-\lambda_3\int_0^T\sqrt{v_u}dW_{v,u} - \frac{\lambda_3^2}{2}\int_0^T v_u du}, \\ \zeta_{2,T} &: = e^{-\lambda_4\int_0^T\sqrt{v_u}dW^{\perp}_{v,u} - \frac{\lambda_4^2}{2}\int_0^T v_u du}. \end{aligned} \tag{2.12}$$

From the volatility process in (2.5) we have

$$\int_t^T \sqrt{v_u}dW_{v,u} = \frac{1}{\delta}\left[v_T - v_t - \kappa\bar{v}(T-t) + \kappa\int_t^T v_u du\right], \tag{2.13}$$

therefore,

$$\mathbb{E}[\zeta_T] = \mathbb{E}[\zeta_{1,T}] = e^{\frac{\lambda_3}{\delta}(\kappa\bar{v}T+v_0)}\mathbb{E}\left[e^{-\frac{\lambda_3}{\delta}v_T - \left(\frac{\lambda_3\kappa}{\delta}+\frac{1}{2}\lambda_3^2\right)\int_0^T v_u du}\right] = e^{\frac{\lambda_3}{\delta}(\kappa\bar{v}T+v_0)}h(1, v_0, T), \tag{2.14}$$

where $h(m, v_t, T-t) = e^{-A(a,b,T-t)-B(a,b,T-t)v_t}$ is the Laplace transform of the couple $(v_T, \int_0^T v_s ds)$ at $a = \frac{\lambda_3}{\delta}m$ and $b = \frac{\lambda_3\kappa}{\delta}m + \frac{1}{2}\lambda_3^2 m - \frac{1}{2}\lambda_4^2 m(m-1)$, see Kraft (2005). The expression for $A(a, b, T-t)$ and $B(a, b, T-t)$ are included in Appendix. $h(1, v_0, T) = e^{-\frac{\lambda_3}{\delta}(\kappa\bar{v}T+v_0)}$, then $\mathbb{E}[\zeta_T] = 1$, hence the process ζ_T is a density.

The pricing kernel in this market is

$$M_t := e^{-rt}\zeta_t,\ M_0 = 1. \tag{2.15}$$

Note that

$$\mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^m \Big| \mathscr{F}_t\right] = \left(e^{\left(-r+\frac{\lambda_3}{\delta}\kappa\bar{v}\right)(T-t)+\frac{\lambda_3}{\delta}v_t}\right)^m h(m, v_t, T-t), \tag{2.16}$$

Under the EMM $\tilde{\mathbb{P}}$, the asset price dynamics in (2.5) are as follows:

$$\begin{aligned} dS_t &= S_t(rdt + \sqrt{v_t}\, d\tilde{W}_{S,t}), \\ dv_t &= \tilde{\kappa}\,(\tilde{v} - v_t)\, dt + \delta\sqrt{v_t}\left(\rho\, d\tilde{W}_{S,t} + \sqrt{1-\rho^2}\, d\tilde{W}^{\perp}_{S,t}\right), \end{aligned} \tag{2.17}$$

where $\tilde{\kappa} = \kappa + \delta\lambda_3$, $\tilde{v} = \frac{\kappa\bar{v}}{\kappa+\delta\lambda_3}$. Hence, the change of measure affects the speed of adjustment $\tilde{\kappa}$ and the long term mean $\tilde{v}$.

The option price can be expressed as $O_t = g(t, S_t, v_t)$. The discounted option price $e^{-rt}O_t$ is a martingale under measure $\tilde{\mathbb{P}}$. Ito's lemma on $e^{-rt}g(t, S_t, v_t)$ under $\tilde{\mathbb{P}}$ gives

$$dO_t = rO_t dt + (g_s S_t + g_v\delta\rho)\sqrt{v_t}d\tilde{W}_{S,t} + g_v\delta\sqrt{1-\rho^2}\sqrt{v_t}d\tilde{W}^{\perp}_{S,t}. \tag{2.18}$$

Let $\pi_{1,t}$ and $\pi_{2,t}$ be a proportion of the wealth invested in the stock and the option respectively, the rest in the risk-free asset. The corresponding wealth process X_t with an initial wealth x_0 satisfies

$$dX_t = X_t\left(rdt + \theta_1\sqrt{v_t}\underbrace{(dW_{S,t} + \lambda_1\sqrt{v_t}dt)}_{d\tilde{W}_{S,t}} + \theta_2\sqrt{v_t}\underbrace{(dW^{\perp}_{S,t} + \lambda_2\sqrt{v_t}dt)}_{d\tilde{W}^{\perp}_{S,t}}\right), \tag{2.19}$$

where

$$\theta_{1,t} = \pi_{1,t} + \frac{g_S S_t + g_v\delta\rho}{O_t}\pi_{2,t},\ \theta_{2,t} = \frac{g_v\delta\sqrt{1-\rho^2}}{O_t}\pi_{2,t}. \tag{2.20}$$

When $\delta \to 0$, the stock price process under measure $\mathbb{P}$ turns into a process similar to a geometric Brownian motion but with a deterministically time-varying volatility such that $\sigma_t \to \sqrt{\bar{v} + e^{-\kappa t}(v_0 - \bar{v})}$.

3 Utility maximization

We consider the following terminal wealth utility maximization problem:

$$\begin{aligned} &\max_{(\pi_t,\phi_t)t\in[0,T]} \mathbb{E}\left[U(X_T)\right], \\ &\text{s.t. } \mathbb{E}\left[M_T X_T\right] = X_0. \end{aligned} \tag{3.1}$$

Following Cox and Huang (1989), the optimal terminal wealth in complete market is given by

$$X_T^* = I(\eta M_T), \tag{3.2}$$

where η is the Lagrangian multiplier and satisfies the budget constraint. I is the inverse function of the first derivative of the utility function U'.

3.1 Utility functions

The utility functions we consider are the CRRA class of utility functions and the SAHARA class of utility functions. The CRRA utility function is defined as follows:

$$U(x) = \begin{cases} \frac{x^{1-\gamma}-1}{1-\gamma} & \gamma \neq 1 \;\&\; \gamma \in R_+ \\ \ln x & \gamma = 1 \end{cases}, \tag{3.3}$$

where γ denotes investors' level of risk aversion, and $x \in R_+$.

The SAHARA utility function has the following form, see Chen et al. (2011):

$$U(x) = \begin{cases} -\frac{1}{\alpha^2-1}\left((x-w_0)+\sqrt{\beta^2+(x-w_0)^2}\right)^{-\alpha}\left((x-w_0)+\alpha\sqrt{\beta^2+(x-w_0)^2}\right) & \alpha \neq 1 \\ \frac{1}{2}\ln\left((x-w_0)+\sqrt{\beta^2+(x-w_0)^2}\right)+\frac{1}{2}\beta^{-2}(x-w_0)\left(\sqrt{\beta^2+(x-w_0)^2}-(x-w_0)\right) & \alpha = 1 \end{cases}, \tag{3.4}$$

where $x \in R$, for a certain scale parameter $\beta > 0$, risk aversion parameter $\alpha > 0$ and threshold wealth $w_0 \in R$. In addition, the reference level at time t is indicated as w_t. The I function is as follows:

$$I(y) = \left(U'\right)^{-1}(y) = \beta \sinh\left(-\frac{1}{\alpha}\ln y - \ln\beta\right) + w_0 = \frac{1}{2}\left(y^{-\frac{1}{\alpha}} - \beta^2 y^{\frac{1}{\alpha}}\right) + w_0 \tag{3.5}$$

with domain $y \in R^+$. Under SAHARA risk preference, the optimal terminal wealth is given by

$$X_T^* = \frac{1}{2}(\eta M_T)^{-\frac{1}{\alpha}} - \frac{1}{2}\beta^2(\eta M_T)^{\frac{1}{\alpha}} + w_T. \tag{3.6}$$

The SAHARA utility function contains the CRRA utility function as a limiting case. Namely, under SAHARA risk preference, when $w_0 = 0$ and $x > 0$, let $\beta \to 0$, then RRA$(x) \to \alpha$, and

$$U(x) \to \begin{cases} 2^{-\alpha}\frac{x^{1-\alpha}}{1-\alpha} & \alpha \neq 1 \;\&\; \alpha \in R_+ \\ \frac{1}{2}\ln x + \frac{1+2\ln 2}{4} & \alpha = 1 \end{cases}, \tag{3.7}$$

i.e. the SAHARA utility function converges to the CRRA utility function apart from an overall change of scale.

3.2 Black-Scholes model

We characterise the optimal wealth strategy under the Black-Scholes model in the following theorem.

Theorem 1. *In case of SAHARA risk preference under the Black-Scholes model, the optimal wealth at time $t \in [0,T]$ is given by*

$$X_t^* = e^{-r(T-t)+\frac{1}{2\alpha^2}\int_t^T \lambda_u^2 du}\beta\, sinh\left(-\frac{1}{\alpha}\ln\left(\eta M_t e^{-r(T-t)+\frac{1}{2}\int_t^T \lambda_u^2 du}\right) - \ln\beta\right) + w_t, \tag{3.8}$$

where w_t is the interim target and $w_t = w_T e^{-r(T-t)}$. η can be found by equating $X_0^* = X_0$. The optimal investment strategy in terms of wealth that should be invested in the risky asset at time t is

$$\pi_t^* X_t^* = \frac{\lambda}{\alpha}\sqrt{\left(e^{-r(T-t)+\frac{1}{2\alpha^2}\int_t^T \lambda_u^2 du}\beta\right)^2 + (X_t^* - w_t)^2}. \tag{3.9}$$

Proof.

$$\begin{aligned} X_t^* &= \mathbb{E}\left[\frac{M_T}{M_t}X_T^* \,\middle|\, \mathscr{F}_t\right] \\ &= \frac{1}{2}(\eta M_t)^{-\frac{1}{\alpha}}\mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{1-\frac{1}{\alpha}} \,\middle|\, \mathscr{F}_t\right] - \frac{1}{2}\beta^2(\eta M_t)^{\frac{1}{\alpha}}\mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{1+\frac{1}{\alpha}} \,\middle|\, \mathscr{F}_t\right] \\ &\quad + w_T\mathbb{E}\left[\frac{M_T}{M_t} \,\middle|\, \mathscr{F}_t\right] \\ &= \frac{1}{2}\iota_{1-\frac{1}{\alpha},t} - \frac{1}{2}\beta^2\iota_{1+\frac{1}{\alpha},t} + w_t \\ &= e^{-r(T-t)+\frac{1}{2\alpha^2}\int_t^T \lambda_u^2 du}\beta \sinh\left(-\frac{1}{\alpha}\ln\left(\eta M_t e^{-r(T-t)+\frac{1}{2}\int_t^T \lambda_u^2 du}\right) - \ln\beta\right) + w_t, \end{aligned} \tag{3.10}$$

where $\iota_{1+m,t} := (\eta M_T)^m\,\mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{1+m} \,\middle|\, \mathscr{F}_t\right] = (\eta M_t)^m e^{-r(1+m)(T-t)+\frac{1}{2}m(1+m)\int_t^T \lambda_u^2 du}$.

Then,

$$\begin{aligned} \frac{dX_t^*}{dM_t} &= -\frac{1}{\alpha}e^{-r(T-t)+\frac{1}{2\alpha^2}\int_t^T \lambda_u^2 du}\beta \cosh\left(-\frac{1}{\alpha}\ln\left(\eta M_t e^{-r(T-t)+\frac{1}{2}\int_t^T \lambda_u^2 du}\right) - \ln\beta\right)\frac{dM_t}{M_t}, \\ \pi_t^* X_t^* &= \frac{\lambda}{\alpha}\sqrt{\left(e^{-r(T-t)+\frac{1}{2\alpha^2}\int_t^T \lambda_u^2 du}\beta\right)^2 + (X_t^* - w_t)^2}. \end{aligned} \tag{3.11}$$

□

Under CRRA risk preference with risk aversion parameter γ, the proportional wealth allocated to the stock is constant, similar to the Merton ratio in Merton (1969). It is given as follows:

$$\pi_t^* = \frac{\lambda}{\gamma}. \tag{3.12}$$

Its counterpart under SAHARA risk preference when $X_t^* \neq 0$, can be written as

$$\pi_t^* = \frac{\lambda}{\alpha_t}, \tag{3.13}$$

where $\alpha_t = \dfrac{\alpha X_t^*}{\sqrt{\left(e^{-r(T-t)+\frac{1}{2\alpha^2}\int_t^T \lambda_u^2 du}\beta\right)^2 + (X_t^* - w_t)^2}}$.

In case $w_t = 0$ and $X_t^* > 0$, let $\beta \to 0$, then $\alpha_t \to \alpha$. This is another way to view CRRA risk preference as a special case of SAHARA risk preference. Note that at time T, $\alpha_T = \dfrac{\alpha X_T^*}{\sqrt{\beta^2 + \left(X_T^* - w_T\right)^2}}$, which has the same functional form as RRA of SAHARA utility function.

For illustrative purpose, we assume that $X_0 = 1$, $\lambda_t = 0.3$ for all $t \in [0, T]$, and the investment horizon T is 12 years. The risk-free rate r is 3%. The parameter α is fixed at 3. The reference level at time T is either $w_T = e^{rT}$ or $w_T = 1.5e^{rT}$, then the corresponding interim reference level at time t is $w_t = e^{rt}$ and $w_t = 1.5e^{rt}$, respectively. The graph below plots the distribution of α_t at time $t = \frac{1}{2}T$ as a function of wealth.

Figure 1: α_t with constant σ.

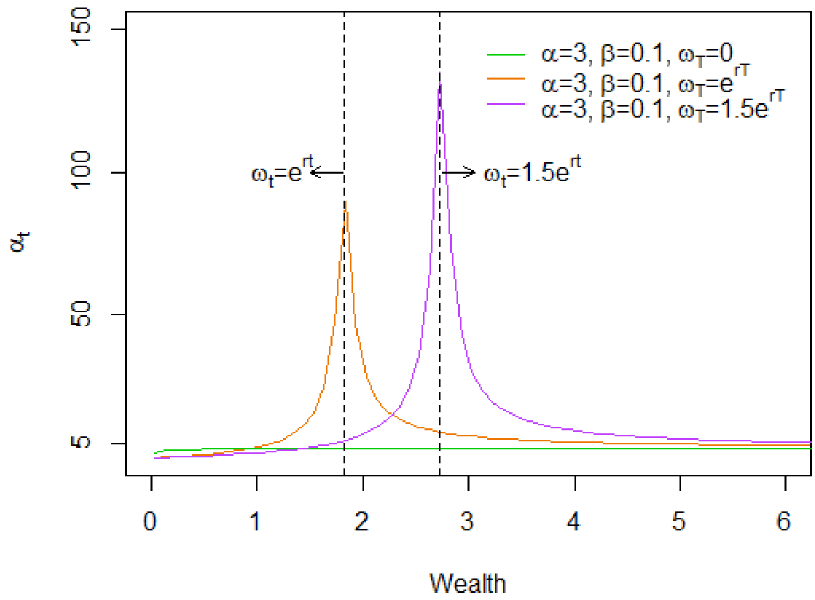


Figure 1 shows that, when $w_t > 0$ (orange and pink lines), α_t varies with wealth. In particular, the value of α_t increases significantly when the wealth level approaches the reference level w_t, whether from below or above, and reaches its highest level around w_t. This indicates investors' tendency of securing the value of the accumulated wealth or minimising the risk of a significant loss relative to the reference level. When the wealth level falls below or rises above the reference level, the value of α_t decreases to zero and α, respectively. This implies that investor will maintain a higher equity allocation with the aim of eliminating the loss in the domain of loss, while with the aim of seeking for extra returns in the domain of gain. Under SAHARA risk preference, investors experience losses more severely than equivalent gains, because the distribution of α_t is steeper in the domain of loss. When $w_t = 0$ (green line), the value of α_t increases with wealth and converges to that of α. The smaller the value of β, the quicker the convergence. Figure 1 also shows that, investors who set the reference level at a higher level, tend to be less risk averse when the wealth level is below this level, while more risk averse when the wealth level is above it. This implies that under SAHARA risk preference, when investors have a higher target, they accept more risks in the domain of loss, and are more willing to protect the accumulated wealth in the domain of gain.

3.3 Heston stochastic volatility model

We characterise the optimal wealth strategy under the Heston model in the following proposition.

Proposition 2. *In case of SAHARA risk preference under the Heston model, the optimal wealth at time* $t \in [0, T]$ *is given by*

$$X_t^* = \frac{1}{2}\epsilon_{1-\frac{1}{\alpha},t} - \frac{1}{2}\beta^2\epsilon_{1+\frac{1}{\alpha},t} + \omega_t, \tag{3.14}$$

where $\epsilon_{1+m,t} := (\eta M_t)^m \, \mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{1+m} \,\middle|\, \mathscr{F}_t\right] = (\eta M_t)^m e^{\left(\left(-r+\frac{\lambda_3}{\delta}\kappa\bar{v}\right)(T-t)+\frac{\lambda_3}{\delta}v_t\right)(1+m)} h(1+m, v_t, T-t)$. η *can be found by setting* $X_0^* = X_0$.

The optimal investment strategy at time $t \in [0, T]$ *is given by*

$$\begin{aligned} \theta_{1.t}^* X_t^* &= -\lambda_1 F(t, v_t, M_t) + \delta\rho L(t, v_t, M_t), \\ \theta_{2,t}^* X_t^* &= -\lambda_2 F(t, v_t, M_t) + \delta\sqrt{1-\rho^2} L(t, v_t, M_t), \\ \pi_{2,t}^* &= \frac{O_t}{g_v \delta\sqrt{1-\rho^2}}\theta_{2,t}^*, \\ \pi_{1,t}^* &= \theta_{1,t}^* - \left(\frac{g_S S_t}{O_t} + \frac{g_v \delta\rho}{O_t}\right)\pi_{2,t}^*, \end{aligned} \tag{3.15}$$

where $a_m := \frac{\lambda_3}{\delta}m$, $b_m := \frac{\lambda_3\kappa}{\delta}m + \frac{1}{2}\lambda_3^2 m - \frac{1}{2}\lambda_4^2 m(m-1)$ *and* $B_m := B(a_m, b_m, T-t)$;
$F(t, v_t, M_t) := -\frac{1}{\alpha}\left(\frac{1}{2}\epsilon_{1-\frac{1}{\alpha},t} + \frac{1}{2}\beta^2\epsilon_{1+\frac{1}{\alpha},t}\right)$;
$L(t, v_t, M_t) := \left(a_{1-\frac{1}{\alpha}} - B_{1-\frac{1}{\alpha}}\right)\frac{1}{2}\epsilon_{1-\frac{1}{\alpha},t} - \left(a_{1+\frac{1}{\alpha}} - B_{1+\frac{1}{\alpha}}\right)\frac{1}{2}\beta^2\epsilon_{1+\frac{1}{\alpha},t}$.

Under CRRA risk preference, the optimal investment strategy is given as follows, see Chen et al. (2018) and (Liu & Pan, 2003):

$$\begin{aligned} &\theta_{1,t}^* = \frac{\lambda_1}{\gamma} - H(T-t)\delta\rho, \quad \theta_{2,t}^* = \frac{\lambda_2}{\gamma} - H(T-t)\delta\sqrt{1-\rho^2}, \\ &\pi_{1,t}^* = \frac{\lambda_1}{\gamma} - \frac{\lambda_2\rho}{\gamma\sqrt{1-\rho^2}} - \pi_{2,t}^*\frac{g_S S_t}{O_t}, \quad \pi_{2,t}^* = \left(\frac{\lambda_2}{\gamma\delta\sqrt{1-\rho^2}} + H(T-t)\right)\frac{O_t}{g_v}, \end{aligned} \tag{3.16}$$

where $H(\tau) := \frac{e^{\chi\tau}-1}{2\chi+(k_1+\chi)(e^{\chi\tau}-1)}\frac{1-\gamma}{\gamma^2}(\lambda_1^2+\lambda_2^2)$, $k_1 := \kappa+\frac{\gamma-1}{\gamma}(\rho\lambda_1+\sqrt{1-\rho^2}\lambda_2)\delta$, and $\chi := \sqrt{k_1^2 + \frac{\gamma-1}{\gamma^2}(\lambda_1^2+\lambda_2^2)\delta^2}$.

3.3.1 Index V

The VIX squared is defined to be the variance swap rate and it is computed as the conditional expectation under a risk-neutral measure. Under the Heston model it is given as follows, see Zhang and Zhu (2006):

$$\text{VIX}_t^2 = \mathbb{E}^{\tilde{\mathbb{P}}}\left[\frac{1}{T-t}\int_t^T v_u du \,\middle|\, \mathscr{F}_t\right] = \frac{1-e^{-\tilde{\kappa}(T-t)}}{\tilde{\kappa}(T-t)}v_t + \tilde{v}\left(1 - \frac{1-e^{-\tilde{\kappa}(T-t)}}{\tilde{\kappa}(T-t)}\right). \tag{3.17}$$

Denote VIX_t^2 by V_t, then

$$\frac{\partial V_t}{\partial S_t} = 0, \quad \frac{\partial V_t}{\partial v_t} = \frac{1-e^{-\tilde{\kappa}(T-t)}}{\tilde{\kappa}(T-t)}. \tag{3.18}$$

This index is delta-neutral as it does not depend on the stock price. To simplify the calculation but without loss of generality, we include this index in our portfolio.

4 Model mis-specification

If investors use a wrong model, they might miscalculate for instance expected returns, volatilities, and correlations among the assets. This can lead to an incorrect allocation of their assets to different risk factors, potentially exposing their portfolio to more or less risk than intended, which can lead to losses or missed opportunities. In this section we deal with a particular model mis-specification where investors wrongly use the Black-Scholes model with deterministic volatility instead of the Heston model. We assume a no-arbitrage financial market in both cases and that the investor matches the variance of the pricing kernel relatives. Under the Heston model the variance of the pricing kernel relative is:

$$\begin{aligned}\mathbb{V}\mathrm{ar}\left[\frac{M_T}{M_t} \,\Big|\, \mathscr{F}_t\right] =& \mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^2 \,\Big|\, \mathscr{F}_t\right] - \left(\mathbb{E}\left[\frac{M_T}{M_t} \,\Big|\, \mathscr{F}_t\right]\right)^2 \\ =& e^{-2r(T-t)}\left(e^{2\lambda_3\delta^{-1}v_t+2\lambda_3\delta^{-1}\kappa\bar{v}(T-t)}e^{-A(a,b,T-t)-B(a,b,T-t)v_t} - 1\right),\end{aligned} \tag{4.1}$$

where $a = 2\lambda_3\delta^{-1}$ and $b = 2\lambda_3\delta^{-1}\kappa + \lambda_3^2 - \lambda_4^2$. Then,

$$e^{\int_t^T \lambda_u^2 du} = e^{2\lambda_3\delta^{-1}v_t+2\lambda_3\delta^{-1}\kappa\bar{v}(T-t)-A(a,b,T-t)-v_tB(a,b,T-t)}. \tag{4.2}$$

The proportional wealth invested in the stock, π_t, is given in equation (3.9) and equation (3.12) for SAHARA risk preference and CRRA risk preference, respectively. Here we equate λ to λ_1. The rest is invested in the risk-free asset. The wealth process will be

$$dX_t = X_t(r + \pi_t\lambda_1 v_t)dt + \pi_t X_t\sqrt{v_t}dW_t. \tag{4.3}$$

The indirect utility for the mis-specified model cannot be computed in closed-form, hence we have to resort to Monte Carlo simulation.

5 Numerical analysis

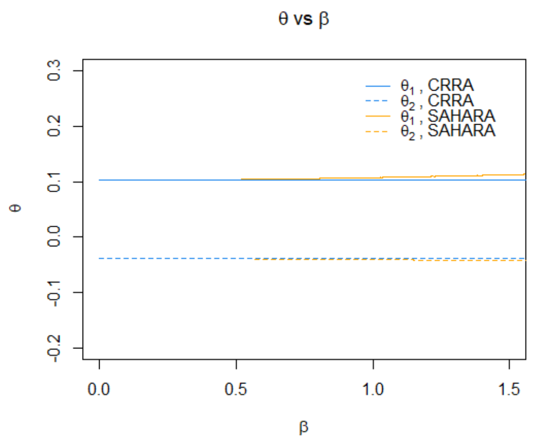
In this section, we explore under the Heston model the investment strategies and the distribution of the outcomes under CRRA and SAHARA risk preferences, as well as the impact of these types of risk preference on the distribution of the terminal wealth when investors wrongly use the Black-Scholes model. Under CRRA risk preference, the investor has risk aversion $\gamma = 3$. Under SAHARA risk preference, the investor has risk aversion $\alpha = 3$; the scale parameter is fixed at $\beta = 0.1$. Following Bakshi and Kapadia (2003) and Liu and Pan (2003), we assume a negative value of the volatility risk premium. Important to note that we will allow λ_2 to vary in our analysis so as to get a better understanding of how different levels of the diffusive volatility risk premium could affect the distribution of the outcomes. For the base-case parameters we use similar values as in Chen et al. (2018):

$$r = 0.03,\ S_0 = 100,\ \kappa = 1,\ \bar{v} = 1.5,\ v_0 = 1,\ \delta = 0.5,\ \lambda_1 = 0.3,\ \lambda_2 = -0.1,\ \rho = -0.4,\ T = 40,\ X_0 = 1.$$

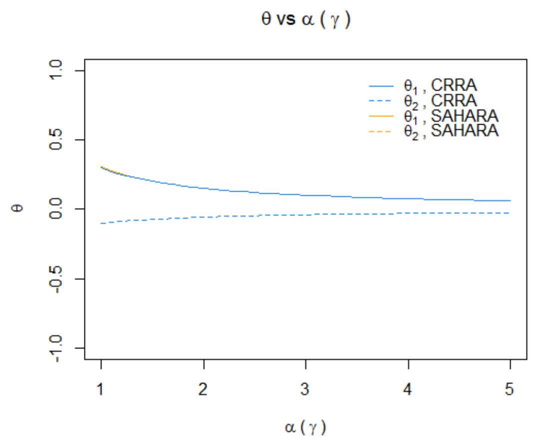
5.1 Impact of β on θ

In this subsection, we explore the impact of the parameter β on the value of θ, i.e. θ_1 and θ_2. For this purpose, we equate w_T to zero. Figure 2 plots the distribution of θ as a function of β under CRRA and SAHARA risk preferences on panel a, and that of θ as a function of the risk aversion on panel b.

Figure 2: Impact of β on θ at time $\frac{1}{2}T$.



(a) SAHARA $\alpha = 3$, $w_T = 0$ and CRRA $\gamma = 3$.



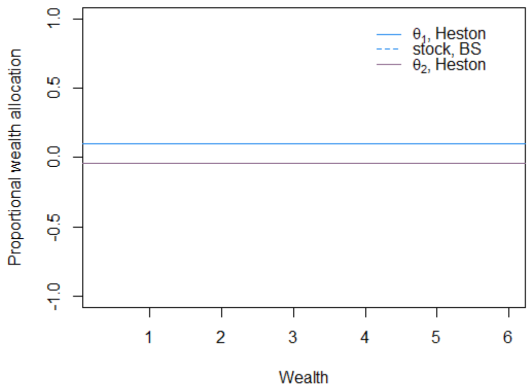
(b) SAHARA $\beta = 0.1$, $w_T = 0$ and CRRA.

Panel a shows that when β takes small values, the corresponding values of θ do not significantly differ under these two types of risk preference. Panel b shows additionally that when β is 0.1 (small), there is almost no difference between the values of the corresponding θ when the value of α and γ vary but stay equal. Panel b also shows that for both risk preferences, the absolute value of θ increases when the investor is less risk averse, i.e. when the risk aversion parameter α or γ takes smaller values.

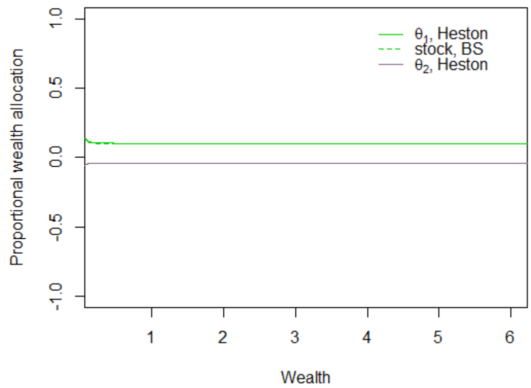
5.2 Investment strategies

In this subsection we present the investment strategies under CRRA and SAHARA risk preferences at time $\frac{1}{2}T$. In particular, we focus on the proportional asset allocation to the fundamental risk factors under the Heston model, θ, and that to the stock under the Black-Scholes model. The results are displayed in Figure 3.

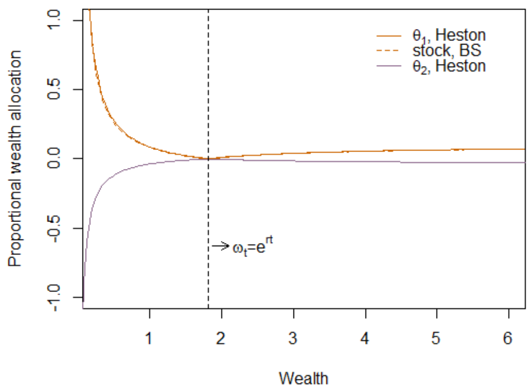
Figure 3: Investment strategies at time $\frac{1}{2}T$.



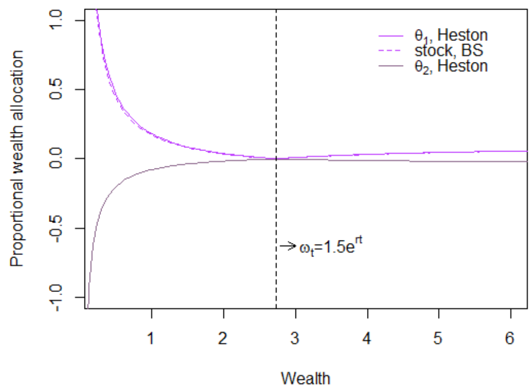
(a) CRRA $\gamma = 3$.



(b) SAHARA $\alpha = 3$, $\beta = 0.1$, $w_T = 0$.



(c) SAHARA $\alpha = 3$, $\beta = 0.1$, $w_T = e^{rT}$.



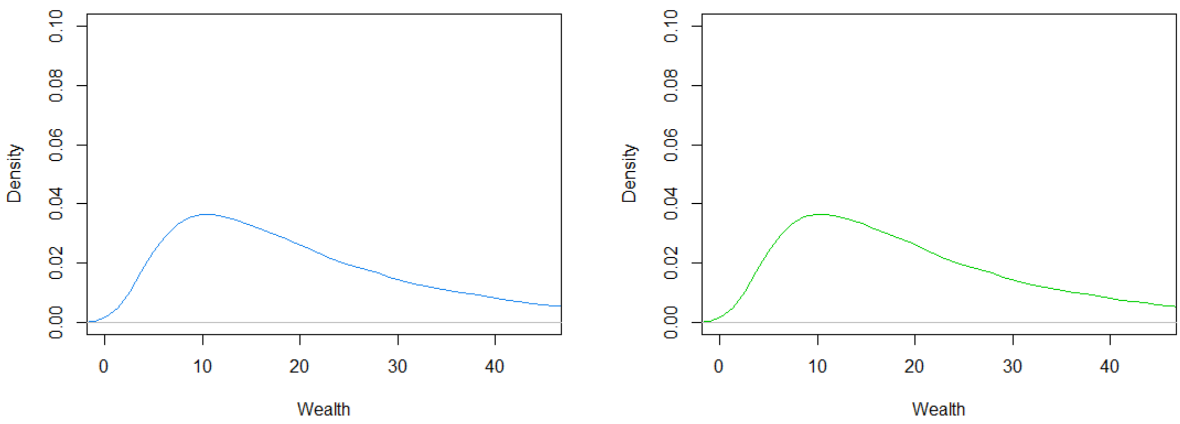
(d) SAHARA $\alpha = 3$, $\beta = 0.1$, $w_T = 1.5e^{rT}$.

Panel a shows that under CRRA risk preference, the proportional asset allocation to the fundamental risk factors or the stock does not vary with wealth. It is interesting to note that the value of θ_1 under the Heston model is similar to that of the asset allocation to the stock under the Black-Scholes model. By analysing equation (3.12) and equation (3.16), we see that this occurs particularly when δ takes small values, i.e. when the market is not very volatile. Panel b shows that given positive wealth, the investment strategy under SAHARA risk preference when the value of β is 0.1 and the reference level is set at zero wealth, is similar to that under CRRA risk preference with the same risk aversion. Panel c and d show that, under SAHARA risk preference when the reference level is positive, the wealth allocation to the fundamental risk factors θ is the least in absolute value around the reference level, which can be interpreted as locking-in the wealth at the reference level; its absolute value increases when the wealth level rises above or falls below the reference level, and this occurs to a lesser extent in the former case than in the latter case, which can be interpreted as becoming overconfident when investors have made gains, while taking extra risks when they have made losses, respectively. The wealth allocation to the stock under the Black-Scholes model closely matches θ_1 under the Heston model.

5.3 Distribution of outcomes

In this subsection we provide the distribution of outcomes under these risk preferences under the Heston model, as well as that when investors wrongly use the Black-Scholes model. For the results, we run 100,000 simulations with one time step per month.

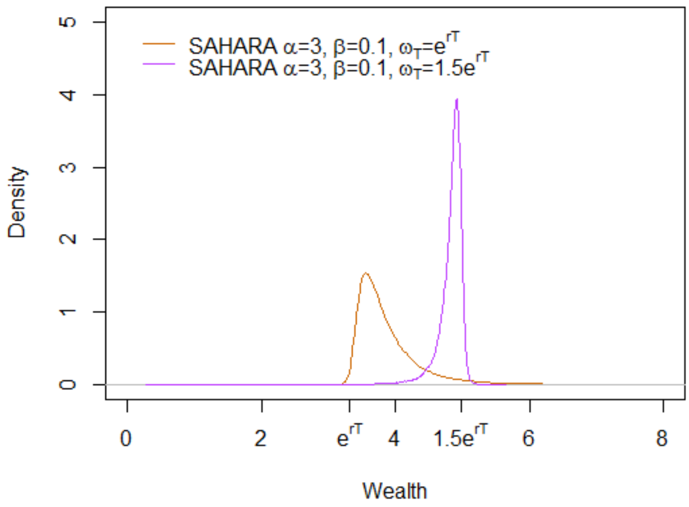
Figure 4: Distribution of terminal wealth under the Heston model part 1.



(a) CRRA $\gamma = 3$.

(b) SAHARA $\alpha = 3$, $\beta = 0.1$, $w_T = 0$.

Figure 5: Distribution of terminal wealth under the Heston model part 2.



(a) SAHARA $\alpha = 3$, $\beta = 0.1$.

Panel a of Figure 4 shows that the distribution of the terminal wealth is widely spread under the CRRA risk preference with risk aversion γ being 3, since there is no inherent mechanism for focusing on a desired target. Panel b of Figure 4 shows that similar results can be generated under the SAHARA risk preference with the same risk aversion, the value of β being small and the reference level being set at zero wealth. By contrast, Figure 5 shows that under SAHARA risk preference, when the reference level w_T is set at e^{rT} or $1.5e^{rT}$, a peak is formed around the reference level. This gives a certain degree of certainty regarding the level of the terminal wealth. To study the distribution of the outcomes, several tail-probabilities are presented below in Table 1.

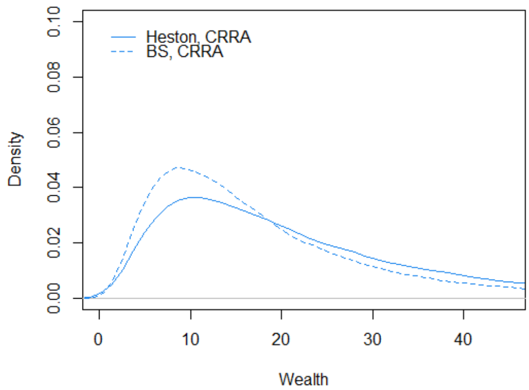
Table 1: Tail-probabilities of the distribution at time T under the Heston model when $\lambda_2 = -0.1$.

	$\gamma = 3$	$\alpha = 3$, $\beta = 0.1$, $w_T = 0$	$\alpha = 3$, $\beta = 0.1$, $w_T = e^{rT}$	$\alpha = 3$, $\beta = 0.1$, $w_T = 1.5e^{rT}$
$\mathrm{P}(X_T \geq e^{rT})$	98.35%	98.33%	99.56%	99.91%
$\mathrm{P}(X_T \geq 1.5e^{rT})$	95.04%	95.00%	3.25%	10.11%
$\mathrm{P}(X_T \geq 2e^{rT})$	90.31%	90.23%	0.22%	0

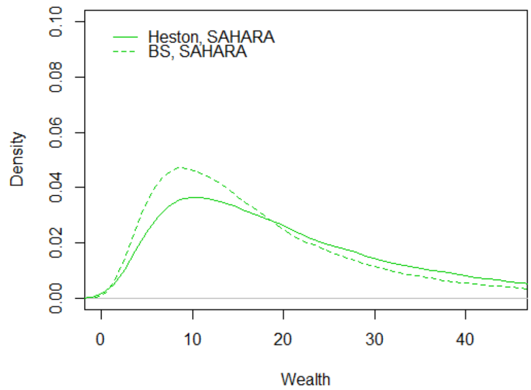
As expected, the distributions shown in Figure 4 have similar tail-probabilities. Under SAHARA risk preference, by increasing the reference level w_T to e^{rT}, approximately 99% of the outcomes are in the range of e^{rT} and $2e^{rT}$, instead of 8% under CRRA risk preference. This leads to a significant increase in the certainty of the terminal wealth actually realised. By increasing the reference level w_T further to $1.5e^{rT}$, the probability of the terminal wealth being not below $1.5e^{rT}$ increases by around 7%. Similar results emerge when the reference level is set at a higher level. In short, with the dynamic target-based strategy under SAHARA risk preference, the likelihood of achieving the desired target can be significantly improved. It is worth mentioning that a high reference level could lead to negative wealth under SAHARA risk preference, nevertheless, the risk of achieving negative wealth can be eliminated by imposing a minimum requirement for the terminal wealth.

Next, we compare the distributions for these types of risk preference under the Heston model to that under the Black-Scholes model. The results are presented in Figure 6.

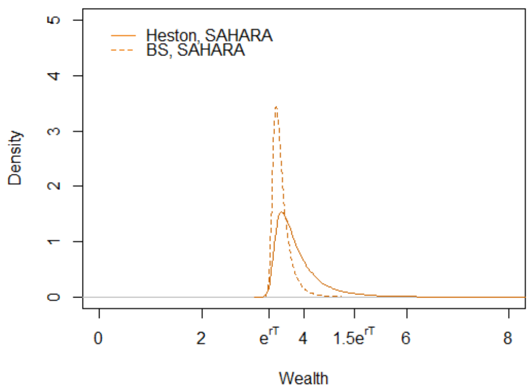
Figure 6: Distribution of terminal wealth when λ_2 is -0.1.



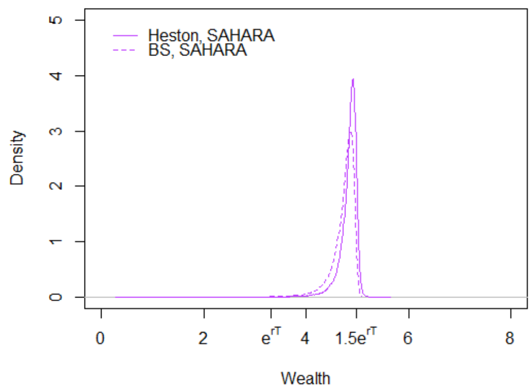
(a) CRRA $\gamma = 3$.



(b) SAHARA $\alpha = 3$, $\beta = 0.1$, $w_T = 0$



(c) SAHARA $\alpha = 3$, $\beta = 0.1$, $w_T = e^{rT}$.



(d) SAHARA $\alpha = 3$, $\beta = 0.1$, $w_T = 1.5e^{rT}$.

For these types of risk preference under the Black-Scholes model, the distributions retain much resemblance of their counterparts under the Heston model, except that the mode of the distribution moves slightly to the left, which is mainly the effect of not-having the advantage of the risk and return tradeoff through investing in derivatives as well as the small absolute value of λ_2. To further study the effect of model mis-specification on the distribution, several tail-probabilities under the Black-Scholes model are provided in Table 2.

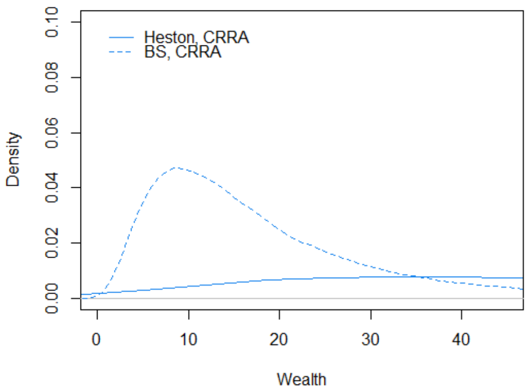
Table 2: Tail-probabilities of the distribution at time T under the Black-Scholes model.

	$\gamma = 3$	$\alpha = 3$, $\beta = 0.1$, $w_T = 0$	$\alpha = 3$, $\beta = 0.1$, $w_T = e^{rT}$	$\alpha = 3$, $\beta = 0.1$, $w_T = 1.5e^{rT}$
$\mathrm{P}(X_T \geq e^{rT})$	97.69%	97.69%	99.03%	99.75%
$\mathrm{P}(X_T \geq 1.5e^{rT})$	92.82%	92.82%	0.06%	2.42%
$\mathrm{P}(X_T \geq 2e^{rT})$	86.00%	86.01%	0	0

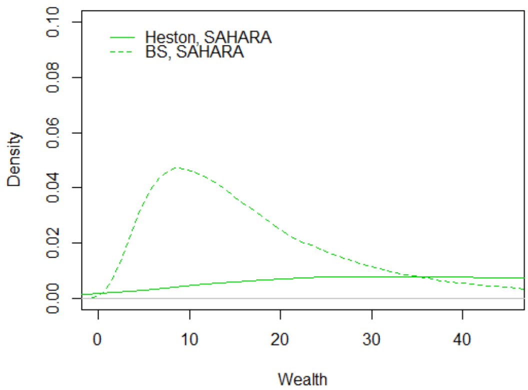
Compared to the results in Table 1, all these probabilities decrease, though the changes are not substantial. Under the Black-Scholes model, the CRRA risk preference with γ being 3 and the SAHARA risk preference with α being 3, β being 0.1 and w_T being 0 lead to similar tail-probabilities, and there is no strong concentration of outcomes about a particular value. Under SAHARA risk preference, when the reference level w_T is e^{rT} or $1.5e^{rT}$, most of the outcomes are in the range of e^{rT} and $1.5e^{rT}$.

We further evaluate the impact of the value of λ_2 on the distribution of optimal wealth by setting $\lambda_2 = -0.3$. The results are summarized in Figure 7 and in Table 3. Figure 7 plots the distribution of the terminal wealth when λ_2 is -0.3 and Table 3 provides several tail-probabilities.

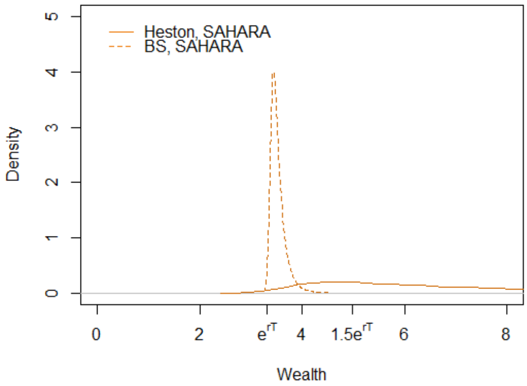
Figure 7: Distribution of terminal wealth when λ_2 is -0.3.



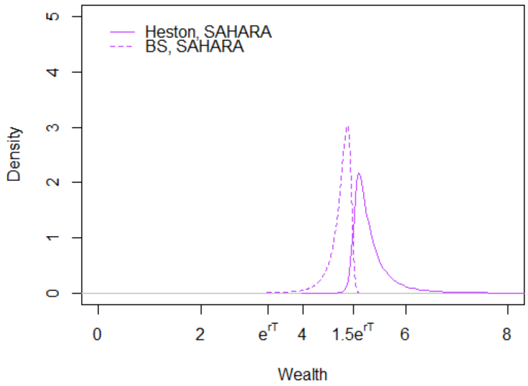
(a) CRRA $\gamma = 3$.



(b) SAHARA $\alpha = 3$, $\beta = 0.1$, $w_T = 0$



(c) SAHARA $\alpha = 3$, $\beta = 0.1$, $w_T = e^{rT}$.



(d) SAHARA $\alpha = 3$, $\beta = 0.1$, $w_T = 1.5e^{rT}$.

Table 3: Tail-probabilities of the distribution at time T under the Heston model when $\lambda_2 = -0.3$.

	$\gamma = 3$	$\alpha = 3$ $\beta = 0.1$ $w_T = 0$	$\alpha = 3$ $\beta = 0.1$ $w_T = e^{rT}$	$\alpha = 3$ $\beta = 0.1$ $w_T = 1.5e^{rT}$
$\mathrm{P}(X_T \geq e^{rT})$	99.86%	99.85%	100%	100%
$\mathrm{P}(X_T \geq 1.5e^{rT})$	99.56%	99.54%	72.70%	93.15%
$\mathrm{P}(X_T \geq 2e^{rT})$	99.11%	99.04%	46.49%	1.65%

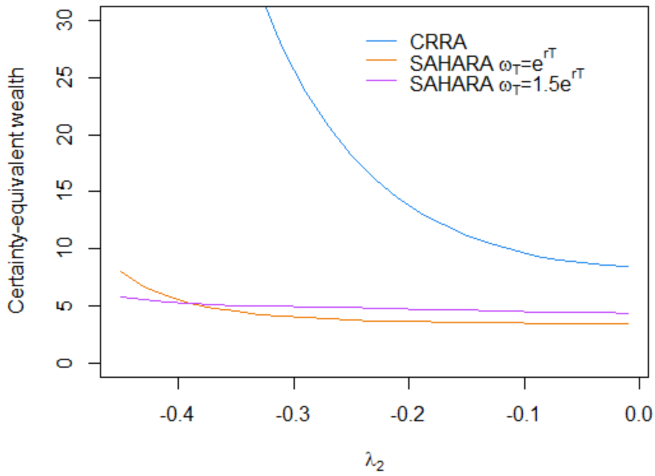
Figure 7 shows that, under the Heston model when λ_2 is decreased from -0.1 to -0.3, the mode of the distribution moves to the right and the distribution gives a more dispersed range of outcomes. In addition, the distribution under SAHARA risk preference with a positive reference level is more positioned to the right of the reference level. Thus, a higher absolute value of λ_2 leads to a higher mean level of the terminal wealth, but also to a significant increase in the uncertainty of the terminal wealth. We notice that among these types of risk preference, the spread of the distribution is on a much smaller scale under the SAHARA risk preference when the reference level at time T is set at e^{rT} or $1.5e^{rT}$. We also notice that by increasing the reference level from e^{rT} to $1.5e^{rT}$, approximately 98% instead of approximately 53% of the outcomes is between e^{rT} and $2e^{rT}$, which implies that the distribution is more concentrated around the reference level when the reference level at time T is $1.5e^{rT}$ rather than when it is e^{rT}.

Table 4: Certainty-equivalent wealth.

	$\lambda_2 = -0.3$				$\lambda_2 = -0.1$				$\lambda_2 = 0$			
	$\gamma = 3$	$\alpha = 3$ $\beta = 0.1$ $w_T = 0$	$\alpha = 3$ $\beta = 0.1$ $w_T = e^{rT}$	$\alpha = 3$ $\beta = 0.1$ $w_T = 1.5e^{rT}$	$\gamma = 3$	$\alpha = 3$ $\beta = 0.1$ $w_T = 0$	$\alpha = 3$ $\beta = 0.1$ $w_T = e^{rT}$	$\alpha = 3$ $\beta = 0.1$ $w_T = 1.5e^{rT}$	$\gamma = 3$	$\alpha = 3$ $\beta = 0.1$ $w_T = 0$	$\alpha = 3$ $\beta = 0.1$ $w_T = e^{rT}$	$\alpha = 3$ $\beta = 0.1$ $w_T = 1.5e^{rT}$
Heston	26.35	25.62	4.29	4.93	9.69	9.65	3.50	4.52	8.45	8.48	3.46	4.39
BS	8.42	8.42	3.42	4.37	8.42	8.42	3.43	4.37	8.42	8.42	3.43	4.37

Table 4 presents the certainty-equivalent wealth for different values of λ_2. When λ_2 is 0, the certainty-equivalent wealth under the Heston is slightly higher than that under the Black-Shcoles models, which implies that there is a benefit in derivative investments even when the diffusive volatility risk premium is zero. This is comparable with one of the findings of Liu and Pan (2003) who examine the certainty-equivalent wealth under the Heston model with access to as well as no access to the derivatives. In the latter case the market is incomplete. Under the SAHARA risk preference with a positive reference level, the certainty equivalent wealth is close to the reference level. When λ_2 is -0.3 or -0.1, the certainty equivalent wealth under the Heston model increases for these types of risk preference. A large increase is observed under the CRRA risk preference and the SAHARA risk preference with the reference level being set at zero wealth, while a slight increase under the SAHARA risk preference when the reference level at time T is e^{rT} or $1.5e^{rT}$. In contrast, the value of λ_2 barely affects the certainty-equivalent wealth under the Black-Scholes model. This confirms the benefit in derivative investments. It also shows that this benefit strongly depends on the volatility risk premium.

Figure 8: Distribution of certainty-equivalent wealth against λ_2.



To provide a quantitative assessment of the portfolio improvement due to the value of λ_2, we use the base-case parameters described earlier and vary the value of λ_2 from 0 to -0.5. Figure 8 plots the distribution of the certainty-equivalent wealth against λ_2 under the Heston model. We see that the certainty equivalent-wealth under CRRA risk preference decreases greatly with λ_2 (< 0), hence it is very sensitive to how volatility risk is priced. Our results also show that under SAHARA risk preference, the certainty equivalent-wealth decreases slightly with λ_2 (< 0) when the reference level at time T is e^{rT}, and the distribution is almost flat when the reference level at time T is $1.5e^{rT}$. Compared to CRRA risk preference, the certainty-equivalent wealth under the SAHARA risk preference with these positive reference levels are much less sensitive to the value of the diffusive volatility risk premium. Therefore, the dynamic target-based investment strategy under SAHARA risk preference leads to a more robust certainty-equivalent wealth when the value of the volatility risk premium varies. With this investment strategy, the loss in the certainty-equivalent wealth could be largely reduced when investors ignore the stochastic nature of the volatility in the Heston model but use the Black-Scholes model instead.

6 Conclusion

We investigate the impact of stochastic volatility on the investment strategies and the distribution of the terminal wealth under CRRA and SAHARA risk preferences. We find that the outcomes generated under the SAHARA risk preference with the value of the parameter β being small and the reference level being set at zero wealth, are similar to that under CRRA risk preference with the same risk aversion. We also find that the dynamic target-based strategy under SAHARA risk preference leads to a distribution of the terminal wealth that is more concentrated around the reference level. Compared to CRRA risk preference, the certainty-equivalent wealth under the SAHARA risk preference with a positive reference level is less sensitive to the value of the diffusive volatility risk premium. When investors ignore the stochastic nature of the volatility in the Heston model and use the Black-Scholes model instead, the distribution of the terminal wealth can be concentrated around the reference level, when investors match the variance of the pricing kernel relatives and set a positive reference level under SAHARA risk preference. Furthermore, the portfolio improvement in terms of certainty-equivalent wealth through investing in derivatives greatly depends on the value of the volatility risk premium under CRRA risk preference, while this dependence is much less under SAHARA risk preference with a positive reference level. Therefore, the target-based investment strategy under SAHARA risk preference leads to a more robust certainty-equivalent wealth which is less sensitive to the value of diffusive volatility risk premium. This is especially useful when there is no common agreement on a reasonable value for the diffusive volatility risk premium. With this strategy, the loss in the certainty-equivalent wealth could be reduced when investors wrongly use the Black-Scholes model instead of the Heston model.

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Appendix

$A(a, b, T-t)$ & $B(a, b, T-t)$

Assume that $dv_t = \kappa(\bar{v} - v_t)dt + \delta\sqrt{v_t}\, dW_t$, where W_t is a standard Brownian motion. Then, given $\mathscr{F}_t$, the conditional Laplace transform of the couple $(v_t, \int_t^T v_s ds)$ at (a, b) is well-defined if

$$a \geq -\frac{\kappa + \sqrt{\kappa^2 + 2\beta\delta^2}}{\delta^2} \text{ and } b \geq -\frac{\kappa^2}{2\delta^2}, \tag{A.1}$$

and is given by

$$\mathrm{E}\left[e^{-av_T - b\int_t^T v_s ds} \,\middle|\, \mathscr{F}_t\right] = e^{-A(a,b,T-t) - v_t B(a,b,T-t)}, \tag{A.2}$$

where A and B are defined as

$$A(a, b, \tau) := -\frac{2\kappa\bar{v}}{\delta^2} \ln\left(\frac{2\chi e^{\frac{(\chi+\kappa)\tau}{2}}}{(\delta^2 a + \kappa)(e^{\chi\tau} - 1) + \chi(e^{\chi\tau} + 1)}\right) \tag{A.3}$$

and

$$B(a, b, \tau) := \frac{a\left(\chi + \kappa + e^{\chi\tau}(\chi - \kappa)\right) + 2b(e^{\chi\tau} - 1)}{(\delta^2 a + \kappa)(e^{\chi\tau} - 1) + \chi(e^{\chi\tau} + 1)} \tag{A.4}$$

with $\chi := \sqrt{\kappa^2 + 2b\delta^2}$, $\tau = T - t$, see Lemma A.4 in Chen et al. (2018) or Proposition 5.1 in Kraft (2005).