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Care-dependent Tontines

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Abstract

With the gradual deepening of aging, the affordability of long-term care (LTC) services in aging societies will become increasingly questionable. Both private stand-alone LTC insurance and care-dependent annuity do not seem to provide efficient solutions, due to their high risk charges. In this article, we propose two ways of combining the long-term care business with retirement tontines, i.e. care-dependent tontines, which shift a part of longevity risk to policyholders and provide increased payments in care-dependent state. We determine the optimal payment structures of these products that maximize the individuals' expected lifetime utility. Using realistic risk loadings based on data from China Health and Retirement Longitudinal Study (CHARLS) and capital requirement of China Risk Oriented Solvency System (C-ROSS), it is shown that both care-dependent tontines present as better choices for individuals, comparing to care-dependent annuities. The results are robust in a different regulation scheme – Solvency II. Our findings contribute to improving the penetration of elderly care insurance with appealing care-dependent insurance products.

Keywords: Tontine; Care-dependent; Lifetime utility.

JEL: G22, G13

1   Introduction

The aging wave has been sweeping the world. With continuous improvement of medical and social security services, people have been living longer in expectation. Although this is great progress for humanity, it raises plenty of aging problems in many countries. For instance, the population aging speed in China is one of the fastest among countries. Since entering an aging era in 2000, the degree of population aging in China has been rising. According to a report published by AIC and IPLE (2020), in 2019, among the Chinese people aged over 60 in the survey area, there were 7% in moderate Activities of Daily Living (ADL)[1] disability status and 4.8% in severe ADL disability. Besides, the prediction statistics from Statista (Statista (2021)) have pointed out that China's population over 60 will rise from 17.4% of the total population in 2020, to 29.9% in 2040 with a rather low fertility rate.

Long-term care (LTC) insurance is born out of the long-term need for care, especially in later life. It assists retirees to cover expenses they may need with care at home or in a facility when they cannot perform daily living activities. Although some types of LTC expenditure have already been covered in the public health/LTC insurance scheme, the increasing number of the disabled elderly has prompted extensive concerns about the future affordability of LTC services (Morrow and Röger (2003)). In this sense, the development of a private LTC insurance market has become even more important and urgent. However, industry practice experiences from OECD countries have proved that the private long-term care insurance is not an effective solution as it only accounts for a small market. Not much LTC insurance is issued to a relatively high percentage of potential buyers at the retirement ages, for the sake of reducing the self-bearing risks of the issuers[2]. These policy exclusions (Brown and Warshawsky (2013)) exactly keep those who need the LTC insurance away. Besides, LTC policies are offered at premiums that are substantially higher than the actuarially fair level. According to Brown and Finkelstein (2009), the loading can be enormously high, particularly when taking account of the lapse of policies. In the U.S. market, it raises the loading of 18 cents on the dollar to 51 cents on the dollar.

On the basis of these facts, Murtaugh et al. (2001) put forward empirically the idea of integrating the life annuity and LTC insurance into one product. Other scholars also

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[1] ADL refers to people's daily self-care activities such as bathing, dressing, eating, continence, toileting, and transferring.

[2] The LTC insurance is intensively underwritten by its issuers (Brown and Warshawsky (2013)).

come up with different ways to combine lifetime annuities and LTC insurance (Brown and Warshawsky (2013), Vidal-Melia et al. (2016), Pitacco (2015), Hoermann and Ruß (2008), Ramsay and Oguledo (2020), Chen et al. (2021), etc.). In fact, by combining the life annuity and LTC coverage, it may make the long-term insurance product available to more people, especially those who are currently rejected by the stand-alone LTC insurance (e.g. those with poor health or unhealthy lifestyles). And also, more available insurance products may contribute to enriching products in the old age insurance market as well as increasing the penetration of elderly care insurance. In addition, it reduces the cost of both stand-alone insurance products. Through the estimation of Murtaugh et al. (2001), such bundled products significantly reduce premiums by 35 percent relative to the stand-alone products.

Even if the care-dependent annuities are typically less expensive than buying life annuity and long-term care products separately, the market for these products is still rather limited. According to Webb (2009), these bundled products only make up less than 10% of the voluntary annuity market in the United States. One reason is that adverse selection still exists in the care-dependent annuity market (Zhou-Richter and Gründl (2011))[3]. Another possible reason is that it is rather challenging for insurers to determine reserves for these products, which might also lead to high administration and risk charges. One possible way for the insurer to transfer a part of these risks to policyholders is to combine the long-term care business with retirement tontines, see e.g. Hieber and Lucas (2020) putting forward a care-dependent tontine scheme. Under the framework of tontines, it is policyholders that bear most of the longevity risks, stated differently, the "mortality credit" of the dead provides more payments for the alive. For the LTC coverage, it requires a higher payment when the policyholder moves to a severely sick state, while the mortality rate in this state will also increase. Thus, the increased care-dependent payment could be viewed as an advance of additional "mortality credit"[4]. The focus of Hieber and Lucas (2020) is on how to distribute the mortality credit such that fairness can be achieved among the individuals.

In contrast to Hieber and Lucas (2020), in this article, we propose two ways of designing

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[3]An adverse selection is said to exist in the care-dependent annuity market when people with both high longevity and morbidity risk are more likely to purchase such products, which will force the insurers to increase their premiums accordingly. The adverse selection in annuities is reduced (Murtaugh et al. (2001), Spillman et al. (2003), Brown and Warshawsky (2013)), but not eliminated in the care-dependent annuity market (Zhou-Richter and Gründl (2011)).

[4]Mortality of the severely sick ones occurs more likely than that of healthy ones, and the corresponding "mortality credits" are larger. The additional "mortality credit" here represents the excess part of sick ones' "mortality credit" over the healthy ones'.

the care-dependent tontines (CDT): (i) consider that all insured members (both healthy ones and severely sick ones) are in one pool; (ii) the insured members are divided into two groups: the healthy, and the severely sick. At each time t, we reallocate the ones whose states get changed into the corresponding groups. Considering the fact that there are three main categories of LTC insurance products currently sold in the private LTC insurance market, i.e. predetermined benefits, reimbursement benefits and service benefits (Denuit et al. (2019)). Our products can be regarded as care insurance products with predetermined benefits. Although we do not explicitly model the care costs in the benefits, in designing our products, we do take account of the fact that people in care-dependent state have increasing liquidity need for care costs. Individuals with various risk aversion can choose those payments maximizing their own expected lifetime utility. In other words, this article stands on the policyholder's side and describes the optimal decisions of payment streams that the policyholder would prefer. Based on fair premium calculations and a utility framework (Chen and Rach (2019), Chen et al. (2020), Chen and Rach (2021)), we determine the structures of the CDTs that maximize the policyholder's expected lifetime utility with constant relative risk aversion and no bequest motives. Furthermore, in this article, we compare different care-dependent products analytically and numerically.

When considering an actuarially fair premium, we find that the care-dependent annuity (CDA) is the best choice among different care-dependent products. Adding care-dependent payoffs to the regular retirement products does not change the preference order of the tontine and the annuity under actuarially fair pricing (Milevsky and Salisbury (2015), Chen et al. (2019)). In order to conduct a realistic comparison among the three care-dependent products, we rely on the data of China Health and Retirement Longitudinal Study (CHARLS) and China Risk Oriented Solvency System (C-ROSS). We compute the risk loadings following C-ROSS for various care-dependent products under consideration. Our results reveal that the two types of CDTs are more attractive than CDA for policyholders, when taking account of the realistic risk loadings. More specifically, we show that the two CDTs lead to lower gross premiums than the CDA product, while ensuring the same expected lifetime utility to the policyholders as the CDA product. Besides, as the pool size grows larger, the advantage of CDTs over CDA becomes more substantial. Further, we find that the two-pool CDT is more attractive to those who are less risk-averse, while one-pool CDT is a better choice for those with larger risk aversion coefficients. Finally, we detect that for insurers, under our baseline parameter setting, constructing a care-dependent tontine in a two-pool structure may be

the most cost-efficient way to add the care-dependent coverage among the three products studied in this article. In order to examine the possible robustness of our results under different regulation regimes, we also compute the risk loadings with reference to the capital requirements of Solvency II, the European insurance regulation framework. Using the baseline parameter setting, it still shows that the CDTs are preferable to CDA, which means our results are to some extent robust across regulation regimes. Our findings provide the insurers with considerable views on the design of care-dependent insurance contracts, which may boost the development of elderly care insurance market and contribute to fulfilling the increasing demand of diversified care insurance products.

The remainder of this article is structured as follows: in Section 2, we introduce our transition probabilities used throughout the entire article. In Section 3, we derive the optimal payoffs of the care-dependent tontines from two different angles respectively. Additionally, for the sake of product comparison later, we also give the optimal payoffs of the care-dependent annuities under our model settings. Section 4 sets up the framework for the product comparison under a realistic setting. And Section 5 presents the numerical analysis, describing the used statistics and results. In the last Section, we conclude the article. Several proofs, a statistical procedure to determine the transition probabilities, the risk loading computation following the capital requirement of Solvency II, and additional tables concerning the sensitivity analysis are listed in the appendix.

2　Transition Probabilities

In this section, we introduce the transition probabilities used throughout the entire article on the basis of a three-state model. The transition probabilities reflect the policyholder's health condition.

Let x be the initial age, and S_{x+t} be a Markovian process describing the development of a single policy in continuous time. Then the LTC insurance is modelled by a multi-state model with state space $\mathcal{S} = \{1 = \text{healthy/active}, 2 = \text{severely sick/care-dependent}, 3 = \text{dead}\}$, and a set of transitions according to Figure 2.1. We assume the policyholder is healthy at the initial age, i.e. $S_x \equiv 1$.

Given that a policyholder is in state i at age y, the transition probabilities of this policyholder being in state j at age $y + t$, are defined as:

$$^{ij}_t p_y = \mathbb{P}\{S_{y+t} = j \mid S_y = i\} \quad i, j \in \mathcal{S}, i \neq j. \tag{2.1}$$
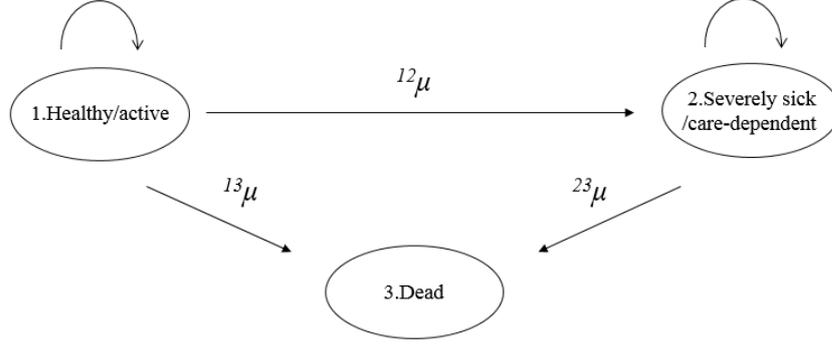
Figure 2.1: State set and transition intensities

The corresponding transition intensities could be written as (Haberman and Pitacco (1998)):

$$^{ij}\mu_y = \lim_{t \to 0} \left(\frac{^{ij}_t p_y}{t} \right), \quad i,j \in \mathcal{S}, i \neq j. \tag{2.2}$$

Furthermore, the probability of a policyholder being in state i at age y and remaining in state i till age $y+t$ is:

$$^{ii}_t p_y = \mathbb{P}\{S_{y+t} = i, \forall t \mid S_y = i\}, \quad i \in \mathcal{S}. \tag{2.3}$$

Given the transition intensities of the multi-state model, we obtain the probabilities (2.1) and (2.3) by Kolmogorov forward differential equations (Haberman and Pitacco (1998)).

$$^{11}_t p_y = \exp \left\{ -\int_0^t \left[^{12}\mu_{y+s} + ^{13}\mu_{y+s} \right] \mathrm{d}s \right\}, \tag{2.4}$$

$$^{12}_t p_y = \int_0^t \left[^{11}_s p_y \cdot ^{12}\mu_{y+s} \cdot ^{22}_{t-s} p_{y+s} \right] \mathrm{d}s, \tag{2.5}$$

$$_t p_y = ^{11}_t p_y + ^{12}_t p_y, \tag{2.6}$$

$$^{22}_t p_y = \exp \left\{ -\int_0^t ^{23}\mu_{y+s} \mathrm{d}s \right\}. \tag{2.7}$$

- $^{11}_t p_y$ is the occupancy probability of a healthy individual aged y staying healthy at age $y+t$;

- $^{12}_t p_y$ denotes the probability of a healthy individual aged y, becoming severely sick between $(0,t)$ and remaining severely sick until t;

- $_t p_y$ is the t-year survival probability of a healthy individual aged y, which includes

the case that he remains healthy for t years and the case that he becomes severely sick during the t-year time;

- while $^{22}_{t}p_y$ represents the occupancy probability of a y-year-old sick individual staying severely sick at age $y + t$.

Here, we disregard the possibility of recovery from the severely sick state, i.e. the transition $\{2 \to 1\}$ is not considered (Pitacco (2016); Hanewald et al. (2019); Chen et al. (2021)).

3 Care-dependent Products

Consider a retiree at age x endowed with a wealth level $v > 0$ at his retirement date (in our framework at time $t = 0$). Following Yaari (1965)'s setting, we assume a continuous-time stream of payments for the care-dependent tontines. In a care-dependent tontine contract, we further assume that the risks are shared among homogeneous policyholders who are assumed to be identical copies of each other. There are n policyholders initially. To construct the care-dependent tontine, we put forward two ways of designing the product: (i) consider that all insured members (both healthy ones and severely sick ones) are in one pool; (ii) the insured members are divided into two groups: the healthy, and the severely sick. We will introduce these two kinds of care-dependent tontines in the following subsections respectively.

3.1 One-pool Care-dependent Tontine

Firstly, we consider all the members in only one pool at each time t. Here the one pool means that only alive and dead persons are identified at each time t, while different payment streams are provided to the healthy and the severely sick respectively. We use the notations $d_1^{oc_1}(t)$ and $d_2^{oc_1}(t)$ for the payment streams to the healthy and the severely sick separately, which are pre-determined when the policy is underwritten. Correspondingly, $b_1^{oc_1}(t)$ and $b_2^{oc_1}(t)$ are the payoffs for the healthy members and the severely sick members. In accordance to the tontine scheme stated in Milevsky and Salisbury (2015), we define the care-dependent tontine payoff for the policyholders in the healthy state as

$$b_1^{oc_1}(t) := \begin{cases} \mathbb{1}_{\{S_{x+t}=1\}} \frac{n d_1^{oc_1}(t)}{N(t)}, & \text{if } N(t) > 0, \\ 0, & \text{else} \end{cases}, \tag{3.1}$$

where $N(t)$ denotes the number of alive policyholders at time t, containing both healthy and severely sick ones.

In another case, when the policyholder is severely sick, we define the care-dependent tontine payoff as:

$$b_2^{oc_1}(t) := \begin{cases} \mathbb{1}_{\{S_{x+t}=2\}} \dfrac{nd_2^{oc_1}(t)}{N(t)}, & \text{if } N(t) > 0, \\ 0, & \text{else} \end{cases}, \qquad (3.2)$$

As individuals usually have a higher demand for liquidity when becoming severely sick, it might be more reasonable to expect $d_2^{oc_1}(t) > d_1^{oc_1}(t)$. In later sections, a risk-averse policyholder will choose $d_1^{oc_1}(t)$ and $d_2^{oc_1}(t)$ optimally to maximize his expected lifetime utility (Equation (3.13)). The results are then given in Theorem 3.1. After these optimal payments are derived, we will examine the relation between $d_1^{oc_1}(t)$ and $d_2^{oc_1}(t)$.

Next, we determine the actuarially fair premium of this care-dependent tontine under an actuarial pricing framework. To focus on the effects of mortality and morbidity risks, for simplicity, we ignore the financial risks[5]. Besides, the policyholder under consideration is assumed to pay a single premium at the beginning of the contract. Thus, we assume that the premium earns a constant and continuously compounded risk-free rate $r \in R$. This assumption of the single premium is made not only for convenience of computation, but also for the fact that single premium has been widely used for retirement products. In 2018, sales of single-premium immediate annuity in the United States amounted to 9.7 billion dollars[6]. Meanwhile, insurers including OneAmerica, Global Atlantic Financial Group, and etc., have already provided the single-premium LTC plus annuity products in the market[7]. The actuarially fair premium for the considered one-pool care-dependent tontine is given by

$$P_0^{oc_1} := \mathbb{E}\left[\int_0^\infty e^{-rt}b_1^{oc_1}(t)\mathrm{d}t + \int_0^\infty e^{-rt}b_2^{oc_1}(t)\mathrm{d}t\right]$$

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[5]When defining the predetermined benefits for the care-dependent tontine products, we neglect the financial market risk. In other words, we do not consider the interest rate risk, nor are the benefits unit-linked. Incorporating equity risk can make the benefits more volatile. If we compare pure mortality-linked products with unit-linked products, it is to expect that the unit-linked products probably are more preferable. However, it is unclear what the preference order will turn out if we compare a unit-linked care-dependent annuity to a unit-linked care-dependent tontine. We leave this question for future research.

[6]LIMRA Secure Retirement Institute: Total Annuity Sales Have Best Quarter in Nearly 10 Years. https://www.limra.com/en/newsroom/news-releases/2019/limra-secure-retirement-institute-total-annuity-sales-have-best-quarter-in-nearly-10-years

[7]https://www.ltcinsuranceconsultants.com/long-term-care-annuity

$$= \mathbb{E}\left[\int_0^\infty e^{-rt}\mathbb{1}_{\{S_{x+t}=1\}}\frac{nd_1^{oc_1}(t)}{N(t)}\mathrm{d}t\right] + \mathbb{E}\left[\int_0^\infty e^{-rt}\mathbb{1}_{\{S_{x+t}=2\}}\frac{nd_2^{oc_1}(t)}{N(t)}\mathrm{d}t\right]$$

$$= \int_0^\infty e^{-rt}\cdot {}_t^{11}p_x \sum_{k=0}^{n-1}\frac{nd_1^{oc_1}(t)}{k+1}\binom{n-1}{k}({}_tp_x)^k(1-{}_tp_x)^{n-1-k}\,\mathrm{d}t$$

$$+ \int_0^\infty e^{-rt}\cdot {}_t^{12}p_x \sum_{k=0}^{n-1}\frac{nd_2^{oc_1}(t)}{k+1}\binom{n-1}{k}({}_tp_x)^k(1-{}_tp_x)^{n-1-k}\,\mathrm{d}t$$

$$= \int_0^\infty e^{-rt}\frac{{}_t^{11}p_x}{{}_tp_x}\sum_{k=1}^{n}\binom{n}{k}({}_tp_x)^k(1-{}_tp_x)^{n-k}\cdot d_1^{oc_1}(t)\mathrm{d}t$$

$$+ \int_0^\infty e^{-rt}\frac{{}_t^{12}p_x}{{}_tp_x}\sum_{k=1}^{n}\binom{n}{k}({}_tp_x)^k(1-{}_tp_x)^{n-k}\cdot d_2^{oc_1}(t)\mathrm{d}t$$

$$= \int_0^\infty e^{-rt}\frac{{}_t^{11}p_x}{{}_tp_x}\left(1-(1-{}_tp_x)^n\right)\cdot d_1^{oc_1}(t)\mathrm{d}t + \int_0^\infty e^{-rt}\frac{{}_t^{12}p_x}{{}_tp_x}\left(1-(1-{}_tp_x)^n\right)\cdot d_2^{oc_1}(t)\mathrm{d}t.$$

$$(3.3)$$

For lines three and four, in the case that the policyholder is still alive, $N(t)$ is at least 1. Thus, the overall pool size satisfies $N(t)-1 \sim \mathrm{Bin}\,(n-1,{}_tp_x)$.

The policyholder's utility is assumed to be given by the following state-dependent utility function

$$U(b_i(t)) := \sum_{i\in\mathcal{S}}u_i(b_i(t)),\qquad (3.4)$$

where b_i is the payoff of care-dependent tontine in state $i, i\in\mathcal{S}$. And $u_i(b_i(t))$, $i\in\mathcal{S}$ is strictly increasing and concave functions in b_i.

It is usually hard to determine a person's utility function exactly. Thus, a possible alternative is to establish one's utility function according to some plausible properties (e.g. people are non-satiated, and typically show decreasing absolute risk aversion in wealth). The power utility satisfies the above two properties and is abundantly used in theoretical research regarding the long-term care insurance (Brown and Finkelstein (2008), Davidoff (2010), Ameriks et al. (2020), Chen et al. (2021), etc.) because of its nice analytical tractability. After setting up the model for utility, one can fit it with real-world data to estimate the relevant parameters (e.g. risk aversion coefficient). For example, Friedman (1974) has estimated the policyholder's relative risk aversion coefficient w.r.t. power utility applying the data from health insurance. Szpiro (1986) uses the property and liability insurance data of the U.S. to estimate the relative risk aversion coefficient. Therefore, we assume that the policyholder evaluates payoffs through a power utility with

a constant relative risk aversion coefficient $\gamma \in (0, \infty), \gamma \neq 1$. Furthermore, considering the fact that different values of γ may affect our results, a sensitivity analysis is given w.r.t. γ later on. Then, for a healthy individual, his utility is

$$u_1(\omega) = \frac{\omega^{1-\gamma}}{1-\gamma}, \tag{3.5}$$

where ω is the payoff for healthy participants.

We introduce a payment weighting factor α, with

$$\alpha = \begin{cases} \in (0, 1], & \text{if } \gamma > 1, \\ > 1, & \text{if } \gamma \in (0, 1) \end{cases}, \tag{3.6}$$

and the utility function for the severely sick is defined as

$$u_2(\omega) = \frac{(\alpha \cdot \omega)^{1-\gamma}}{1-\gamma}. \tag{3.7}$$

We have known from the existing literature that the severe sickness may increase the marginal utility of payments (e.g. Evans and Viscusi (1991)). Moreover, for the purpose of designing attractive products, we would like to compensate the severely sick people with higher payments. In this sense, the marginal utility in the severely sick state is assumed to be not less than that in the healthy state. Note that the marginal utility in the healthy state and the severely sick state are $\frac{\partial u_1(\omega)}{\partial \omega} = \omega^{-\gamma}$ and $\frac{\partial u_2(\omega)}{\partial \omega} = \alpha^{1-\gamma}\omega^{-\gamma}$ respectively. In order to achieve this assumption, $\alpha^{1-\gamma} \geq 1$ needs to be met. Thus, different values of α are needed, due to the negative and positive sign of $(1-\gamma)$ for $\gamma > 1$ and $\gamma \in (0, 1)$ respectively. For $\alpha = 1$, the utility function of the severely sick is identical to that of the healthy.

Here, for this one-pool case, the objective function can be written as

$$\sup_{d_1^{oc_1}(t), d_2^{oc_1}(t)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(u_1\left(\frac{nd_1^{oc_1}(t)}{N(t)}\right) \mathbb{1}_{\{S_{x+t}=1\}} + u_2\left(\frac{nd_2^{oc_1}(t)}{N(t)}\right) \mathbb{1}_{\{S_{x+t}=2\}}\right) dt\right]$$

$$\text{s.t. } P_0^{oc_1} \leq v, \tag{3.8}$$

where ρ is the individual's subjective discount rate and v the initial wealth the policyholder owns to invest in the care-dependent tontine. In Theorem 3.1, we solve the optimization problem (3.8).

Theorem 3.1. *Assume that the policyholder's preferences for payments can be described*

by (3.8). *For a one-pool care-dependent tontine product, maximizing the expected state-dependent lifetime utility* (3.8), *the optimal payment stream functions are given by*

$$d_1^{oc_1*}(t) = \frac{e^{1/\gamma(r-\rho)t}}{(\lambda_{oc_1}^*)^{1/\gamma}} \cdot \frac{(\kappa_{n,\gamma}({}_tp_x))^{1/\gamma}}{(1 - (1 - {}_tp_x)^n)^{1/\gamma}}, \tag{3.9}$$

$$d_2^{oc_1*}(t) = \alpha^{1/\gamma-1} \cdot \frac{e^{1/\gamma(r-\rho)t}}{(\lambda_{oc_1}^*)^{1/\gamma}} \cdot \frac{(\kappa_{n,\gamma}({}_tp_x))^{1/\gamma}}{(1 - (1 - {}_tp_x)^n)^{1/\gamma}}, \tag{3.10}$$

$$\Rightarrow d_2^{oc_1*} = \alpha^{1/\gamma-1} \cdot d_1^{oc_1*}, \tag{3.11}$$

where $\kappa_{n,\gamma}({}_tp_x) = \sum_{k=1}^{n} \binom{n}{k} \left(\frac{k}{n}\right)^\gamma ({}_tp_x)^k (1 - {}_tp_x)^{n-k}$.

Furthermore, the optimal Lagrangian multiplier $\lambda_{oc_1}^* > 0$ *is given by*

$$\lambda_{oc_1}^* = \left(\frac{1}{v}\left(\int_0^\infty e^{\left(\frac{r-\rho}{\gamma} - r\right)t} \cdot \frac{\left(\frac{{}_t^{11}p_x}{{}_tp_x} \cdot \kappa_{n,\gamma}({}_tp_x)\right)^{1/\gamma}}{\left(\frac{{}_t^{11}p_x}{{}_tp_x} \cdot (1 - (1 - {}_tp_x)^n)\right)^{1/\gamma-1}} \mathrm{d}t \right.\right.$$

$$\left.\left. + \int_0^\infty e^{\left(\frac{r-\rho}{\gamma} - r\right)t} \cdot \alpha^{1/\gamma-1} \cdot \frac{\left(\frac{{}_t^{12}p_x}{{}_tp_x} \cdot \kappa_{n,\gamma}({}_tp_x)\right)^{1/\gamma}}{\left(\frac{{}_t^{12}p_x}{{}_tp_x} \cdot (1 - (1 - {}_tp_x)^n)\right)^{1/\gamma-1}} \mathrm{d}t \right)\right)^\gamma. \tag{3.12}$$

And then, the expected state-dependent lifetime utility is given by

$$U_0^{oc_1*} = \int_0^\infty e^{-\rho t} \frac{{}_t^{11}p_x}{{}_tp_x} \sum_{k=1}^{n} \binom{n}{k} \left(\frac{k}{n}\right)^\gamma ({}_tp_x)^k (1 - {}_tp_x)^{n-k} \cdot u_1(d_1^{oc_1*}(t))\mathrm{d}t$$

$$+ \int_0^\infty e^{-\rho t} \frac{{}_t^{12}p_x}{{}_tp_x} \sum_{k=1}^{n} \binom{n}{k} \left(\frac{k}{n}\right)^\gamma ({}_tp_x)^k (1 - {}_tp_x)^{n-k} \cdot u_2(d_2^{oc_1*}(t))\mathrm{d}t. \tag{3.13}$$

As can be seen in (3.11), α and γ play crucial roles in the relation between $d_2^{oc_1*}$ and $d_1^{oc_1*}$. For $\alpha = 1$, we return to the original tontine setting of Chen et al. (2019) and $d_2^{oc_1*} = d_1^{oc_1*}$; our optimal tontine payments take the same form, with the exclusive difference that the critical Lagrangian multiplier $\lambda_{oc_1}^*$ depends on the various transition probabilities.

As shown by Equation (3.11), the optimal care-dependent tontine payment $d_2^{oc_1*}$ for the severely sick is $\alpha^{1/\gamma-1}$ times of $d_1^{oc_1*}$ for the healthy. According to the different value arranges of α when $\gamma > 1$ and $\gamma \in (0,1)$ (see (3.6)), it always holds $\alpha^{1/\gamma-1} > 1$. It

means, the sick members will receive more payments, which coincides with our aim – to provide higher liquidity to cover the medical care cost.

3.2 Two-pool Care-dependent Tontine

In another case, we distinguish two groups of policyholders at each time $t > 0$, i.e. the pool of healthy members, and the pool of severely sick members. At each time t, we reallocate the ones who become severely sick into the corresponding pool.

Then the payoff for the healthy members is defined as:

$$
b_1^{oc2}(t) := \begin{cases} \mathbb{1}_{\{S_{x+t}=1\}} \frac{nd_1^{oc2}(t)}{N_1(t)}, & \text{if } N_1(t) > 0, \\ 0, & \text{else} \end{cases} . \tag{3.14}
$$

And the payoff for the severely sick members is defined as:

$$
b_2^{oc2}(t) := \begin{cases} \mathbb{1}_{\{S_{x+t}=2\}} \frac{nd_2^{oc2}(t)}{N_2(t)}, & \text{if } N_2(t) > 0, \\ 0, & \text{else} \end{cases} . \tag{3.15}
$$

Here $N_1(t)$ represents the number of healthy policyholders, and $N_2(t)$ is that in the pool of the severely sick.

For a two-pool care-dependent tontine, the actuarially fair premium is given by:

$$
\begin{aligned}
P_0^{oc2} &:= \mathbb{E}\left[\int_0^\infty e^{-rt} b_1^{oc2}(t)\mathrm{d}t + \int_0^\infty e^{-rt} b_2^{oc2}(t)\mathrm{d}t \right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-rt} \mathbb{1}_{\{S_{x+t}=1\}} \frac{nd_1^{oc2}(t)}{N_1(t)}\mathrm{d}t \right] + \mathbb{E}\left[\int_0^\infty e^{-rt} \mathbb{1}_{\{S_{x+t}=2\}} \frac{nd_2^{oc2}(t)}{N_2(t)}\mathrm{d}t \right] \\
&= \int_0^\infty e^{-rt}\, {}^{11}_t p_x \sum_{k=0}^{n-1} \frac{nd_1^{oc2}(t)}{k+1} \binom{n-1}{k} \left({}^{11}_t p_x\right)^k \left(1 - {}^{11}_t p_x\right)^{n-1-k}\mathrm{d}t \\
&\quad + \int_0^\infty e^{-rt}\, {}^{12}_t p_x \sum_{k=0}^{n-1} \frac{nd_2^{oc2}(t)}{k+1} \binom{n-1}{k} \left({}^{12}_t p_x\right)^k \left(1 - {}^{12}_t p_x\right)^{n-1-k}\mathrm{d}t \\
&= \int_0^\infty e^{-rt} \sum_{k=1}^{n} \binom{n}{k} \left({}^{11}_t p_x\right)^k \left(1 - {}^{11}_t p_x\right)^{n-k} \cdot d_1^{oc2}(t)\mathrm{d}t \\
&\quad + \int_0^\infty e^{-rt} \sum_{k=1}^{n} \binom{n}{k} \left({}^{12}_t p_x\right)^k \left(1 - {}^{12}_t p_x\right)^{n-k} \cdot d_2^{oc2}(t)\mathrm{d}t \\
&= \int_0^\infty e^{-rt} \left(1 - \left(1 - {}^{11}_t p_x\right)^n\right) \cdot d_1^{oc2}(t)\mathrm{d}t
\end{aligned}
$$

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$$+ \int_0^\infty e^{-rt} \left(1 - \left(1 - {}^{12}_t p_x\right)^n\right) \cdot d_2^{oc_2}(t)\mathrm{d}t. \tag{3.16}$$

In the third line, conditional on $S_{x+t} = 1$, there is at least one alive member in the healthy pool. Then the healthy pool size is distributed as $N_1(t) - 1 \sim \mathrm{Bin}\left(n - 1, {}^{11}_t p_x\right)$. In addition, in line four, conditional on $S_{x+t} = 2$, the number in the severely sick pool is not less than 1. Then $N_2(t) - 1 \sim \mathrm{Bin}\left(n - 1, {}^{12}_t p_x\right)$.

Here again we write down the objective function:

$$\sup_{d_1^{oc_2}(t), d_2^{oc_2}(t)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(u_1\left(\frac{nd_1^{oc_2}(t)}{N_1(t)}\right)\mathbb{1}_{\{S_{x+t}=1\}} + u_2\left(\frac{nd_2^{oc_2}(t)}{N_2(t)}\right)\mathbb{1}_{\{S_{x+t}=2\}}\right)\mathrm{d}t\right]$$

$$\text{s.t. } P_0^{oc_2} \leq v. \tag{3.17}$$

In Theorem 3.2, we solve the objective function (3.17).

Theorem 3.2. *Assume that the policyholder's preferences for payments can be described by (3.17). For a two-pool care-dependent tontine product, maximizing the expected state-dependent lifetime utility (3.17), the optimal payment stream functions are given by*

$$d_1^{oc_2*}(t) = \frac{e^{1/\gamma(r-\rho)t}}{(\lambda_{oc_2}^*)^{1/\gamma}} \cdot \frac{\left(\kappa_{n,\gamma}({}^{11}_t p_x)\right)^{1/\gamma}}{\left(1 - \left(1 - {}^{11}_t p_x\right)^n\right)^{1/\gamma}}, \tag{3.18}$$

$$d_2^{oc_2*}(t) = \alpha^{1/\gamma-1} \cdot \frac{e^{1/\gamma(r-\rho)t}}{(\lambda_{oc_2}^*)^{1/\gamma}} \cdot \frac{\left(\kappa_{n,\gamma}({}^{12}_t p_x)\right)^{1/\gamma}}{\left(1 - \left(1 - {}^{12}_t p_x\right)^n\right)^{1/\gamma}}. \tag{3.19}$$

The optimal Lagrangian multiplier $\lambda_{oc_2}^* > 0$ is given by

$$\lambda_{oc_2}^* = \left(\frac{1}{v}\left(\int_0^\infty e^{\left(\frac{r-\rho}{\gamma}-r\right)t}\right.\right.$$

$$\left.\left. \cdot \left[\frac{\left(\kappa_{n,\gamma}({}^{11}_t p_x)\right)^{1/\gamma}}{\left(1 - \left(1 - {}^{11}_t p_x\right)^n\right)^{1/\gamma-1}} + \frac{\alpha^{1/\gamma-1} \cdot \left(\kappa_{n,\gamma}({}^{12}_t p_x)\right)^{1/\gamma}}{\left(1 - \left(1 - {}^{12}_t p_x\right)^n\right)^{1/\gamma-1}}\right]\mathrm{d}t\right)\right)^\gamma. \tag{3.20}$$

And then, the expected state-dependent lifetime utility is given by

$$U_0^{oc_2} = \int_0^\infty e^{-\rho t}\sum_{k=1}^n \binom{n}{k}\left(\frac{k}{n}\right)^\gamma \left({}^{11}_t p_x\right)^k \left(1 - {}^{11}_t p_x\right)^{n-k} \cdot u_1(d_1^{oc_2*}(t))\mathrm{d}t$$

$$+ \int_0^\infty e^{-\rho t}\sum_{k=1}^n \binom{n}{k}\left(\frac{k}{n}\right)^\gamma \left({}^{12}_t p_x\right)^k \left(1 - {}^{12}_t p_x\right)^{n-k} \cdot u_2(d_2^{oc_2*}(t))\mathrm{d}t. \tag{3.21}$$

In comparison with the work of Hieber and Lucas (2020), which mainly focuses on the insurer's perspective, we start from the policyholder's angle. Our first design of care-dependent tontine coincides with the 3-state case of Hieber and Lucas (2020), i.e. we both pool together two types of people (the healthy and the severely sick). The difference is how we generate the optimal payoffs. Under the setting of Hieber and Lucas (2020), the authors compensate the sick members' extra mortality credits to themselves owing to higher mortality, thus the payoff for the severely sick gets higher than that for the healthy; additionally, the payoff for the severely sick will vary with the extra mortality credits. While in our case, a rational policyholder will choose the optimal payoffs that maximize his utility, under the budget constraint. In this sense, pricing is not the main purpose of this paper. The payoff for the severely sick is determined by the policyholder's payment weighting factor and the overall utility. It gets higher than the payoff for the healthy due to the payment weighting factor.

3.3 Care-dependent Annuity

In order to compare different care-dependent products, we write down the optimal payoff functions of a care-dependent annuity using our model framework. For a policyholder that is healthy, he will receive a regular annuity payoff of $c_1^{ac}(t) > 0$. Once he becomes care-dependent, a different payoff $c_2^{ac}(t) > 0$ will be provided. It is also expected that $c_2^{ac}(t) > c_1^{ac}(t)$ for a higher liquidity is desired when the policyholder gets severely sick. This will be examined later.

Before going towards the optimal payoffs, let us first write down the actuarially fair premium for the care-dependent annuity:

$$
\begin{aligned}
P_0^{ac} &:= \mathbb{E}\left[\int_0^\infty e^{-rt}\mathbb{1}_{\{S_{x+t}=1\}}c_1^{ac}(t)\mathrm{d}t\right] + \mathbb{E}\left[\int_0^\infty e^{-rt}\mathbb{1}_{\{S_{x+t}=2\}}c_2^{ac}(t)\mathrm{d}t\right] \\
&= \int_0^\infty e^{-rt}\,{}^{11}_tp_x c_1^{ac}(t)\mathrm{d}t + \int_0^\infty e^{-rt}\,{}^{12}_tp_x c_2^{ac}(t)\mathrm{d}t.
\end{aligned}
\tag{3.22}
$$

Analogously, the policyholder's utility is given by

$$
\sup_{c_1^{ac}(t),c_2^{ac}(t)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(u_1\left(c_1^{ac}(t)\right)\mathbb{1}_{\{S_{x+t}=1\}} + u_2\left(c_2^{ac}(t)\right)\mathbb{1}_{\{S_{x+t}=2\}}\right)\mathrm{d}t\right]
$$
$$
\text{s.t. } P_0^{ac} \le v.
\tag{3.23}
$$

Then we maximize the utility optimization problem (3.23) by Theorem 3.3.

Theorem 3.3. *For a care-dependent annuity product, the optimal payoffs are given by*

$$c_1^{ac*}(t) = \left(\lambda_{ac}^* \cdot e^{(\rho-r)t}\right)^{-1/\gamma} \tag{3.24}$$

$$c_2^{ac*}(t) = \alpha^{1/\gamma-1} \cdot \left(\lambda_{ac}^* \cdot e^{(\rho-r)t}\right)^{-1/\gamma}. \tag{3.25}$$

By budget constraint, we obtain

$$\lambda_{ac}^* = \left(\frac{1}{v} \int_0^\infty e^{\left(\frac{r-\rho}{\gamma}-r\right)t}({}^{11}_t p_x + \alpha^{1/\gamma-1} \cdot {}^{12}_t p_x)\mathrm{d}t\right)^\gamma. \tag{3.26}$$

The policyholder's expected state-dependent lifetime utility is given by

$$U_0^{ac*} = \int_0^\infty e^{-\rho t} \, {}^{11}_t p_x \cdot u_1(c_1^{ac*}(t))\mathrm{d}t + \int_0^\infty e^{-\rho t} \, {}^{12}_t p_x \cdot u_2(c_2^{ac*}(t))\mathrm{d}t. \tag{3.27}$$

Similarly as in one-pool care-dependent tontine case, $\alpha = 1$ leads to a regular annuity, which does not lead to an increase in payoff when becoming care-dependent, i.e. $c_1^{ac*}(t) = c_2^{ac*}(t)$.

In case that actuarially fair premiums are adopted for care-dependent tontines and care-dependent annuities as in Equations (3.3), (3.16) and (3.22), the relation between the care-dependent annuities and care-dependent tontines can be explored theoretically.

Proposition 3.4. *By results reached from problems (3.8), (3.17), and (3.23), we have*

$$U_0^{ac*} \geq U_0^{oc_1*}, \tag{3.28}$$

$$U_0^{ac*} \geq U_0^{oc_2*}. \tag{3.29}$$

Proof. See Appendix A.4. □

Adding care-dependency payoffs to the regular retirement products does not change the preference order of the tontine and the annuity under actuarially fair pricing. The optimal care-dependent annuities always deliver a higher expected lifetime utility level than the optimal care-dependent tontines. However, between the two care-dependent tontines, no clear relation can be detected.

4   Product Comparison under A Realistic Setting

So far, we have obtained the optimal payoffs as well as the corresponding utility of each product with actuarially fair premiums on a net basis. In the following, we take the realistic risk loadings into account, in order to explore how the attractiveness of the care-dependent tontines and care-dependent annuities would change in the real world[8]. We compute the gross premium which consists of the net premium and the risk loading (Section 4.1), and the expected life-time utility of each product. A product leading to a higher utility level (but costing more) is not necessarily better than a product leading to a lower utility level (but costing less). In Section 4.2, we will show how to make the considered care-dependent products reach the same expected utility level, such that we can focus on the comparison of the gross premiums of the products.

4.1   Calculation of Risk Loadings

By ignoring any administration or acquisition charges, we assume a single charged risk loading, i.e. $F_0^{oc_1} \geq 0$ for the 1-pool care-dependent tontine, $F_0^{oc_2} \geq 0$ for the 2-pool care-dependent tontine and $F_0^{ac} \geq 0$ for care-dependent annuity. Then we can specify the initial gross premium for each product by

$$\hat{P}_0^{oc_1} = P_0^{oc_1} + F_0^{oc_1}, \tag{4.1}$$

$$\hat{P}_0^{oc_2} = P_0^{oc_2} + F_0^{oc_2}, \tag{4.2}$$

$$\hat{P}_0^{ac} = P_0^{ac} + F_0^{ac}. \tag{4.3}$$

Now, in order to determine the risk loading, we further consider the longevity risk and dis-

---

[8]With reference to Chen et al. (2019), one could notice that when adding a reasonable risk loading for longevity risk, tontines become more attractive than annuities. However, it is not the case without a consideration about risk loading.

ease risk[9] following China Risk Oriented Solvency System (C-ROSS)[10] regulation about minimum capital requirement in China (see CBIRC (2020b)).

According to the Insurance Company Solvency Supervision Rules No.5, the minimum capital requirement in retirement insurance section is computed by a scenario-comparison method (CBIRC (2020b)). It is worthwhile to mention that the base scenario assumptions are used by the insurers to calculate their best-estimate liabilities. The capital requirement is the change of the present value (PV) between the unfavorable scenario and the base scenario, and it should not be negative, i.e.

$$MC = \max\left(PV_{unf} - PV_{bas}, 0\right), \tag{4.4}$$

where MC is the minimum capital requirement for overall insurance risk in retirement insurance business. The reinsurance factor is left out here, and PV_{bas} represents the PV under base scenario assumptions at time 0, and PV_{unf} is that under the unfavorable scenario at time 0. As there are actually two types of risks here, we will need to calculate the minimum capital under each kind of risks separately. We denote the MC_{long} as the minimum capital requirement for longevity risk and MC_{morb} for disease incidence risk. Then, they can be computed by

$$MC_{long} = \max\left(PV_{long} - PV_{bas}, 0\right), \tag{4.5}$$

$$MC_{morb} = \max\left(PV_{morb} - PV_{bas}, 0\right), \tag{4.6}$$

with PV_{long} and PV_{morb} representing the PV in unfavorable situation of longevity and

---

[9]As for the disease risk, in fact, it contains not only the disease incidence risk, but also the disease deterioration risk; these two types of disease risks are correlated (CBIRC (2020b)). Nevertheless, dealing with this may distract from our main points that we want to convey in this part. As our focus in this section is to illustrate the possible choice between two care-dependent tontines and the care-dependent annuity in view of a gross premium. Under actuarially fair premium setting, results of Proposition 3.4 have shown the care-dependent annuity is always more attractive than either of the care-dependent tontine. Thus, we ignore the disease deterioration risk faced by the severely sick members in the following contents. Disease incidence risk refers to the risk of insurance companies suffering unexpected losses due to the actual experience of the disease incidence being higher than expected. Disease deterioration risk means the risk that the disease deterioration trend is higher than expected and finally causes non-expected losses to the insurers.

[10]On 13 February 2015, the China Insurance Regulatory Commission (now called China Banking and Insurance Regulatory Commission (CBIRC)) released the rules of the new solvency regime, which is known as China Risk Oriented Solvency System (C-ROSS). It adopts a regulatory framework of 'three pillars', reshapes it according to the characteristics of China's insurance market to ensure that it is viable and reflects the realities of the emerging market, and makes sure it is comparable with other representative solvency regimes in the world (e.g. Solvency II in Europe and Risk-based Capital Phase 2 in Singapore) in terms of its three-pillar structure and specific regulatory standards and requirements.

disease incidence risk, respectively. The longevity risk and disease incidence risk are uncorrelated according to CBIRC (2020b). Thus we get the overall required minimum insurance capital by

$$MC_{ins} = \sqrt{MC_{long}^2 + MC_{morb}^2}.$$ (4.7)

Refering to CBIRC (2020b), the assumptions for one-year probability under the unfavorable scenario are defined to be the assumptions for one-year probability under the base scenario multiplied by certain shock factors. The unfavorable scenario assumption is equal to the base scenario assumption $\times (1 + SF)$, where SF is the unfavorable scenario factor, the proportional shift upward or downward of the underlying assumption (e.g. the best-estimate survival probabilities).

In the aspect of longevity risk, SF of longevity risk is based on the proportional shift downward of the base mortality assumption at all future policy dates. We denote the unfavorable scenario factor of longevity risk by SF_t^{long}, and it takes the values according to the policy duration as follows (CBIRC (2020b)):

$$SF_t^{long} = \begin{cases} (1 - 3\%)^t - 1 & 0 < t \leq 10 \\ (1 - 3\%)^{10} \cdot (1 - 2\%)^{t-10} - 1 & 10 < t \leq 20 \\ (1 - 3\%)^{10} \cdot (1 - 2\%)^{10} \cdot (1 - 1\%)^{t-20} - 1 & 20 < t \leq 30 \\ (1 - 3\%)^{10} \cdot (1 - 2\%)^{10} \cdot (1 - 1\%)^{10} - 1 & t > 30 \end{cases}.$$ (4.8)

In the aspect of disease incidence risk, the unfavorable scenario factor SF_t^{morb} is based on the proportional shift upward of the base morbidity assumption at all future policy dates. We set it to be 20%, which is stated by CBIRC (2020b), i.e.

$$SF_t^{morb} = 20\%, \quad \forall t.$$ (4.9)

In the unfavorable scenario, we denote p^{long} for probability in consideration with a longevity risk and p^{morb} is applied to represent the unfavorable probability with the disease incidence risk. Then, for different care-dependent products, the PVs of cash flows for different scenarios at time 0 are given by

$$PV_i^{oc_1} = \int_0^\infty e^{-rt} \frac{_{t}^{11}p_x^i}{_t p_x^i} \left(1 - \left(1 - {_t p_x^i}\right)^n\right) \cdot d_1^{oc_1*}(t)\mathrm{d}t$$
$$+ \int_0^\infty e^{-rt} \frac{_{t}^{12}p_x^i}{_t p_x^i} \left(1 - \left(1 - {_t p_x^i}\right)^n\right) \cdot d_2^{oc_1*}(t)\mathrm{d}t,$$ (4.10)

$$PV_i^{oc_2} = \int_0^\infty e^{-rt} \left(1 - \left(1 - {}_{t}^{11}p_x^i \right)^n \right) \cdot d_1^{oc_2*}(t) \mathrm{d}t$$

$$+ \int_0^\infty e^{-rt} \left(1 - \left(1 - {}_{t}^{12}p_x^i \right)^n \right) \cdot d_2^{oc_2*}(t) \mathrm{d}t, \tag{4.11}$$

$$PV_i^{ac} = \int_0^\infty e^{-rt} \, {}_{t}^{11}p_x^i c_1^{ac*}(t) \mathrm{d}t + \int_0^\infty e^{-rt} \, {}_{t}^{12}p_x^i c_2^{ac*}(t) \mathrm{d}t. \tag{4.12}$$

$$i = bas, \quad long, \quad morb,$$

where $i = bas$ represents the base scenario assumption, $i = long$ is the unfavorable scenario of longevity risk, and $i = morb$ is the unfavorable scenario of disease incidence risk.

According to the Insurance Company Solvency Supervision Rules No.3 (CBIRC (2020a)), the risk margin RM actually measures the difference of future liabilities in an unfavorable scenario and a base scenario with a regulation proportion, which reflects as a risk compensation to the insurers or re-insurers. Thus, we take it as an alternative to risk loading for longevity and disease incidence risks (Chen et al. (2019); Bauer et al. (2010)). Then, according to CBIRC (2020a), the risk margin RM is defined by

$$RM = MC_{ins} \cdot \frac{G^{-1}(85\%)}{G^{-1}(99.5\%)}, \tag{4.13}$$

where MC_{ins} is the overall required minimum capital defined by (4.7), and $G^{-1}(x\%)$ represents the quantile of a normal distribution under probability $x\%$, and $x = 85$ is prescribed by the regulator (CBIRC (2020a)). $G(\cdot)$ is the distribution of best-estimate liabilities[11]. Furthermore, the initial single loading for longevity and disease incidence risks is then set to be $F_0 = RM$.

In this paper, the risk loading in our setting is purely driven by solvency capital requirement. In the real world, typically a higher risk loading than required by the solvency regulation is charged.

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[11]The definition of $G(\cdot)$ is not given clearly in CBIRC (2020a) as it is still being tested by the insurance industry. Here we take it as the distribution of best-estimate liabilities with reference to the quantile method for risk margin computation (Zheng et al. (2013)). In addition, as determining the distribution of best-estimate liabilities is not our focus in this article, dealing with the additional issues might cause too much distraction from the key points we try to convey. Thus, we apply the standard normal distribution, and get $\frac{G^{-1}(85\%)}{G^{-1}(99.5\%)} \approx 0.403$. We also use different normal distributions and get corresponding coefficients, but it turns out that different coefficients $\frac{G^{-1}(85\%)}{G^{-1}(99.5\%)}$ have little impact on our conclusions.

4.2 Product Comparison with Utility Indifference Number

Taking account of the risk loadings for the various care-dependent products, we cannot purely compare the expected discounted lifetime utility resulting from the corresponding optimal payoffs. A product leading to a higher utility level might cost in total more initially, i.e. the gross premium of this product might be higher. To further compare the attractiveness of two care-dependent tontines and the care-dependent annuity and ease our comparison, we first make one dimension of the two quantities (utility and gross premium) identical. More specifically, we compute the number of care-dependent tontines that one should purchase, in order to obtain the same utility yielded by one care-dependent annuity. Q_1 and Q_2 are used to denote the number of one-pool care-dependent tontines and two-pool ones respectively (as has been applied by Chen et al. (2019)). Then we have:

$$
\begin{aligned}
U_0^{ac} &= \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(u_1\left(Q_1\frac{nd_1^{oc_1}(t)}{N(t)}\right)\mathbb{1}_{\{S_{x+t}=1\}} + u_2\left(Q_1\frac{nd_2^{oc_1}(t)}{N(t)}\right)\mathbb{1}_{\{S_{x+t}=2\}}\right)\mathrm{d}t\right] \\
&= Q_1^{1-\gamma}\cdot U_0^{oc_1},
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
U_0^{ac} &= \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(u_1\left(Q_2\frac{nd_1^{oc_2}(t)}{N_1(t)}\right)\mathbb{1}_{\{S_{x+t}=1\}} + u_2\left(Q_2\frac{nd_2^{oc_2}(t)}{N_2(t)}\right)\mathbb{1}_{\{S_{x+t}=2\}}\right)\mathrm{d}t\right] \\
&= Q_2^{1-\gamma}\cdot U_0^{oc_2}.
\end{aligned}
\tag{4.15}
$$

The Q_1 and Q_2 are solved by:

$$
Q_1 = \left(\frac{U_0^{ac}}{U_0^{oc_1}}\right)^{\frac{1}{1-\gamma}},
\tag{4.16}
$$

$$
Q_2 = \left(\frac{U_0^{ac}}{U_0^{oc_2}}\right)^{\frac{1}{1-\gamma}}.
\tag{4.17}
$$

Q_1 and Q_2 respectively represent the number of 1-pool CDT and 2-pool CDT with actuarially fair premium $P_0^{oc_1} = P_0^{oc_2} = v = 10000$ that one needs to purchase, in order to receive the same expected discounted lifetime utility as from a CDA with actuarially fair premium $P_0^{ac} = v = 10000$. In other words, the policyholder becomes indifferent between one CDA and Q_1 1-pool CDTs (or Q_2 2-pool CDTs). Among the care-dependent products with identical expected discounted lifetime utility, the one with lowest comparable gross premium would be most attractive to the policyholder. Hence, we next will compare the gross premium of Q_1 1-pool care-dependent tontines (i.e. $Q_1 \cdot \hat{P}^{oc_1}$), Q_2

2-pool care-dependent tontines (i.e. $Q_2 \cdot \hat{P}^{oc_2}$) with one care-dependent annuity (i.e. $\hat{P}^{ac}$).

To quantify this part of results, we shall rely on numerical techniques. In this case, we apply the data from China Health and Retirement Longitudinal Study (CHARLS) and rely on China Risk Oriented Solvency System (C-ROSS) to calculate the realistic risk loadings and make product comparison.

5 Numerical Analysis

In this section, we compare the three different care-dependent products numerically. First, we estimate the transition probabilities. After that, we make the comparison among various care-dependent products from two aspects: (i) we compute the corresponding risk loadings, (ii) and we generate the utility indifference numbers to compare the attractiveness of different products. Finally, sensitivity analyses are given, to examine how the results will change with different inputs (i.e. risk aversion coefficient γ, payment weighting factor α, and risk-free rate r).

5.1 Estimates of Transition Probabilities

To estimate the transition probabilities, we use the data from China Health and Retirement Longitudinal Study (CHARLS). This database covers the period from 2011 to 2018, and has 4 waves in total. It contains information of Chinese residents ages 45 and older. The baseline national wave of CHARLS is being fielded in 2011 and includes about 10,257 households and 17,708 individuals in 150 counties/districts and 450 villages/resident committees. The survey data includes detailed demographic characteristics (such as age, education level, marital status, etc.), family economic status, health status (self-evaluated health status, chronic disease status, fundamental living ability and cognitive ability, etc.), participation in medical insurance, medical service utilization and community basic information.

The state of health is defined by the number of elderly people who lose their ability to perform daily activities. The ability of daily living (ADL) is originally proposed by Katz et al. (1963) and has been widely used in academia. There are 6 indicators of ADL: eating, dressing, bathing, getting into and out of bed, using the toilet, and controlling urination and defecation. Consistent with the basic situation in Hu et al. (2016)'s research

and Chinese LTC insurance practice[12], we define individuals as severely sick if they have difficulty with three or more (i.e., 3+) ADLs.

To fully utilize the available data, we construct an unbalanced panel dataset with sample weights considered and missing records deleted. Among four waves of the survey 2011, 2013, 2015, 2018, each individual should have at least two consecutive observations.

To calculate the transition intensities among different age groups, we first calculate the crude transition intensities with reference to the work of Hanewald et al. (2019). When going towards a specific transition situation, the crude transition intensity for an individual aged y is given by

$$\widetilde{{}^{ij}\mu}_y = \frac{{}^{ij}C_y}{{}^{i}Y_y}, \quad i,j \in \mathcal{S}, i \neq j. \tag{5.1}$$

where ${}^{ij}C_y$ represents the total transition number from state i to state j at age y. And ${}^{i}Y_y$ describes the overall years of exposure (e.g. sum of years for which the y-year-old healthy individuals stay healthy). More concretely, the crude transition intensity of healthy to severely sick at age 60, is defined as the total number of transitions from healthy to severely sick at age 60 divided by the total number of years of risk exposure for the 60-year-old at a healthy state. The crude transition intensity directly reflects the number of transitions from the previous state to the next state in unit time. The higher the crude transition intensity, the greater the probability of occurrence of the corresponding transition.

On the basis of resulting crude transition intensities, we then apply a GLM approach (see Appendix A.5) to smooth the trend of transition intensity going with age (Haberman and Renshaw (2009), Fong and Feng (2016)). By Equations (2.4), (2.5) and (2.7) defined in Section 2, we approximate the t-year probabilities ${}^{11}_tp_y$, ${}^{12}_tp_y$ and ${}^{22}_tp_y$ by age and gender. As gender difference is not our focus in this article, we only demonstrate the results for the 60-year-old males in the following contents. Figure 5.1 displays the estimated t-year probabilities for a male aged 60.

The courses of the depicted curves follow the natural expectations: in Figure 5.1, the occupancy probability of being healthy ${}^{11}_tp_{60}$ starts from one, goes down with t and approaches zero for larger t as the policyholder is going to die at an older age. When

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[12]LTC insurance products in many Chinese insurance companies (e.g. China Life Insurance Company, Kunlun Healthy Insurance Company and etc.), define that one becomes hard doing three or more ADLs as the trigger condition of making LTC payments.
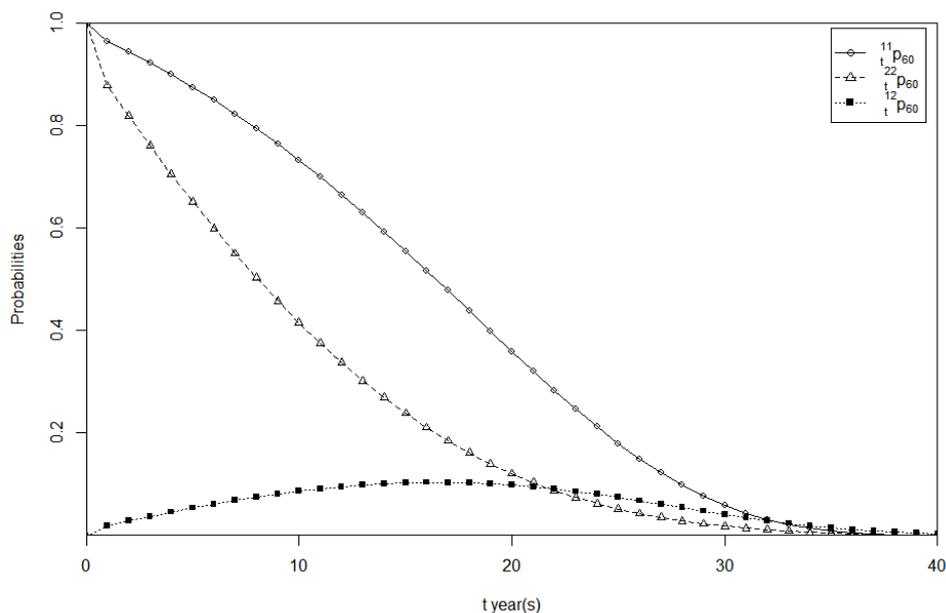
Figure 5.1: t-year probabilities of a 60-year-old male ($^{11}_tp_{60}$, $^{22}_tp_{60}$ and $^{12}_tp_{60}$).

looking at the curve with hollow triangles, we notice that the occupancy probability of being severely sick $^{22}_tp_{60}$ descends rapidly in the first two decades and tends to be smooth to zero when he is about to die. Last but not least, the third curve describes the t-year transition probability of a 60-year-old male, i.e $^{12}_tp_{60}$. The transition probability goes up at first then down for longer years. As transition probability is in fact affected by forces from two different directions (see Equation (2.5)): on the one hand, the increasing probability of becoming severely sick with age, tends to pull up the transition probability; while on the other hand, the transition probability is brought down by the decreasing occupancy probability of being severely sick, or say, the decreasing survival probability in severely sick state. In the about first two decades, the first type of force dominates; while later, the second type of force plays a leading role in the overall influence to the probability $^{12}_tp_{60}$.

5.2   Care-dependent Tontine *vs* Care-dependent Annuity

In this subsection, we calculate the risk loadings for different care-dependent products, and use utility indifference numbers to show policyholders' preference for these products in a realistic scenario. But before that, we need to fix some further parameters (see Table 5.1).

With reference to the 5-year average growth rate of CPI in China (Statistics Bureau of

Variable	Notation	Value
Net premium	$P_0^j = v, j = oc_1, oc_2, ac$	10000
Pool size	n	1000
Risk-free rate	r	0.02
Subjective discount rate	ρ	0.02
Risk aversion coefficient	γ	2
Initial age	x	60
Payment weighting factor	α	0.5

Table 5.1: Additional parameters for further computation.

the P.R.China (2020)), we fix the risk-free rate r at 2%. Then we let $\rho = r$ at first, and the cases where $\rho > r$ and $\rho < r$ will be further examined (see e.g. Chen et al. (2021)). As for the risk aversion coefficient, we set $\gamma = 2$ firstly (see e.g. Havranek et al. (2015)), and we will vary it later for sensitivity analysis.

5.2.1 Risk Loadings

Referring to our calculation method for risk loadings (see Section 4.1), as well as the baseline parameters (see Table 5.1), we can compute the minimum capital requirements (MCs) and risk loadings ($F_0's$) for different care-dependent products under consideration. The risk loadings for each care-dependent product are shown in Table 5.2. Generally, the risk loadings for care-dependent tontine products are much lower than that of a care-dependent annuity. The care-dependent annuity requires the highest risk loading, which amounts to a relevant share of 5.62% ($= \frac{595.118}{10000+595.118}$) of the initial contribution. It could be explained from two aspects: (i) the insurer completely bears the longevity risk for this annuity-like product; (ii) this big amount of risk loading partly comes from the unfavorable assumption for longevity risk (see Equation (4.8)) provided by CBIRC (2020b). For instance, SF_t^{long} takes a large value as time goes by[13], which means it is assumed there is a huge longevity risk in the long run.

We also observe the risk loading of the 1-pool CDT, i.e. 37.589, accounts for 0.37% ($= \frac{37.589}{10000+37.589}$) of the gross premium. This is substantially greater than that of the 2-pool CDT, which is negligible. The difference comes from the first row in the Table 5.2. (i) The 1-pool CDT requires a larger minimum capital for longevity risk (MC^{long}).

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[13]According to Equation (4.8), when $t > 30$, $SF_t^{long} = (1-3\%)^{10} \cdot (1-2\%)^{10} \cdot (1-1\%)^{10} - 1 = -45.5\%$.

Let us take a closer look at the liability structures of these two CDTs (see Equation (4.10) and (4.11)). With a strike from longevity risk, the changed probabilities will have a greater impact on the 1-pool CDT than the 2-pool one. The term $(1 - (1 - p)^n)$ tends to 1 as n becomes larger[14]. Thus, the longevity risk does not substantially affect the two-pool CDT, especially in a large pool case. (ii) The 1-pool CDT also requires a larger minimum capital for disease incidence risk (MC^{morb}). The insurer takes over a relatively high disease incidence risk for 1-pool CDT. While for the 2-pool CDT, on one side, the increase of disease incidence risk causes $_t^{11}p_x^{morb}$ to decline over years; on the other side, an increased disease incidence risk brings about a greater $_t^{12}p_x^{morb}$. In our case, the effect of disease incidence risk gets cancelled out and even drives down the future cash flows. By definition of the corresponding minimum capital (4.6), the minimum capital for disease incidence risk ($MC_{oc_2}^{morb}$) of the 2-pool CDT is equal to zero (This explanation also holds for $MC_{ac}^{morb} = 0$ of CDA).

$MC_{oc_1}^{long}$	$MC_{oc_2}^{long}$	MC_{ac}^{long}	$MC_{oc_1}^{morb}$	$MC_{oc_2}^{morb}$	MC_{ac}^{morb}
51.6	0.633	1477	77.7	0	0
$MC_{oc_1}^{ins}$	$MC_{oc_2}^{ins}$	MC_{ac}^{ins}	$F_0^{oc_1}$	$F_0^{oc_2}$	F_0^{ac}
93.3	0.633	1477	37.589	0.255	595.118

Table 5.2: Minimum capitals and risk loadings for various care-dependent products.

In addition to risk loading computation w.r.t. the Chinese regulation regime (C-ROSS), we also carry out the relevant computation under a different regulation regime, i.e. the capital requirements of Solvency II, the insurance regulation in EU countries. The risk loadings obtained by the Solvency II do not deviate much from those by C-ROSS. The care-dependent annuity still requires the highest risk loading, while under the baseline parameter setting, the two-pool care-dependent tontine requires the least (see Appendix A.6).

Regarding different pool sizes, we also choose the baseline parameters from Table 5.1 to compute the corresponding risk loadings. Table 5.3 shows that for small pool sizes, say $n = 10$, the loadings for the CDTs are relatively large. The advantage of the CDTs over the CDA is less substantial. A larger n leads to less risk loadings for both one-pool CDT and two-pool CDT, but that of the one-pool CDT stops decreasing when the pool size n reaches some certain level. We could observe that its risk loading remains the

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[14]Here, p represents the probability, e.g. $_t p_x^{long}$, $_t^{11}p_x^{long}$.

same when $n \geq 1000$. This is due to the fact that the term $(1 - (1 - p)^n)$ tends to 1 as n becomes larger as mentioned at an early place. There is no other term related to n in the calculation for the one-pool CDT's risk loading, which means the influence of n fades away as n grows. In case of two-pool CDT, for a substantially large pool size, the loading for care-dependent tontines is negligible.

n	$F_0^{oc_1}$	$F_0^{oc_2}$	F_0^{ac}
10	144	270	595
100	41	14.9	595
500	37.7	0.978	595
1000	37.6	0.255	595
2000	37.6	0.0648	595
5000	37.6	0.0071	595

Table 5.3: Risk loading F_0 for different pool sizes n using the baseline parameter setting from Table 5.1. Net premium $P_0^{oc_1} = P_0^{oc_2} = P^{ac} = v = 10000$, subjective discount rate $\rho = 0.02$, risk-free rate $r = 0.02$, initial age $x = 60$, risk aversion coefficient $\gamma = 2$, and payment weighting factor $\alpha = 0.5$.

5.2.2 Product Comparison with Utility Indifference Number

As introduced in Section 4.2, we will compute the utility indifference numbers Q_1 and Q_2 for one-pool and two-pool care-dependent tontine separately. Using parameters assumed in Table 5.1, we obtain Q_1, Q_2 and comparable gross premiums for different care-dependent products:

$$Q_1 = 1.000896 \text{ and } Q_2 = 1.003010,$$
$$Q_1 \cdot \hat{P}^{oc_1} = 1.000896 \times (10000 + 37.589) = 10046.583,$$
$$Q_2 \cdot \hat{P}^{oc_2} = 1.003010 \times (10000 + 0.255) = 10030.356,$$
$$\hat{P}^{ac} = 10000 + 595.118 = 10595.118.$$

Obviously, from results above, it is easy to find out that the 2-pool CDT becomes the best choice for the policyholder and then is the 1-pool CDT, using the baseline parameters. The CDA's gloss is taken off by its high risk loading. To conclude, incorporating an initial risk loading required by the insurance regulator, we find that under our baseline setting, the care-dependent tontines are much more attractive than care-dependent annuities for both cases, and the 2-pool CDT gets more appealing than the 1-pool one.

In the following, we explore the sensitivity of different inputs, i.e. risk aversion coefficient γ, payment weighting factor α, and risk-free rate r respectively. Based on the utility indifference number, we calculate the comparable gross premiums for different care-dependent products, which is seen as the basis for people to choose a product. Generally, as can be seen from Table 5.4 and Table 5.5, it is noted that CDTs are always better choices than CDA.

We first numerically examine the sensitivity of the policyholder's risk aversion, i.e. γ, and other parameters are taken from the baseline setting table (i.e. Table 5.1). We notice from Theorem 3.1, 3.2 and 3.3, that γ exerts an impact on the policyholder's utility as well as risk loadings through different payoffs. For example, changing γ brings about the changes in the payment streams for the severely sick ($d_2^{oc_1}$, $d_2^{oc_2}$, and d_2^{ac}) and the utility indifference number (Q_1 and Q_2), thus it affects the comparable gross premiums of care-dependent tontines. In the case that $\gamma > 1$ and $\alpha \in (0,1)$, the rows of Table 5.4 reveal that the more risk-averse policyholders regard CDTs as a little bit more expensive than those who are less risk-averse. To our surprise, we find that for those who are assigned a bigger risk aversion coefficient, e.g. $\gamma \geq 8$ when $\alpha = 0.5$, the 1-pool CDT becomes the best option while 2-pool CDT comes in second in terms of choices. In the case that $\gamma \in (0,1)$ and $\alpha > 1$, the comparable gross premium of the 1-pool CDT drops with γ. With different value ranges of γ and α, the payment streams for the severely sick vary in different direction with γ[15]. The variation in the payment stream leads to changes in the risk loading, and then the comparable gross premium. The risk aversion coefficient γ plays a critical role in the payment functions of these two CDTs, thus a slight change of γ may bring about relatively large difference in comparable gross premiums.

Next, let us fix the risk aversion coefficient and look at the columns of Table 5.4. And also, the other parameters are taken from the baseline setting table (i.e. Table 5.1). Generally speaking, the change of the payment weighting factor does not lead to 2-pool CDT being less attractive. The payment weighting factor α has an effect on the comparable gross premiums of CDTs through utility indifference numbers and risk loadings. It can be detected that a greater α slightly decreases the comparable gross premiums of CDTs when $\gamma > 1$ and $\alpha \in (0,1)$, while increases those in the case that $\gamma \in (0,1)$ and $\alpha > 1$. Likewise, the difference comes from the fact that the payment streams for the severely sick

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[15]By Equation (3.10), (3.19) and (3.25), when $\gamma > 1$ and $\alpha \in (0,1)$, greater γ leads to a rise in the term $\alpha^{1/\gamma-1}$, while a decline in $\alpha^{1/\gamma-1}$ when $\gamma \in (0,1)$ and $\alpha > 1$. More concretely, $\frac{\partial \alpha^{1/\gamma-1}}{\partial \gamma} = -\frac{1}{\gamma^2} \cdot \ln \alpha \cdot \alpha^{1/\gamma-1}$, with $\alpha^{1/\gamma-1} > 0$ and $-\frac{1}{\gamma^2} < 0$. Thus, the monotony depends on the value of α.

vary in different directions with α, when α lies in different value ranges (see Equation (3.10), (3.19) and (3.25)). In the case that $\gamma > 1$ and $\alpha \in (0,1)$, $\alpha^{1/\gamma-1}$ goes down with α. Thus, corresponding payments for the severely sick drops with α [16]. Further, risk loadings reduce as α increases. In the case that $\gamma \in (0,1)$ and $\alpha > 1$, $\alpha^{1/\gamma-1}$ goes up with α. Thus, the corresponding payments for the severely sick and then risk loadings increase with α; meanwhile, the advantage of the 2-pool CDT becomes more substantial [17]

Finally, as mentioned before, we are about to analyze the cases that $r < \rho$ and $r > \rho$. With $\rho = 2\%$ keeping fixed, we vary the risk-free interest rate r by 0 to 0.05, and comparable gross premiums correspondingly. By Table 5.5, analogously, the comparable gross premiums of CDTs decrease in risk-free interest rate r when $\gamma > 1$ and $\alpha \in (0,1)$, and increase with r when $\gamma \in (0,1)$ and $\alpha > 1$. However, this effect is moderate.

	$Q_1 * \hat{P}_0^{oc_1}$			$Q_2 * \hat{P}_0^{oc_2}$			$\hat{P}_0^{ac}$		
	$\alpha \in (0,1)$, $\quad \gamma > 1$								
α \ γ	2	5	8	2	5	8	2	5	8
0.2	10113.74	10224.02	10269.52	10039.65	10126.93	10211.94	10729.28	10916.33	10969.9
0.5	10046.59	10088.6	10110.18	10030.36	10082.98	10133.68	10595.12	10650.96	10666.18
0.8	10019.41	10038.94	10052.78	10026.6	10066.86	10105.5	10541.05	10555.52	10559.25
	$\alpha > 1$, $\quad \gamma \in (0,1)$								
α \ γ	0.2	0.5	0.8	0.2	0.5	0.8	0.2	0.5	0.8
1.5	10266.52	10047.26	10013.38	10006.57	10008.14	10010.84	11070.1	10610.15	10538.91
2	10533.97	10087.92	10021.03	10010.43	10009.6	10011.28	11641.77	10692.76	10554.36
5	10840.26	10265.98	10048.25	10014.86	10015.99	10012.83	12301.26	11064.11	10609.34

Table 5.4: Sensitivity analysis w.r.t risk aversion coefficient γ and payment weighting factor α. Other parameters are taken from the baseline setting table (i.e. Table 5.1). More concretely, net premium $P_0^{oc_1} = P_0^{oc_2} = P_0^{ac} = v = 10000$, pool size $n = 1000$, risk-free rate $r = 0.02$, subjective discount rate $\rho = 0.02$, and initial age $x = 60$.

---

[16] But the payments for the severely sick are still higher than those for the healthy.

[17] $\frac{\partial \alpha^{1/\gamma-1}}{\partial \alpha} = (\frac{1}{\gamma} - 1)\alpha^{1/\gamma-2}$, with $\alpha^{1/\gamma-2} > 0$. Thus, the monotony depends on the value of γ.

r	$Q_1 * \hat{P}_0^{oc_1}$	$Q_2 * \hat{P}_0^{oc_2}$	$\hat{P}_0^{ac}$	$Q_1 * \hat{P}_0^{oc_1}$	$Q_2 * \hat{P}_0^{oc_2}$	$\hat{P}_0^{ac}$
	$\gamma = 2, \quad \alpha = 0.5$			$\gamma = 0.5, \quad \alpha = 2$		
0.00	10050.93	10033.5	10690.53	10076.65	10008.1	10515.29
0.01	10048.7	10031.86	10641.07	10082.07	10008.79	10597.38
0.02	10046.59	10030.36	10595.12	10087.92	10009.6	10692.76
0.03	10044.59	10028.97	10552.47	10094.2	10010.56	10803.27
0.04	10042.7	10027.7	10512.9	10100.92	10011.7	10930.82
0.05	10040.91	10026.53	10476.22	10108.06	10013.04	11077.43

Table 5.5: Sensitivity analysis w.r.t risk-free rate r. Other parameters are taken from the baseline setting table (i.e. Table 5.1). Net premium $P_0^{oc_1} = P_0^{oc_2} = P_0^{ac} = v = 10000$, pool size $n = 1000$, subjective discount rate $\rho = 0.02$, initial age $x = 60$, risk aversion coefficient $\gamma = 2$, and payment weighting factor $\alpha = 0.5$.

5.3 Further Results

Regarding the risk loadings computed above, we can further explore how much it costs to add the care-dependent coverage to the original retirement products. In this case, we split the risk loading of a care-dependent tontine/annuity into the risk loading of optimal tontine/annuity (in the form of optimal tontine/annuity introduced by Chen et al. (2019)) and cost of care-dependent coverage. By setting the weighting factor $\alpha = 1$ for 1-pool CDT and CDA, while keeping the other parameters the same in Table 5.1, we get the risk loading required by optimal tontine for 0.011, and optimal annuity for 518.502 (see Table 5.6). Subtracting the risk loading of optimal tontine/annuity from that of care-dependent tontine/annuity, the cost for care-dependent coverage of each product is clear. The results indicate that for the insurers, under the baseline parameter setting, constructing a care-dependent tontine in a two-pool structure may be the most cost-efficient way to add the care-dependent coverage among the three insurance products under consideration. Due to the rather high cost of care-dependnet coverage, the insurer of the one-pool care-dependent tontines do not only play an administrative role as regular tontines, but serves more as a regular care insurance provider.

6 Conclusion and Discussion

This article comes up with two different ways to evaluate the care-dependent tontines with a policyholder's view. We first compute the actuarially fair premiums and then based on a utility framework, we determine the optimal payment stream structures of

Risk Loading of 1-pool CDT	37.589
Risk Loading of 2-pool CDT	0.255
Risk Loading of CDA	595.118
Risk Loading of Optimal Tontine	0.011
Risk Loading of Optimal Annuity	518.502
Cost of Care-dependent Coverage (1-pool CDT)	37.578
Cost of Care-dependent Coverage (2-pool CDT)	0.244
Cost of Care-dependent Coverage (CDA)	76.616

Table 5.6: Cost of care-dependent coverage. Other parameters are taken from the baseline setting table (i.e. Table 5.1). Net premium $P_0^{oc1} = P_0^{oc2} = P_0^{ac} = v = 10000$, pool size $n = 1000$, subjective discount rate $\rho = 0.02$, risk-free rate $r = 0.02$, initial age $x = 60$, risk aversion coefficient $\gamma = 2$, and payment weighting factor $\alpha = 0.5$.

the care-dependent tontines that maximize the policyholder's expected lifetime utility. Further, by taking account of the realistic risk loadings, we are able to make product comparison in a setting closer to the realistic world. Our results reveal that by purchasing a care-dependent tontine, the policyholders benefit from lower risk loadings comparing with the care-dependent annuity proposed by predecessors. The results are robust when following the capital requirement of Solvency II, the insurance regulation in EU countries. In addition, two-pool care-dependent tontines draw attractions to the policyholder with a smaller risk aversion coefficient, while one-pool care-dependent tontines are more appealing to the more risk-averse policyholders. Further, we find that for insurers, under the baseline parameter setting, constructing a care-dependent tontine in a one-pool structure may cost more for adding the care-dependent coverage compared with the two-pool case. The (care) insurance for the old age is an increasingly concerning topic for now and future, our findings enrich the related existing literature and are helpful to improve the penetration of elderly care insurance with appealing care-dependent insurance products.

In this article, in order to get analytical forms of the product structure as well as simplify the analysis, we construct our products by pooling the policyholders of homogeneous cohorts. It will also be interesting to go with pooling heterogeneous cohorts from the policyholder's side. We will leave it for future research.

A Appendix

A.1 Proofs for Care-dependent Tontines in One-pool Case

To obtain the optimal payout functions $d_1^{oc_1}(t)$ and $d_2^{oc_1}(t)$ to the care-dependent tontine, we maximize the expected state-dependent utility (3.8) subject to (3.3). First, we compute the policyholder's expected utility $U_0^{oc_1}$. In case of power utility, $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$, $\gamma > 0$ and $\gamma \neq 1$. We find that

$$
\begin{aligned}
U_0^{oc_1} &= \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(u_1\left(\frac{n d_1^{oc_1}(t)}{N(t)} \right) \mathbb{1}_{\{S_{x+t}=1\}} + u_2\left(\frac{n d_2^{oc_1}(t)}{N(t)} \right) \mathbb{1}_{\{S_{x+t}=2\}} \right) \mathrm{d}t \right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(u_1\left(\frac{n d_1^{oc_1}(t)}{N(t)} \right) \mathbb{1}_{\{S_{x+t}=1\}} \right) \mathrm{d}t \right] + \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(u_2\left(\frac{n d_2^{oc_1}(t)}{N(t)} \right) \mathbb{1}_{\{S_{x+t}=2\}} \right) \mathrm{d}t \right] \\
&= \int_0^\infty e^{-\rho t} \mathbb{E}\left[{}_t^{11}p_x u_1\left(\frac{n d_1^{oc_1}(t)}{N(t)} \right) \right] \mathrm{d}t + \int_0^\infty e^{-\rho t} \mathbb{E}\left[{}_t^{12}p_x u_2\left(\frac{n d_2^{oc_1}(t)}{N(t)} \right) \right] \mathrm{d}t \\
&= \int_0^\infty e^{-\rho t} \frac{{}_t^{11}p_x}{{}_tp_x} \sum_{k=1}^n \binom{n}{k} \left(\frac{k}{n} \right)^\gamma ({}_tp_x)^k (1 - {}_tp_x)^{n-k} \cdot u(d_1^{oc_1}(t)) \mathrm{d}t \\
&\quad + \alpha^{1-\gamma} \cdot \int_0^\infty e^{-\rho t} \frac{{}_t^{12}p_x}{{}_tp_x} \sum_{k=1}^n \binom{n}{k} \left(\frac{k}{n} \right)^\gamma ({}_tp_x)^k (1 - {}_tp_x)^{n-k} \cdot u(d_2^{oc_1}(t)) \mathrm{d}t. \qquad \text{(A.1)}
\end{aligned}
$$

Then, we write down the Lagrangian function as follows. For easy reading, we simplify the notations $d_1^{oc_1}(t)$ and $d_2^{oc_1}(t)$ as $d_1(t)$, $d_2(t)$ respectively.

$$
\begin{aligned}
&L(d_1(t), d_2(t), \lambda_{oc_1}) \\
&= \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(u_1\left(\frac{n d_1(t)}{N(t)} \right) \mathbb{1}_{\{S_{x+t}=1\}} + u_2\left(\frac{n d_2(t)}{N(t)} \right) \mathbb{1}_{\{S_{x+t}=2\}} \right) \mathrm{d}t \right] \\
&\quad + \lambda_{oc_1}(v - P_0^{oc_1}) \\
&= \int_0^\infty e^{-\rho t} \frac{{}_t^{11}p_x}{{}_tp_x} \sum_{k=1}^n \binom{n}{k} \left(\frac{k}{n} \right)^\gamma ({}_tp_x)^k (1 - {}_tp_x)^{n-k} \cdot u(d_1(t)) \mathrm{d}t \\
&\quad + \alpha^{1-\gamma} \int_0^\infty e^{-\rho t} \frac{{}_t^{12}p_x}{{}_tp_x} \sum_{k=1}^n \binom{n}{k} \left(\frac{k}{n} \right)^\gamma ({}_tp_x)^k (1 - {}_tp_x)^{n-k} \cdot u(d_2(t)) \mathrm{d}t \\
&\quad + \lambda_{oc_1}\left(v - \left(\int_0^\infty e^{-rt} \frac{{}_t^{11}p_x}{{}_tp_x} (1 - (1 - {}_tp_x)^n) \cdot d_1(t) \mathrm{d}t \right. \right. \\
&\quad \left. \left. + \int_0^\infty e^{-rt} \frac{{}_t^{12}p_x}{{}_tp_x} (1 - (1 - {}_tp_x)^n) \cdot d_2(t) \mathrm{d}t \right) \right). \qquad \text{(A.2)}
\end{aligned}
$$

In order to obtain the maximum of Lagrangian function, we take derivatives with respect

to $d_1(t)$ and $d_2(t)$, i.e.

$$\frac{\partial L(d_1(t), d_2(t), \lambda_{oc_1})}{\partial d_1(t)}$$

$$= e^{-\rho t}\frac{\frac{11}{t}p_x}{_tp_x} \sum_{k=1}^{n} \binom{n}{k} (_tp_x)^k (1 - _tp_x)^{n-k} \left(\frac{nd_1(t)}{k}\right)^{-\gamma}$$

$$- \lambda_{oc_1} \cdot e^{-rt}\frac{\frac{11}{t}p_x}{_tp_x} (1 - (1 - _tp_x)^n) \stackrel{!}{=} 0, \tag{A.3}$$

$$\frac{\partial L(d_1(t), d_2(t), \lambda_{oc_1})}{\partial d_2(t)}$$

$$= \alpha^{1-\gamma} \cdot e^{-\rho t}\frac{\frac{12}{t}p_x}{_tp_x} \sum_{k=1}^{n} \binom{n}{k} (_tp_x)^k \cdot (1 - _tp_x)^{n-k} \left(\frac{nd_2(t)}{k}\right)^{-\gamma} \tag{A.4}$$

$$- \lambda_{oc_1} \cdot e^{-rt}\frac{\frac{12}{t}p_x}{_tp_x} (1 - (1 - _tp_x)^n) \stackrel{!}{=} 0. \tag{A.5}$$

Denote $\kappa_{n,\gamma}(_tp_x) = \sum_{k=1}^{n} \binom{n}{k} \left(\frac{k}{n}\right)^{\gamma} (_tp_x)^k (1 - _tp_x)^{n-k}$.

The Lagrangian function takes the global optima when

$$d_1^{oc_1*}(t) = \frac{e^{1/\gamma(r-\rho)t}}{(\lambda_{oc_1}^*)^{1/\gamma}} \cdot \frac{(\kappa_{n,\gamma}(_tp_x))^{1/\gamma}}{(1 - (1 - _tp_x)^n)^{1/\gamma}}, \tag{A.6}$$

$$d_2^{oc_1*}(t) = \alpha^{1/\gamma-1} \cdot \frac{e^{1/\gamma(r-\rho)t}}{(\lambda_{oc_1}^*)^{1/\gamma}} \cdot \frac{(\kappa_{n,\gamma}(_tp_x))^{1/\gamma}}{(1 - (1 - _tp_x)^n)^{1/\gamma}}. \tag{A.7}$$

The $\lambda_{oc_1}^* > 0$ is chosen satisfying the budget constraint, i.e.

$$\lambda_{oc_1}^* = \left(\frac{1}{v}\left(\int_0^{\infty} e^{\left(\frac{r-\rho}{\gamma}-r\right)t} \cdot \frac{\left(\frac{\frac{11}{t}p_x}{_tp_x} \cdot \kappa_{n,\gamma}(_tp_x)\right)^{1/\gamma}}{\left(\frac{\frac{11}{t}p_x}{_tp_x} \cdot (1 - (1 - _tp_x)^n)\right)^{1/\gamma-1}} dt\right.\right.$$

$$\left.\left.+ \int_0^{\infty} e^{\left(\frac{r-\rho}{\gamma}-r\right)t} \cdot \alpha^{1/\gamma-1} \cdot \frac{\left(\frac{\frac{12}{t}p_x}{_tp_x} \cdot \kappa_{n,\gamma}(_tp_x)\right)^{1/\gamma}}{\left(\frac{\frac{12}{t}p_x}{_tp_x} \cdot (1 - (1 - _tp_x)^n)\right)^{1/\gamma-1}} dt\right)\right)^{\gamma}. \tag{A.8}$$

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A.2 Proofs for Care-dependent Tontines in Two-pool Case

When we distinguish two pools at each time $t > 0$, the policyholder's state-dependent expected utility $U_0^{oc_2}$ could be written as

$$
\begin{aligned}
U_0^{oc_2} &= \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(u_1\left(\frac{nd_1^{oc_2}(t)}{N_1(t)}\right)\mathbb{1}_{\{S_{x+t}=1\}} + u_2\left(\frac{nd_2^{oc_2}(t)}{N_2(t)}\right)\mathbb{1}_{\{S_{x+t}=2\}}\right)\mathrm{d}t\right] \\
&= \int_0^\infty e^{-\rho t}\sum_{k=1}^n \binom{n}{k}\left(\frac{k}{n}\right)^\gamma \left({}^{11}_t p_x\right)^k \left(1 - {}^{11}_t p_x\right)^{n-k}\cdot u(d_1^{oc_2}(t))\mathrm{d}t \\
&\quad + \alpha^{1-\gamma}\cdot\int_0^\infty e^{-\rho t}\sum_{k=1}^n \binom{n}{k}\left(\frac{k}{n}\right)^\gamma \left({}^{12}_t p_x\right)^k \left(1 - {}^{12}_t p_x\right)^{n-k}\cdot u(d_2^{oc_2}(t))\mathrm{d}t. \quad \text{(A.9)}
\end{aligned}
$$

Afterwards, we write down the Lagrangian function. Here, we also simplify the notations $d_1^{oc_2}(t)$ and $d_2^{oc_2}(t)$ as $d_1(t)$, $d_2(t)$ respectively.

$$
\begin{aligned}
&L(d_1(t), d_2(t), \lambda_{oc_2}) \\
&= \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(u_1\left(\frac{nd_1(t)}{N_1(t)}\right)\mathbb{1}_{\{S_{x+t}=1\}} + u_2\left(\frac{nd_2(t)}{N_2(t)}\right)\mathbb{1}_{\{S_{x+t}=2\}}\right)\mathrm{d}t\right] + \lambda_{oc_2}(v - P_0^{oc_2}) \\
&= \int_0^\infty e^{-\rho t}\kappa_{n,\gamma}({}^{11}_t p_x)u(d_1(t))\mathrm{d}t + \alpha^{1-\gamma}\cdot\int_0^\infty e^{-\rho t}\kappa_{n,\gamma}({}^{12}_t p_x)u(d_2(t))\mathrm{d}t \\
&\quad + \lambda_{oc_2}\left(v - \left(\int_0^\infty e^{-rt}\left(1 - \left(1 - {}^{11}_t p_x\right)^n\right)d_1(t)\mathrm{d}t + \int_0^\infty e^{-rt}\left(1 - \left(1 - {}^{12}_t p_x\right)^n\right)d_2(t)\mathrm{d}t\right)\right).
\end{aligned}
$$
$$\text{(A.10)}$$

As denoted, $\kappa_{n,\gamma}({}^{ij}_t p_x) = \sum_{k=1}^n \binom{n}{k}\left(\frac{k}{n}\right)^\gamma \left({}^{ij}_t p_x\right)^k \left(1 - {}^{ij}_t p_x\right)^{n-k}$, $i, j \in \mathcal{S}$.

Then we take derivatives with respect to $d_1(t)$ and $d_2(t)$ to maximize the Lagrangian function

$$
\begin{aligned}
\frac{\partial L(d_1(t), d_2(t), \lambda_{oc_2})}{\partial d_1(t)} &= e^{-\rho t}\kappa_{n,\gamma}({}^{11}_t p_x)\left(d_1(t)\right)^{-\gamma} \\
&\quad - \lambda_{oc_2}e^{-rt}\left(1 - \left(1 - {}^{11}_t p_x\right)^n\right) \overset{!}{=} 0, \quad\quad\quad \text{(A.11)} \\
\frac{\partial L(d_1(t), d_2(t), \lambda_{oc_2})}{\partial d_2(t)} &= \alpha^{1-\gamma}\cdot e^{-\rho t}\kappa_{n,\gamma}({}^{12}_t p_x)\left(d_2(t)\right)^{-\gamma} \\
&\quad - \lambda_{oc_2}e^{-rt}\left(1 - \left(1 - {}^{12}_t p_x\right)^n\right) \overset{!}{=} 0. \quad\quad\quad \text{(A.12)}
\end{aligned}
$$

The Lagrangian function takes the global optima when

$$d_1^*(t) = \frac{e^{1/\gamma(r-\rho)t}}{(\lambda_{oc_2}^*)^{1/\gamma}} \cdot \frac{\left(\kappa_{n,\gamma}({}_t^{11}p_x)\right)^{1/\gamma}}{\left(1 - (1 - {}_t^{11}p_x)^n\right)^{1/\gamma}}, \tag{A.13}$$

$$d_2^*(t) = \alpha^{1/\gamma - 1} \cdot \frac{e^{1/\gamma(r-\rho)t}}{(\lambda_{oc_2}^*)^{1/\gamma}} \cdot \frac{\left(\kappa_{n,\gamma}({}_t^{12}p_x)\right)^{1/\gamma}}{\left(1 - (1 - {}_t^{12}p_x)^n\right)^{1/\gamma}}. \tag{A.14}$$

The $\lambda_{oc_2}^* > 0$ is chosen satisfying the budget constraint, i.e.

$$\lambda_{oc_2}^* = \left(\frac{1}{v}\left(\int_0^\infty e^{\left(\frac{r-\rho}{\gamma} - r\right)t}\right.\right.$$
$$\left.\left.\cdot \left[\frac{\left(\kappa_{n,\gamma}({}_t^{11}p_x)\right)^{1/\gamma}}{\left(1 - (1 - {}_t^{11}p_x)^n\right)^{1/\gamma - 1}} + \frac{\alpha^{1/\gamma - 1} \cdot \left(\kappa_{n,\gamma}({}_t^{12}p_x)\right)^{1/\gamma}}{\left(1 - (1 - {}_t^{12}p_x)^n\right)^{1/\gamma - 1}}\right] \mathrm{d}t\right)\right)^\gamma. \tag{A.15}$$

A.3 Proofs for Care-dependent Annuities

For simplicity, we use $c_1(t)$ and $c_2(t)$ to represent $c_1^{ac}(t)$ and $c_2^{ac}(t)$ separately. Then, the Lagrangian function is given by

$$L(c_1(t), c_2(t), \lambda_{ac})$$
$$= \int_0^\infty e^{-\rho t} {}_t^{11}p_x \cdot u(c_1(t))\mathrm{d}t + \alpha^{1-\gamma} \cdot \int_0^\infty e^{-\rho t} {}_t^{12}p_x \cdot u(c_2(t))\mathrm{d}t$$
$$+ \lambda_{ac} \cdot \left(v - \int_0^\infty e^{-rt} {}_t^{11}p_x c_1(t)\mathrm{d}t - \int_0^\infty e^{-rt} {}_t^{12}p_x c_2(t)\mathrm{d}t\right). \tag{A.16}$$

Next, by optimal conditions $\partial L(c_1(t), c_2(t), \lambda_{ac})/\partial c_1(t) \overset{!}{=} 0$ and $\partial L(c_1(t), c_2(t), \lambda_{ac})/\partial c_2(t) \overset{!}{=} 0$, the optimal payoffs for care-dependent annuities are yielded, i.e.

$$c_1^{ac*}(t) = \left(\lambda_{ac}^* \cdot e^{(\rho - r)t}\right)^{-1/\gamma}, \tag{A.17}$$

$$c_2^{ac*}(t) = \alpha^{1/\gamma - 1} \cdot \left(\lambda_{ac}^* \cdot e^{(\rho - r)t}\right)^{-1/\gamma}. \tag{A.18}$$

Through the budget constraint, we obtain

$$\lambda_{ac}^* = \left(\frac{1}{v}\int_0^\infty e^{\left(\frac{r-\rho}{\gamma} - r\right)t}({}_t^{11}p_x + \alpha^{1/\gamma - 1} \cdot {}_t^{12}p_x)\mathrm{d}t\right)^\gamma. \tag{A.19}$$

A.4 Proofs for Utility Inequalities

By Equations (3.27), (3.13) and (3.21), and let $P_0^{ac} = P_0^{oc_1} = P_0^{oc_2} = v$, then we have

$$
\begin{aligned}
U_0^{ac*} &= \int_0^\infty e^{-\rho t}\,{}_t^{11}p_x \cdot u_1(c_1^{ac*}(t))\mathrm{d}t + \int_0^\infty e^{-\rho t}\,{}_t^{12}p_x \cdot u_2(c_2^{ac*}(t))\mathrm{d}t \\
&= \frac{1}{1-\gamma}\int_0^\infty e^{-\rho t}\,{}_t^{11}p_x \cdot (c_1^{ac*}(t))^{1-\gamma}\,\mathrm{d}t + \frac{\alpha^{1/\gamma-1}}{1-\gamma}\int_0^\infty e^{-\rho t}\,{}_t^{12}p_x \cdot (c_2^{ac*}(t))^{1-\gamma}\,\mathrm{d}t \\
&= \frac{\lambda_{ac}^* v}{1-\gamma}, \tag{A.20}
\end{aligned}
$$

$$
\begin{aligned}
U_0^{oc_1*} &= \int_0^\infty e^{-\rho t}\frac{{}_t^{11}p_x}{{}_t p_x}\sum_{k=1}^n \binom{n}{k}\left(\frac{k}{n}\right)^\gamma ({}_t p_x)^k (1-{}_t p_x)^{n-k}\cdot u_1(d_1^{oc_1*}(t))\mathrm{d}t \\
&\quad + \int_0^\infty e^{-\rho t}\frac{{}_t^{12}p_x}{{}_t p_x}\sum_{k=1}^n \binom{n}{k}\left(\frac{k}{n}\right)^\gamma ({}_t p_x)^k \cdot (1-{}_t p_x)^{n-k}\cdot u_2(d_2^{oc_1*}(t))\mathrm{d}t \\
&= \frac{1}{1-\gamma}\int_0^\infty e^{-\rho t}\frac{{}_t^{11}p_x}{{}_t p_x}\kappa_{n,\gamma}({}_t p_x)\,(d_1^{oc_1*})^{1-\gamma}\,\mathrm{d}t + \frac{\alpha^{1/\gamma-1}}{1-\gamma}\int_0^\infty e^{-\rho t}\frac{{}_t^{12}p_x}{{}_t p_x}\kappa_{n,\gamma}({}_t p_x)\,(d_2^{oc_1*})^{1-\gamma}\,\mathrm{d}t \\
&= \frac{\lambda_{oc_1}^* v}{1-\gamma}, \tag{A.21}
\end{aligned}
$$

$$
\begin{aligned}
U_0^{oc_2*} &= \int_0^\infty e^{-\rho t}\sum_{k=1}^n \binom{n}{k}\left(\frac{k}{n}\right)^\gamma ({}_t^{11}p_x)^k (1-{}_t^{11}p_x)^{n-k}\cdot u_1(d_1^{oc_2*}(t))\mathrm{d}t \\
&\quad + \int_0^\infty e^{-\rho t}\sum_{k=1}^n \binom{n}{k}\left(\frac{k}{n}\right)^\gamma ({}_t^{12}p_x)^k \cdot (1-{}_t^{12}p_x)^{n-k}\cdot u_2(d_2^{oc_2*}(t))\mathrm{d}t \\
&= \frac{1}{1-\gamma}\int_0^\infty e^{-\rho t}\kappa_{n,\gamma}({}_t^{11}p_x)\,(d_1^{oc_2*})^{1-\gamma}\,\mathrm{d}t + \frac{\alpha^{1/\gamma-1}}{1-\gamma}\int_0^\infty e^{-\rho t}\kappa_{n,\gamma}({}_t^{12}p_x)\,(d_2^{oc_2*})^{1-\gamma}\,\mathrm{d}t \\
&= \frac{\lambda_{oc_2}^* v}{1-\gamma}. \tag{A.22}
\end{aligned}
$$

To prove (3.28), we only need to demonstrate by

$$
U_0^{ac*} \geq U_0^{oc_1*} \quad \Leftrightarrow \quad
\begin{cases}
\lambda_{ac}^* \geq \lambda_{oc_1}^*, & \text{if } \gamma \in (0,1) \\
\lambda_{ac}^* \leq \lambda_{oc_1}^*, & \text{if } \gamma > 1
\end{cases}
. \tag{A.23}
$$

Before going towards the proofs, we define

$$
a := \left(\frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1\right),
$$
$$
b := (1, 1, \cdots, 1, 1).
$$

Such that

$$\| a \|_{L^\infty} = 1, \qquad \| a \|_{L^1} = p,$$
$$\| b \|_{L^\infty} = 1, \qquad \| b \|_{L^1} = 1 - (1-p)^n.$$

the L^γ-norm of x is written as

$$\| x \|_{L^\gamma} := \left[\sum_{k=1}^{n} x_k^\gamma \binom{n}{k} p^k (1-p)^{n-k} \right]^{1/\gamma}. \tag{A.24}$$

Here we take the proof for (3.28) as an example:

Proof. For $\gamma > 1$, define $\tilde{\gamma} = \frac{\gamma}{\gamma-1}$, s.t. $\frac{1}{\gamma} + \frac{1}{\tilde{\gamma}} = 1$.

$$\lambda_{ac}^* \le \lambda_{oc_1}^*$$

$$\Leftrightarrow {}^{11}_t p_x + \alpha^{1/\gamma-1} \cdot {}^{12}_t p_x \le \frac{\left(\frac{{}^{11}_t p_x}{{}_t p_x} \cdot \kappa_{n,\gamma}({}_t p_x) \right)^{1/\gamma}}{\left(\frac{{}^{11}_t p_x}{{}_t p_x} \cdot (1 - (1 - {}_t p_x)^n) \right)^{1/\gamma-1}} + \alpha^{1/\gamma-1} \cdot \frac{\left(\frac{{}^{12}_t p_x}{{}_t p_x} \cdot \kappa_{n,\gamma}({}_t p_x) \right)^{1/\gamma}}{\left(\frac{{}^{12}_t p_x}{{}_t p_x} \cdot (1 - (1 - {}_t p_x)^n) \right)^{1/\gamma-1}}$$

$$\Leftrightarrow {}^{11}_t p_x + \alpha^{1/\gamma-1} \cdot {}^{12}_t p_x \le \left({}^{11}_t p_x + \alpha^{1/\gamma-1} \cdot {}^{12}_t p_x \right) \cdot \frac{(1 - (1 - {}_t p_x)^n)}{{}_t p_x} \cdot \frac{(\kappa_{n,\gamma}({}_t p_x))^{1/\gamma}}{(1 - (1 - {}_t p_x)^n)^{1/\gamma}}$$

$$\Leftrightarrow {}_t p_x \le (\kappa_{n,\gamma}({}_t p_x))^{1/\gamma} \cdot (1 - (1 - {}_t p_x)^n)^{1-1/\gamma}$$

$$\xleftrightarrow{Hoelder Inequality} {}_t p_x = ||a||_{L^1} \le ||a||_{L^\gamma} ||b||_{L^{\tilde{\gamma}}}.$$

And for $\gamma \in (0,1)$,

$$\lambda_{ac}^* \ge \lambda_{oc_1}^*$$
$$\Leftrightarrow {}_t p_x \ge \| a \|_{L^\gamma} \| b \|_{L^{\tilde{\gamma}}}$$
$$\xleftrightarrow{Hoelder Inequality} {}_t p_x = \| a \|_{L^1} \ge \| a \|_{L^\gamma} \| b \|_{L^{\tilde{\gamma}}}.$$

□

Analogously, for (3.29), it can be proved that

$$U_0^{ac*} \ge U_0^{oc_2*} \quad \Leftrightarrow \quad \begin{cases} \lambda_{ac}^* \ge \lambda_{oc_2}^*, & \text{if } \gamma \in (0,1) \\ \lambda_{ac}^* \le \lambda_{oc_2}^*, & \text{if } \gamma > 1 \end{cases} . \tag{A.25}$$

A.5   GLM Approximation

After obtaining the crude transition intensities, we use a GLM model to smooth them. We build up estimations for mentioned $^{12}\mu_y$, $^{13}\mu_y$, and $^{23}\mu_y$. To avoid lengthy, here we take $^{12}\mu_y$ for example.

GLM model

The GLM model is composited by three parts: the link function, linear predictor, and probability distribution.

Link function: Here we adopt a log link function $g(\cdot)$ (Hanewald et al. (2019)).

$$g(^{12}\mu_y) = \ln(^{12}\mu_y) = \eta_y, \tag{A.26}$$

where η_y is the linear predictor.

Linear predictor: Suppose the transition intensity is only relevant to age and gender. For each gender, the linear predictor can be estimated by

$$\eta_y = \sum_{m=0}^{k} \beta_m y^m = \beta_0 + \beta_1 x + \beta_2 y^2 + \cdots + \beta_k y^k. \tag{A.27}$$

With reference to Fong and Feng (2016), we set $k \leq 3$ to avoid model overfitting.

Probability distribution: We further assume the transition intensity is constant for each one-year age group in a given time interval. Suppose transition number follows an independent Poisson distribution between two consecutive survey waves, i.e.

$$^{12}C_y \sim Poisson(^1Y_y \cdot {}^{12}\mu_y), \tag{A.28}$$

where $^{12}C_y$ is the transition number that one transiting from healthy to severely sick at age y. And ^1Y_y represents the year(s) of risk exposure in healthy state at age y.

Estimation and Model Selection

Moreover, we use Maximization Likelihood Estimation (MLE) to approximate the coefficients β_m. The log-likelihood function is given by

$$\ln F = \sum_{y} -{}^1Y_y\, {}^{12}\mu_y + {}^{12}C_y \ln\left({}^1Y_y\, {}^{12}\mu_y\right) + A, \tag{A.29}$$

where A is a constant and F represents the likelihood function.

Finally, we use AIC and BIC criteria to decide the optimal k. Based on the estimated β_m, we can forecast the transition intensity for every agegroup, then the transition probabilities can be further computed.

A.6 Risk Loading Computation w.r.t. Solvency II

We further compute the risk loadings following the capital requirements of Solvency II. Applying a cost of capital (CoC) ratio of 0.06 which is currently used by Solvency II (see, e.g. EIOPA (2014)), we get the risk loadings for different care-dependent products. Similar to the results computed under the requirement of C-ROSS, care-dependent annuity charges the highest risk loading, while under the baseline parameter setting (see Table 5.1), the two-pool care-dependent tontine still requires the least (see Table A.1).

CoC	$F_0^{oc_1}$	$F_0^{oc_2}$	F_0^{ac}
0.06	58.6	0.38	653

Table A.1: Risk loadings for various care-dependent products w.r.t the capital requirement of Solvency II, using the baseline parameter setting from Table 5.1. Net premium $P_0^{oc_1} = P_0^{oc_2} = P_0^{ac} = v = 10000$, pool size $n = 1000$, subjective discount rate $\rho = 0.02$, risk-free rate $r = 0.02$, initial age $x = 60$, risk aversion coefficient $\gamma = 2$, and payment weighting factor $\alpha = 0.5$.

As already known from subsection 5.2.2, under the baseline parameter setting (Table 5.1), the utility indifference number $Q_1 = 1.000896$ and $Q_2 = 1.003010$. Thus, combined with risk loadings in Table A.1, the comparable gross premiums for different care-dependent products are given by

$$Q_1 \cdot \hat{P}^{oc_1} = 1.000896 \times (10000 + 58.6) = 10067.613,$$
$$Q_2 \cdot \hat{P}^{oc_2} = 1.003010 \times (10000 + 0.38) = 10030.481,$$
$$\hat{P}^{ac} = 10000 + 653 = 10653.$$

From the above results we can learn that regarding the comparable gross premiums, people's preference order for these three care-dependent products stays the same as previously computed under the C-ROSS, i.e. two-pool CDT $>$ one-pool CDT $>$ CDA. This holds of course only for given parameters in Table 5.1.

A.7 Additional Tables of Sensitivity Analysis

How do the risk loadings vary with different coefficients (α, γ, r)?

	$F_0^{oc_1}$			$F_0^{oc_2}$			F_0^{ac}		
	$\alpha \in (0,1), \quad \gamma > 1$								
α \ γ	2	5	8	2	5	8	2	5	8
0.2	103.09943	192.84282	218.4911	0.2477858	0.3088327	0.3682938	729.2807	916.3349	969.8961
0.5	37.58904	65.09438	72.68008	0.2552838	0.3308881	0.4024555	595.1179	650.9564	666.1752
0.8	11.06583	18.20009	20.07213	0.2583212	0.3390165	0.4148757	541.0479	555.5173	559.2484
	$\alpha > 1, \quad \gamma \in (0,1)$								
α \ γ	0.2	0.5	0.8	0.2	0.5	0.8	0.2	0.5	0.8
1.5	264.9986	44.9172	10.00598	0.191409	0.2187796	0.2288318	1070.098	610.1457	538.9051
2	531.6991	85.31641	17.58199	0.1657416	0.2148421	0.2280768	1641.7743	692.7645	554.3612
5	837.1008	262.20912	44.52996	0.1376781	0.1975901	0.2253912	2301.2606	1064.105	609.3379

Table A.2: Sensitivity analysis of risk loadings w.r.t risk aversion coefficient γ and payment weighting factor α. Other parameters are taken from the baseline setting table (i.e. Table 5.1). Net premium $P_0^{oc_1} = P_0^{oc_2} = P_0^{ac} = v = 10000$, pool size $n = 1000$, risk-free rate $r = 0.02$, subjective discount rate $\rho = 0.02$, and initial age $x = 60$.

r	$F_0^{oc_1}$	$F_0^{oc_2}$	F_0^{ac}	$F_0^{oc_1}$	$F_0^{oc_2}$	F_0^{ac}
	$\gamma = 2, \quad \alpha = 0.5$			$\gamma = 0.5, \quad \alpha = 2$		
0.00	40.28555	0.339966	690.5326	74.80001	0.1196744	515.2861
0.01	38.91467	0.2948221	641.068	79.88006	0.160809	597.377
0.02	37.58904	0.2552838	595.1179	85.31641	0.2148421	692.7645
0.03	36.30882	0.2207213	552.4653	91.1043	0.2852828	803.2663
0.04	35.07392	0.1905645	512.9012	97.23027	0.3763815	930.8164
0.05	33.88404	0.1642992	476.225	103.67105	0.4932094	1077.4339

Table A.3: Sensitivity analysis of risk loadings w.r.t risk-free rate r. Other parameters are taken from the baseline setting table (i.e. Table 5.1). Net premium $P_0^{oc_1} = P_0^{oc_2} = P_0^{ac} = v = 10000$, pool size $n = 1000$, subjective discount rate $\rho = 0.02$, initial age $x = 60$, risk aversion coefficient $\gamma = 2$, and payment weighting factor $\alpha = 0.5$.

How do the utility indifference numbers vary with different coefficients (α, γ, r)?

	Q_1			Q_2		
	$\alpha \in (0,1), \quad \gamma > 1$					
γ / α	2	5	8	2	5	8
0.2	1.001053	1.003059	1.004994	1.00394	1.012662	1.021157
0.5	1.000896	1.002335	1.003723	1.00301	1.008264	1.013327
0.8	1.000833	1.00207	1.003265	1.002634	1.006652	1.010508
	$\alpha > 1, \quad \gamma \in (0,1)$					
γ / α	0.2	0.5	0.8	0.2	0.5	0.8
1.5	1.000149	1.000233	1.000337	1.000638	1.000792	1.001061
2	1.000215	1.000258	1.000344	1.001027	1.000938	1.001105
5	1.000292	1.000368	1.000371	1.001472	1.001579	1.00126

Table A.4: Sensitivity analysis of utility indifference numbers w.r.t risk aversion coefficient γ and payment weighting factor α. Other parameters are taken from the baseline setting table (i.e. Table 5.1). Net premium $P_0^{oc_1} = P_0^{oc_2} = P_0^{ac} = v = 10000$, pool size $n = 1000$, risk-free rate $r = 0.02$, subjective discount rate $\rho = 0.02$, and initial age $x = 60$.

r	Q_1	Q_2	Q_1	Q_2
	\multicolumn{2}{c}{$\gamma = 2, \quad \alpha = 0.5$}	\multicolumn{2}{c}{$\gamma = 0.5, \quad \alpha = 2$}	
0.00	1.00106	1.003316	1.000184	1.000798
0.01	1.000974	1.003157	1.000217	1.000863
0.02	1.000896	1.00301	1.000258	1.000938
0.03	1.000825	1.002875	1.000307	1.001027
0.04	1.00076	1.002751	1.000365	1.001132
0.05	1.0007	1.002636	1.000434	1.001255

Table A.5: Sensitivity analysis of utility indifference numbers w.r.t risk-free rate r. Other parameters are taken from the baseline setting table (i.e. Table 5.1). Net premium $P_0^{oc_1} = P_0^{oc_2} = P_0^{ac} = v = 10000$, pool size $n = 1000$, subjective discount rate $\rho = 0.02$, initial age $x = 60$, risk aversion coefficient $\gamma = 2$, and payment weighting factor $\alpha = 0.5$.

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