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Risk assessment of longevity risk under Solvency II: Comparing the one-year VaR framework with a terminal VaR framework

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Risk assessment of longevity risk under Solvency II: Comparing the one-year VaR framework with a terminal VaR framework

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Abstract

Solvency II prescribed two shock methodologies to determine the Solvency capital requirement for longevity risk, namely the 99.5% one-year VaR and the standard formula. However, since longevity risk lies in the long-term trend of mortality rates, it is more fitting to measure longevity risk over a multi-year horizon instead of a one-year horizon. The terminal VaR approach or in other words, the multi-year approach, measures longevity risk over a multi-year horizon as opposed to the one-year VaR. The aim of this thesis is to investigate and compare the shock methodologies prescribed by Solvency II and the terminal VaR approach and to see if the one-year VaR meets the multi-year requirements. For the implementation of the shock methodologies, the Lee-Carter and the Li-Lee model are used for the forecasting of mortality rates.

The results obtained indicate that the one-year VaR does not meet the multi-year requirements when considering the whole portfolio. However, when considering two different funds with different age groups, this is not necessarily the case. We find that for older individuals that have already retired the one-year VaR does meet the multi-year requirement.

KEYWORDS: longevity risk, Solvency II, solvency capital requirement, stochastic mortality modeling, Lee-Carter, Lee-Li, Value-at-Risk, run-off approach, terminal VaR

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Contents

1 Introduction

Insurance companies and pension funds are exposed to changes in the value of their liabilities caused by longevity risk. Due to the increasing life expectancy in the Netherlands over the past decades and also due to the expectation that life expectancy will continue to rise, the costs and thus the liabilities of the pension funds are also expected to increase. One way to deal with the increasing liabilities is to increase the premium of the contracts. In the case of the Netherlands, the pension system has responded by increasing the retirement age from 65 to 67 in 2014. Estimating the increases in life expectancy is not a problem, but the main issue is the uncertainty around these increases in life expectancy, caused by the uncertainty in mortality projections. This risk is defined as longevity risk. As a consequence of having uncertainties in mortality projections, future pension payments will also become uncertain, leading to the risk that the pension fund might not have enough capital for the future pension payments. Longevity risk can actually be decomposed into macro-longevity risk and micro-longevity risk. Macro-longevity risk refers to the systematic risk that is not diversifiable, which result from the uncertainty around future mortality rates. On the other hand, micro-longevity risk refers to the idiosyncratic risk, resulting from non-systematic deviations from a person's expected remaining lifetime. This risk is diversifiable, i.e. by increasing the number of participants in the portfolio. Throughout this thesis, we will only focus on macro-longevity risk, i.e. the systematic risk.

The way longevity risk is modeled depends on how future mortality rates are modeled, since it lies in the long-term trend taken by mortality rates. The most popular approach in the literature to model mortality is by using an extrapolative model. In this approach, mortality rates are projected into the future using historical data. In this thesis, two extrapolative models will be employed, namely the Lee and Carter (1992) model, which is known for its simplicity and reasonable accuracy, and the Li and Lee (2005) model, an extension of the Lee-Carter model which is selected by the Royal Dutch Actuarial Association for Projection Table AG2018. With these stochastic models, the best estimates from both models are obtained. The best estimate is referred to the central projection of the future trend. Next is to model the uncertainty around the best estimate, i.e. longevity trend risk. Solvency II plays a major role for insurance companies when modeling longevity risk in Europe.

As of 2016, Solvency II has been officially implemented and it requires insurance companies to have a risk based capital, so in other words extra capital for unexpected losses. This is referred to as the Solvency Capital Requirement (SCR). Under Solvency II, the definition of the SCR is as follows: "the potential amount of own funds that would be consumed by unexpected large events whose probability of occurrence within a one-year time frame is 0.5%". The SCR can either be calculated using the standard formula, which is based on a one time permanent shock on all mortality rates, or an internal model that has been approved by the insurer's supervisor or lastly, a combination of both, i.e. a partial internal model. It is shown by Gylys and Šiaulys (2019) that the standard formula is often used by small and medium sized companies, since it was constructed in a way to represent the risk profile of the average insurer and can also be seen as a benchmark. On the other hand, the internal model consists of a stochastic model which models the uncertainty around the mortality rates. For insurance companies, the most popular way to model the uncertainty is by basing it on a one year 99.5% value-at-risk methodology since it is consistent with the definition of the SCR as defined under Solvency II. Mostly larger firms can invest in an internal model, since it is much more costly. Both these methods are widely used for quantifying longevity risk and quite a few literature have been dedicated on a one year value-at-risk approach, i.e. Richards et al. (2013), Plat (2011), etc. (see following section). However, Richards et al. (2013) argued that some risks, including longevity risk, do not fit naturally in a one year value-at-risk framework, since it is a risk that unfolds over many years. It makes more sense to take a multi-year framework when dealing with longevity risk. This leads us to the following questions; Can the one-year value-at-risk approach meet the multi-year requirement? How does it compare to a multi-year approach?

The aim of this thesis is to compare the three shock methodologies, i.e. the standard formula introduced by Solvency II, a one-year VaR methodology and the multi-year VaR, so-called terminal VaR. This is done by comparing the SCRs that these shock methodologies generate. The one-year VaR and terminal VaR will be computed using stochastic mortality models. First, we will consider the Lee-Carter model for the forecasting of Dutch mortality rates, since again, it is simple and gives reasonable results. With this, the trend as well as age-specific parameters are estimated for the Dutch population. After obtaining the model estimates, forecasting of future mortality rates can be done. The central projection, also referred to as the best estimate, of the mortality rates is used to compute the best estimate of the liabilities (BEL).

When considering the case of the one-year VaR approach, the methodology introduced by Richards et al. (2013) is used. This approach includes generating a random next year mortality data point after estimating the model, and then using this data point as the newly obtained data for the next year. Then, a refit is done with the new data set, giving us new model estimates and then the best estimate is forecasted as of the following year. As a result, mortality will not only drop in the first year, but will also be updated in subsequent years. The present value of the liabilities is then computed for that specific simulation. Finally this whole process is repeated multiple times and the 99.5% quantile is taken as the 99.5% VaR. With this, the SCR is then calibrated according to the definition that Börger (2010) provided. The terminal VaR approach is much simpler compared to the chosen one-year VaR framework. It is the prediction interval of the best estimate. So, in other words, multiple scenarios are simulated as of the current year and and let them develop over the future time horizon up to the terminal year. These scenarios can be seen as actual developments of future mortality. Then the present value of the liabilities are calculated for each simulation. Finally, the 99.5% quantile is taken of the present value of the liabilities, which gives us the terminal VaR. The SCR is then computed as the difference between the terminal VaR and the BEL. After this is done for the Lee-Carter model, the Li-Lee model is considered for the forecasting and simulations of future mortality rates. When simulating the uncertainty around the best estimate, we will only take into account longevity trend risk, i.e. the uncertainties present in the trend.

When comparing the results, it is expected that the SCR of the multi-year VaR of both models will be greater than the SCR of both the standard formula and the one-year VaR of both models, since both the standard formula and the one-year VaR are only meant to capture the risk for one year. It is also interesting to see how the simple Lee-Carter model differs from the Li-Lee model, which is more accurate to model Dutch mortality. It is also expected that the SCR of the one-year VaR of both models are lower compared to that of the standard formula.

This thesis is structured as follows: Section 2.1 discusses literature on mortality modeling, with focus on stochastic models and section 2.2 discusses literature models designed to model longevity risk under solvency II. In section 3, the preliminaries are described, where some notations and definitions are introduced. In chapter 4, mortality modeling is discussed, where in section 4.1 and in section 4.2, the Lee-Carter and the Li-Lee model is discussed, respectively. In chapter 5, the main focus is longevity risk where in section 5.1, Solvency II is introduced together with the specification on how to calculate the SCR. In section 5.2, the one-year VaR framework, terminal VaR framework and the standard formula are explained and discussed. In section 5.3, the results of the three shock methodologies are shown and investigated. And we give our conclusions and summary in section 6.

2 Literature review

For the calibration of the SCR for longevity risk using a VaR framework, i.e. either a one-year or multi-year framework, stochastic modeling of mortality is required. Thus we first do a literature review on different stochastic mortality models in the literature. After discussing several stochastic models, we do a literature review on several models that are designed to model longevity risk under Solvency II.

2.1 Mortality Modeling

When forecasting mortality rates into the future, three broad approaches can be used according to Booth and Tickle (2008). The first one is the expectation approach, the second one is the explanation approach and the third one is the extrapolation approach. In the case of the expectation approach, mortality forecasting is based on the judgement and opinion of experts. In the explanation approach, on the other hand, mortality rates are forecasted either by means of a cause of death model or an explanatory model. As for the extrapolation approach, it consists of using historical regularities observed in age patterns as well as mortality trends over time for the forecasting of mortality rates. It assumes that future trends are basically a continuation of the historical trends. One thing to consider is that some mortality forecasting methods may contain aspects of more than one approach. The extrapolation approach is considered more objective and more likely to give accurate forecasts in the long-term compared to the other two approaches.

Most developments of mortality forecasting happened in the field of the extrapolation approach, where standard statistical methods are applied. One of the most popular extrapolative models out there is the Lee-Carter model, developed by Lee and Carter (1992). This dynamic mortality model is a two-factor mortality model which takes age and period into account. It models the mortality for a single population as follows: $\log \mu_{x,t} = \alpha_x + \beta_x \kappa_t + \epsilon_{x,t}$, where $\mu_{x,t}$ is the force of mortality, the α_x refers to the age pattern of mortality averaged over time, β_x is the sensitivity to κ_t, which in turn describes the change in mortality over time. As for $\epsilon_{x,t}$, it is the error term, which reflects the age-period effect that is not captured by the model. These parameters are estimated using the singular value decomposition (SVD). When forecasting mortality rates, the age-specific parameters are kept fixed, while extrapolating the time-varying parameter using standard time series methods. There are a couple of advantages the Lee-Carter model has to offer according to Booth and Tickle (2008): it is a simple model, i.e. it only has one time-varying parameter, and gives reasonable forecast trends and allows for changing age pattern of mortality.

Ever since the Lee-Carter model had been published, it received a number of criticism and since then, many extensions and variants of the Lee-Carter model has been developed. Quite a few papers proposed alternatives to improve the fit of the model. For example, Brouhns et al. (2002) assume that the number of deaths have a poisson distribution, since the number of deaths should be an integer random variable. They later on determine the model parameters of the Lee-Carter model, i.e. α_x, β_x and κ_t, by using conditional maximum likelihood estimation as opposed to using the SVD for estimation. There is also Lee and Miller (2001), who made the adjustment to use actual (observed) mortality rates in the jump-off year when forecasting future mortality rates instead of fitted rates and in this way, jump-off bias is avoided. Booth et al. (2002) optimizes the fit of the model by basing the forecast on the optimal fitting period, which in turn is determined by a goodness of fit of the model. Other than improving the fit of the model, another adjustment that has been proposed by Li et al. (2013) is the introduction of a subtle rotation in the projected age pattern of mortality decline, i.e. β_x, when life expectancy is expected to be high over a long projection horizon. This is to deal with the much criticised Lee-Carter assumption that β_x is fixed.

Another variant of the Lee-Carter model is the Cairns-Blake-Dowd (CBD) model introduced by Cairns et al. (2006). This model includes two stochastic processes that represents the two time indices of the model, where the first one affects mortality at all ages equally and the second one affects mortality at higher ages much more than lower ages. It models the logit of the one-year mortality probabilities (logit transformation) as a linear function of age, i.e. logit $q_{x,t} = k_t^{(1)} + k_t^{(2)}(x - \bar{x}) + \epsilon_{x,t}$, where $k_t^{(1)}$, the intercept, and $k_t^{(2)}$, the slope, are the two aforementioned stochastic processes, $\bar{x}$ is the mean age of the age interval that is chosen and $\epsilon_{x,t}$ is the error term. According to Pitacco et al. (2009), since the CBD model includes two time factors, it is able to capture imperfect correlation in mortality rates at different ages from one year to another, unlike the Lee-Carter model, which only includes one time factor. The Lee-Carter model has also been extended such that in can include a cohort effect, making it a three factor model, with the goal of improving the forecasting performance. Such a model is referred to as an age-period-cohort (APC) model, which was first introduced by Renshaw and Haberman (2006). They are particularly designed to fit higher ages. The mortality rate is modeled as follows: $\log(\mu_{x,t}) = \alpha_x + \beta_x^{(0)} i_{i-x} + \beta_x^{(1)} \kappa_t$, where $\beta_x^{(0)} i_{i-x}$ refers to the cohort effect. Cairns et al. (2009) also introduced cohorts effect in the CBD model, i.e. logit $q_{x,t} = k_t^{(1)} + k_t^{(2)}(x - \bar{x}) + \gamma_{t-x}^{(3)} + \epsilon_{x,t}$, where $\gamma_{t-x}^{(3)}$ is the cohort effect. They also introduce another model which includes a cohort effect and a quadratic term of the age effect. Plat (2009) proposed a model, which combines some nice features of different models, i.e. the Lee-Carter model, the Renshaw-Haberman and Cairns versions of the CBD model, with the aim of discarding the disadvantages they have. This way the model captures the cohort effect and it also becomes suitable for all age ranges.

Currently, the model that has been chosen by the The Royal Dutch Actuarial Association, Actuarieel Genootschap (2018), to forecast Dutch mortality rates is the Li-Lee model proposed by Li and Lee (2005). This is a multi-population forecasting model. The purpose for the development of the Li-Lee model is to avoid divergence in future mortality rates between populations that have commonalities. This is done by first identifying the central tendencies within the group, which is the joint trend process κ_t, also referred to as the common trend process, and then identifying the group specific trend process κ_t^g, which is the group's deviation from the common trend process in the short term. Enchev et al. (2017) reviews and compares four multi-population models, which include two Li-Lee model extensions. There are many more literature out there on modeling and forecasting mortality rates and also many researchers that made reviews of these different methods and extensions, i.e. Booth and Tickle (2008), Cairns et al. (2009), Cairns et al. (2011), Shang et al. (2011), Janssen (2018) and more.

2.2 Solvency II Framework: Longevity risk

Ever since the Solvency II directive has been published, it has been the topic of research with the main focus on longevity risk. Within this framework, the SCR is defined as the amount of capital the insurer must have as a reserve such that it covers 99.5% of situations which might arise over a horizon of one year. The way it is calibrated is through a one-year 99.5% VaR framework. Richards et al. (2013) proposed a new value-at-risk framework and also discussed two other approaches to reserving longevity risk, namely the stressed trend approach and the mortality-shock method. The stressed trend approach, also referred to as the run-off approach is a long-term stress projection applied over the lifetime of, for example a pensioner. It is the confidence envelope for the central projection over many years, where the central projection is derived from the maximum-likelihood estimate. The mortality-shock method refers to the standard-formula approach where the current and future projected mortality rates fall by 20%, which is specified in QIS5 (CEIOPS, 2010). The VaR framework that Richards et al. (2013) proposed makes use of a stochastic model to simulate mortality rates one year in the future. Then the simulated mortality rates are added as data, which in turn is used to refit the model and see how the central projection is affected. This process is repeated many times and the 99.5% quantile is taken of the liabilities. Jarner and Møller (2015) also uses a similar VaR methodology for the Danish partial internal model for longevity risk. First they model the national mortality rates using a stochastic model and then simulate the mortality rates one year ahead. With these simulated national mortality rates, sector-specific and company-specific one-year ahead mortality rates are calibrated. The stress is then determined by taking the 99.5% quantile of the liabilities calculated by the simulated mortality rates.

The stochastic models used in the aforementioned papers are categorised as so-called spot models, i.e. models that have instantaneous rates such as mortality rates as output. These type of models requires simulation of sample paths for the empirical derivation of ${}_\tau p_{g,x,t}$. Most models in the previous section fall under this category. Börger (2010) on the other used a different type of stochastic model, i.e. so-called forward models, which they used from Bauer et al. (2008). These type of models have the multi-year survival probabilities as output. And unlike spot models, forward models specify directly a distribution for ${}_\tau p_{g,x,t}$. The main advantage that forward models have is that they avoid nested simulations. The drawback is that it is more complex and more difficult to interpret.

Plat (2011) proposed a new stochastic trend model as opposed to using the typical spot models, where the trend is fixed and which only model the realized mortality. This way, mortality trend risk is modeled directly. The trend is defined by a reduction factor λ_x, which is age-specific. This reduction factor is estimated by using historical mortality rates. After obtaining these estimates, they can be used as input for modeling mortality. In this case, they are plugged in a 3-factor model, which is based on the model introduced by Plat (2009). The resulting time series estimates are then jointly modeled for both gender in a 6-factor time-series model. With this, simulations are done to obtain projected mortality rates for one year ahead. From each simulation, the liabilities are calculated and then the one-year VaR is taken. Another approach that directly models mortality trend risk is proposed by Börger et al. (2014). They include a stochastic trend component in their model so that changes in the long-term trend can be modeled. Also they model mortality rates unlike Plat (2011), who models reduction factors. The added stochastic trend process can also be implemented in other stochastic mortality model and is not only applicable in a one year setting but also in a run-off setting.

Other than using the VaR as a risk measure, one can consider analyzing the effects of other risk measures as well. This is because the VaR is not sub-additive, i.e. it does not reward diversification. Also, the VaR does not take into account the shape of the tail beyond the confidence level, which means that it does not consider worst case scenarios. Boonen (2017) analyzed the effects on several risk factors, one of them being longevity risk, of the total SCR if the Solvency II SCR calibration is based on expected shortfall instead of VaR.

3 Preliminaries

In this section, we briefly discuss some notations and definitions that play a role in the modeling and forecasting of mortality trends.

3.1 Notations and Definitions

First, let us consider an individual aged x in calendar year t with gender $g \in G$={male,female}. The changes in mortality will be analyzed as a function of both age and time for each gender. Hence, we define the following,

- $T_{x,t}^{(g)}$ is the remaining lifetime of an individual aged x, gender g at the beginning of year t. This means that the individuals will die in year $t + T_{x,t}^{(g)}$ with age $x + T_{x,t}^{(g)}$.
- $q_{x,t}^{(g)}$ is the one-year death probability of an individual aged x, gender g, in year t. This individual dies before becoming age $x+1$, i.e. $q_{x,t}^{(g)} = \mathbb{P}(T_{x,t}^{(g)} \leq 1)$.
- $p_{x,t}^{(g)}$ is the one-year survival probability of an individual aged x, gender g, in year t, i.e. $p_{x,t}^{(g)} = \mathbb{P}(T_{x,t}^{(g)} > 1)$. In other words, we have

$$p_{x,t}^{(g)} = 1 - q_{x,t}^{(g)}. \tag{3.1}$$

- $\mu_{x,t}^{(g)}$ is the force of mortality of an individual aged x in year t, gender g, formally defined as

$$\mu_{x,t}^{(g)} = \lim_{\Delta \to 0} \frac{P(x < T_{0,t-x}^{(g)} \leq x + \Delta | T_{0,t-x} > x)}{\Delta} = \lim_{\Delta \to 0} \frac{P(0 < T_{x,t}^{(g)} \leq \Delta | T_{x,t} > 0)}{\Delta}. \tag{3.2}$$

It is assumed that the age-specific forces of mortality are constant within bands of age and time. To be more precise, given an integer age x and calendar year t, it holds that

$$\mu_{x+\xi,t+\nu}^{(g)} = \mu_{x,t}^{(g)} \text{ for } \xi \geq 0 \text{ and } \ \nu < 1. \tag{3.3}$$

Since we assume that equation (3.3) holds, we have

$$p_{x,t}^{(g)} = \exp(-\mu_{x,t}^{(g)}), \tag{3.4}$$

for interger age x and calendar year t.

Next, we introduce the notion of the exposure-to-risk. It refers to the total number of 'person-years' in a population over calendar year t. It can be seen as the average number of individuals in the population over a calendar year adjusted to the amount of time they are alive in the population. To be more precise, let $\tau_i^{(g)}$ be the the amount of time that individual i with gender g is alive in $[t, t+1)$ with age $[x, x+1)$. Then the exposure to risk is

$$E_{x,t}^{(g)} = \sum_{i=1}^{n_{xt}^{(g)}} \tau_i^{(g)}, \tag{3.5}$$

where $n_{xt}^{(g)}$ is the number of individuals aged x, gender g in year t. Next, let $D_{x,t}^{(g)}$ denote the number of deaths in a population during year t, i.e. $[t, t+1)$, for person aged x at the beginning of year t with gender g. Pitacco (2009) shows that the maximum likelihood estimator $\hat{m}_{x,t}^{(g)}$ for the force of mortality $\mu_{x,t}^{(g)}$ is

$$\hat{\mu}_{x,t}^{(g)} = \frac{D_{x,t}^{(g)}}{E_{x,t}^{(g)}} = \hat{m}_{x,t}^{(g)}. \tag{3.6}$$

The $\hat{m}_{x,t}^{(g)}$'s are also referred to as (unsmoothed) mortality rates.

Next, we give the formal definition following McNeil et al. (2005) of the value-at-risk (VaR), which is a risk measure that is widely used in regulatory practice. For a given portfolio and given a confidence level $\alpha \in (0,1)$ the VaR of the portfolio at confidence α is given by the smallest number x such that the probability that the loss X is greater than x is lower of equal to $(1-\alpha)$. Thus, the VaR can be defined as:

$$VaR_{\alpha}(X) = \min\{x \in \mathbb{R} \; | P(X > x) \leq 1 - \alpha\}, \tag{3.7}$$

which is the α-quantile of the loss-distribution in probabilistic terms.

4 Mortality Modeling

In this thesis, two extrapolative stochastic models are considered to forecast future Dutch mortality rates. The first one that is discussed is the Lee-Carter model introduced by Lee and Carter (1992), which is discussed in section 4.1 and the second one is the Li-Lee model introduced by Li and Lee (2005), which is discussed in section 4.2.

4.1 The Lee-Carter Model

As introduced in the literature review, according to Booth and Tickle (2008), the Lee-Carter model is a two-factor stochastic model that is a function of age and period that uses historical data to forecast future mortality rates. The Lee-Carter is very well-known and is used for its simplicity and for the very fact that it performs quite well.

4.1.1 Data

The data of the Netherlands is taken from the Human Mortality Database (HMD). To model Dutch mortality rates, the total death per year and total exposure per year is needed. This is exactly what can be found on the HMD. For each gender g, age x and year t we can obtain the total number of deaths, $D_{x,t}^{(g)}$, as well as the total number of exposures, $E_{x,t}^{(g)}$, in the Netherlands. Since the data available on the HMD is up to year 2016, we obtain the data for 2017 and 2018 from CBS (Statline). The data range for the Netherlands is selected such that it starts from 1970 up until the most recent year of which CBS has the total number of deaths, which is currently 2018. The year 1970 is chosen, such that it is aligned with the Projections Table AG2018 data range. The reason why year 1970 is chosen, is because ever since then, a stable development is observed in the mortality probabilities for both men and women of some West-European countries. Thus, the number of years used as historical data in this time period is 48. The age range available in the HMD start from age 0 up to 110+. However, the selected age range will be $x \in \{0, 1, \ldots, 90\}$ so that it can match the chosen age set in the Projections Table AG2018. For older ages, the Kannistö extrapolation is applied. This will be discussed later on.

4.1.2 Model and Calibration Common Parameters

The Lee-Carter model specifies the logarithm of mortality rates for the Netherlands as:

$$\log \mu_{x,t}^{(g)NL} = \alpha_x^{(g)NL} + \beta_x^{(g)NL} \kappa_t^{(g)NL}, \tag{4.1}$$

where $\mu_{g,x,t}^{NL}$ denotes the force of mortality of the Netherlands for a person with age $x \in X = \{x_1 = 0, x_2 = 1, \ldots, x_m = 90\}$, for gender $g \in \{\text{males}, \text{females}\}$ in year $t \in \mathcal{T} = \{t_1 = 1970, t_2 = 1971, \ldots, t_n = 2018\}$. Note that the force of mortality is modeled separately for each gender. To simplify the notation, we will exclude the subscript "NL" on the parameters. $\alpha_x^{(g)}$ and $\beta_x^{(g)}$ are age specific constants and $\kappa_t^{(g)}$ the time varying index. The constant $\alpha_x^{(g)}$ is the base shape across age of the mortality profile. $\beta_x^{(g)}$ is the sensitivity of age x to $\kappa_t^{(g)}$, so in other words it tells us which rate decreases slowly and which one decreases rapidly when there is a change in $\kappa_t^{(g)}$. Now to estimate these parameters, we need to impose additional constraints. If model (4.1) does not have constraints on the parameters, then the model is not identified. The following constraints ensure that there is identification:

$$\sum_{t=t_1}^{t_n} \kappa_t^{(g)} = 0 \text{ and } \sum_{x=x_1}^{x_m} \beta_x^{(g)} = 1. \tag{4.2}$$

The maximum likelihood estimators of the Dutch force of mortality $\hat{\mu}_{x,t}^{(g)}$, i.e. the Dutch mortality rates, is obtained as in equation (3.6) using the observed number of deaths and exposure at age x, for gender g during year t from the data set. Then the Dutch mortality rates are

$$\hat{\mu}_{x,t}^{(g)} = \frac{D_{x,t}^{(g)}}{E_{x,t}^{(g)}}. \tag{4.3}$$

The model then becomes:

$$\log \hat{\mu}_{x,t}^{(g)} = \alpha_x^{(g)} + \beta_x^{(g)}\kappa_t^{(g)} + \epsilon_{x,t}^{(g)}, \tag{4.4}$$

where $\epsilon_{x,t}^{(g)}$ denotes the error terms which refer to the age-specific historical influences which were not captured by the model. The error terms are assumed to be independent over x, t and g with mean zero and homoskedastic variance $\sigma_{\epsilon,g}^2$. The common parameters are then calibrated using least squares estimation. This is done fitting equation (4.4) to the observed force of mortality by using the singular value decomposition. After obtaining the estimates of the common parameters $\hat{\alpha}_x^{(g)}$, $\hat{\beta}_x^{(g)}$ and $\hat{\kappa}_t^{(g)}$, Lee and Carter (1992) proposed that the $\hat{\kappa}_t^{(g)}$ should be adjusted such that in each year t, the observed number of deaths are reproduced.

As mentioned before, in the classical Lee-Carter model, the error terms are assumed to be homoskedastic, i.e. the errors have the same variance over all ages x and time t. This is due to the assumption that the errors are normally distributed, which in turn lead to less precise estimates of the mortality rates for very old ages, since it ignores the fact that the mortality rates are much more volatile at older ages. The high variability is caused by the lack of observations of people being alive. To deal with this problem, Brouhns et al.(2002) assumed that the number of deaths, $D_{x,t}^{(g)}$ is assumed to be modeled from a Poisson distribution. This is also plausible, since the number of deaths should be an integer random variable. Hence, we get:

$$D_{g,x,t} \sim \text{Poisson}(\lambda_{g,x,t}) \quad \text{with} \quad \lambda_{g,x,t} = E_{g,x,t}\mu_{g,x,t} \tag{4.5}$$

With $\mu_{g,x,t} = \exp(\alpha_{g,x} + \beta_{g,x}\kappa_{g,t})$. Then the following log-likelihood function, which is based on the Poisson distribution assumption, is maximized:

$$\max_{\{\alpha_{g,x},\beta_{g,x},\kappa_{g,t}\}} L(\alpha,\beta,\kappa) = \sum_{x=x_1}^{x_m}\sum_{t=t_1}^{t_n} D_{g,x,t}\log(\lambda_{g,x,t}) - \lambda_{g,x,t} - \log(D_{g,x,t}!), \tag{4.6}$$

which is also subject to the same constraints as the original Lee-Carter model, i.e. equation (4.2). To solve this maximization problem, the Newton-Raphson algorithm is implemented. With this algorithm the estimates of the common parameters are obtained recursively. To get the parameter estimates, the system that needs to be solved is by taking the partial derivative of the total squared deviance and setting it to zero. Following the notation of Pitacco et al. (2009) and initializing the parameter estimates $\hat{\alpha}_{g,x}^{(0)} = 0$, $\hat{\beta}_{g,x}^{(0)} = 1$ and $\hat{\kappa}_{g,t}^{(0)} = 0$, where random values can also be used for initialization, we get:

$$\hat{\alpha}_{g,x}^{(k+1)} = \hat{\alpha}_{g,x}^{(k)} - \frac{\sum_{t=t_1}^{t_n}(D_{g,x,t} - E_{g,x,t}\exp(\hat{\alpha}_{g,x}^{(k)} + \hat{\beta}_{g,x}^{(k)}\hat{\kappa}_{g,t}^{(k)}))}{-\sum_{t=t_1}^{t_n} E_{g,x,t}\exp(\hat{\alpha}_{g,x}^{(k)} + \hat{\beta}_{g,x}^{(k)}\hat{\kappa}_{g,t}^{(k)})} \tag{4.7}$$

$$\hat{\kappa}_{g,t}^{(k+1)} = \hat{\kappa}_{g,t}^{(k)} - \frac{\sum_{x=x_1}^{x_m}(D_{g,x,t} - E_{g,x,t}\exp(\hat{\alpha}_{g,x}^{(k+1)} + \hat{\beta}_{g,x}^{(k)}\hat{\kappa}_{g,t}^{(k)}))\hat{\beta}_{g,x}^{(k)}}{-\sum_{x=x_1}^{x_m} E_{g,x,t}\exp(\hat{\alpha}_{g,x}^{(k+1)} + \hat{\beta}_{g,x}^{(k)}\hat{\kappa}_{g,t}^{(k)})(\hat{\beta}_{g,x}^{(k)})^2} \tag{4.8}$$

$$\hat{\beta}_{g,x}^{(k+1)} = \hat{\beta}_{g,x}^{(k)} - \frac{\sum_{t=t_1}^{t_n}(D_{g,x,t} - E_{g,x,t}\exp(\hat{\alpha}_{g,x}^{(k+1)} + \hat{\beta}_{g,x}^{(k)}\hat{\kappa}_{g,t}^{(k+1)}))\hat{\kappa}_{g,t}^{(k)}}{-\sum_{t=t_1}^{t_n} E_{g,x,t}\exp(\hat{\alpha}_{g,x}^{(k+1)} + \hat{\beta}_{g,x}^{(k)}\hat{\kappa}_{g,t}^{(k+1)})(\hat{\kappa}_{g,t}^{(k)})^2} \tag{4.9}$$

Here k is the number of iterations. To stop this algorithm, a specific criterion need to be met. In this case, we continue to iterate if the improvement in log-likelihood exceeds 0.0001. So the moment that the improvement is less than or equal to 0.0001, then the algorithm stops. For this thesis, the Lee-Carter alteration of Brouhns et al. (2002) is chosen for the calibration of the common parameters. When modeling the time series $\kappa_{g,t}$, Lee and Carter (1992) use a random walk with drift to generate mortality projections. This will be discussed in the following section.

The common parameter estimates for the Lee-Carter model are shown in figures 1 to 3. The model is applied separately to Dutch males and females. The chosen age range is again from 0 to 90 and in the years 1970 to 2018. The estimates were calibrated by solving the optimization problem of the Poisson likelihood in equation (4.6) by utilizing the aforementioned Newton-Raphson algorithm. First, when observing the pattern of the historical average of the logarithm of the mortality rates, α_x, it can be seen that for both male and female that mortality during the first year of a new born, i.e. $x = 0$, is relatively high. It then decreases very fast for children up until age 10. Afterwards the logarithm of the mortality rates has almost a linearly increase up to age 90. For the ages in the early twenties, a hump can be seen for both male and female. This is also called the accident hump, where deaths are mainly caused by accidents. Males are more likely to live more dangerously compared to females at those ages, hence why the hump is much higher. Next, in figure 3, the development of the historical mortality trend can be observed for male and female respectively. From this it can be seen that both mortality trends are decreasing. Between year 1975 and year 2000, the estimated mortality trend for females has a faster decline compared to the estimated trend for males. This is over all ages. However, after year 2000, the estimated trend for males starts to decline faster than that of females. Over the whole horizon, it can be concluded that the estimated trend for females looks more like a straight line compared to the estimated trend for males. As we established that the mortality trends are decreasing, the sensitivity parameter β_x indicate how each age group reacts to the decreasing trend. With a decreasing trend, a higher β_x indicates a stronger or faster decrease for people in age group x. If β_x is low, this indicates the opposite, i.e. slower decrease. The estimate of the sensitivity parameter for both male and female can be seen in figure 2. This leads to the conclusion that mortality for older males increase at a slower rate than for younger males and for the oldest ages there are no improvement at all. For the females on the other hand, the mortality for younger females increase at a faster rate, but also around age 75 there is a faster increase that cannot be ignored.

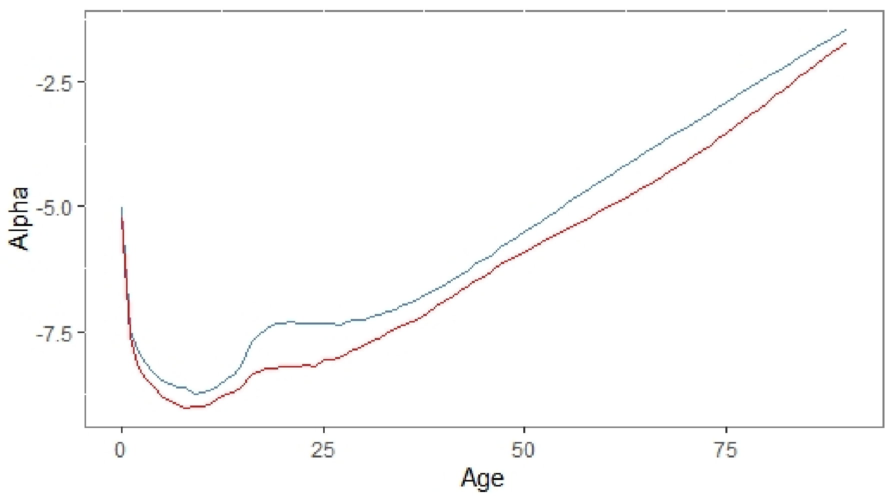

Figure 1: The Lee-Carter estimates of α_x for both male (blue) and female (red) of the Dutch population from 1970 up to and including 2018.

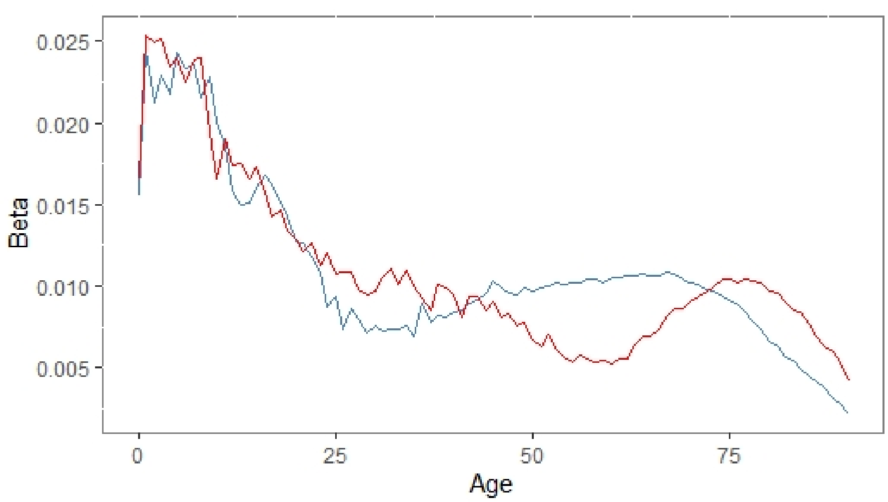

Figure 2: The Lee-Carter estimates of β_x for both male (blue) and female (red) of the Dutch population from 1970 up to and including 2018.

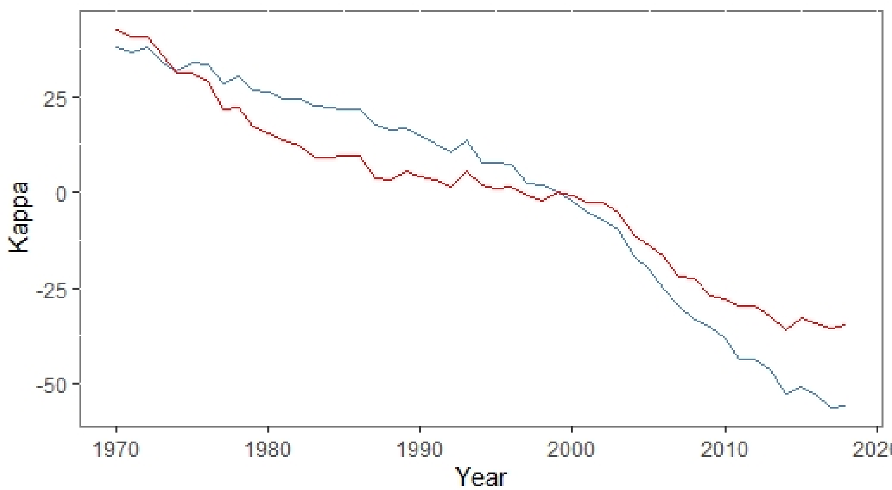

Figure 3: The Lee-Carter estimates of κ_t for both male (blue) and female (red) of the Dutch population from 1970 up to and including 2018.

After obtaining the common parameters estimates, the next step is to forecast mortality rates into the future.

4.1.3 Forecasting Mortality

When projecting mortality rates into the future, the time-varying variable is the one that needs to be projected into the future. As briefly mentioned in the previous section, the trend processes $\kappa_{g,t_1}, \ldots, \kappa_{g,t_n}$ are modeled as a random walk with drift (RWD), which is defined as

$$\kappa_{g,t} = \kappa_{g,t-1} + \theta_g + e_{g,t}, \tag{4.10}$$

with

$$e_{g,t} = \begin{pmatrix} e_{M,t} \\ e_{F,t} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2_{M,e} & r\sigma_{M,e}\sigma_{F,e} \\ r\sigma_{M,e}\sigma_{F,e} & \sigma^2_{F,e} \end{pmatrix} \right), \tag{4.11}$$

where $t = t_2, ..., t_n$ and θ_g denotes the drift term. The gender specific drift term refers to the annual average change in $\kappa_{g,t}$. The trend processes for both male and female are jointly estimated, which in turn generates errors which are normally distributed with mean zero and covariance matrix, denoted by C, i.e.

$$C = \begin{pmatrix} \sigma^2_{M,e} & r\sigma_{M,e}\sigma_{F,e} \\ r\sigma_{M,e}\sigma_{F,e} & \sigma^2_{F,e}, \end{pmatrix}$$

where $\sigma^2_{M,e}$ refers to the variance of the errors of males and $\sigma^2_{F,e}$ refers to the variance of the errors of females. Then $r\sigma_{M,e}\sigma_{F,e}$ is the product of the standard deviations of male and female with the correlation coefficient r. In this way, correlation between the errors are captured, which in turn captures the correlation between the genders. We assume different trend processes for the genders, because there are still differences between male and female, i.e. men started smoking earlier than women or take gender specific diseases into account. To jointly estimate the trend processes, the seemingly unrelated regression (SUR) is used, which in turn generates the estimates of the gender specific drift term, $\hat{\theta_g}$ and covariance matrix, $\hat{C}$. The RWD is also known as ARIMA(0,1,0), which belongs to the autoregressive integrated moving average (ARIMA) class of time series modeling. These type of models make use of the history and past shocks to explain the time series.

After fitting the time series model, we can use the obtained estimates to project future mortality rates. From equation (4.10), we estimate the future trend processes as follows:

$$\hat{\kappa}_{g,t_n+k} = \kappa_{g,t_n} + k\hat{\theta}_g + \sum_{j=1}^{k} e_{g,k}, \tag{4.12}$$

where k is the number of years projected in the future. The point forecast of the trend parameter, which will also be referred to as the best estimate (BE) of the future trend parameters is the expectation of the future trend conditional on the past trends, i.e.:

$$\hat{\kappa}^{BE}_{g,t_n+k} = \mathbb{E}[\hat{\kappa}_{g,t_n+k} | \kappa_{g,t_1}, \kappa_{g,t_2}, \ldots, \kappa_{g,t_n}] = \kappa_{g,t_n} + k\hat{\theta}_g. \tag{4.13}$$

Hence, the best estimate of the future trends per gender g has intercept κ_{g,t_n} and slope θ_g on projection horizon k.

In reality, the actual forecast is subject to both stochastic and parameter uncertainty. The stochastic uncertainty, also known as volatility risk, refers to the ARIMA standard error, i.e. $e_{g,t}$. On the other hand, the parameter uncertainty, also known as parameter risk, refers to the ARIMA parameters themselves, i.e. the drift estimate $\hat{\theta}_g$. To construct an appropriate prediction interval, both uncertainties need to be taken into account.

First, we consider stochastic uncertainty only and thus fixing the drift term to $\hat{\theta}_g$. To construct a prediction interval, M trajectories are simulated of the stochastic error terms, which are drawn from the normal distribution in equation (5.38). And for each of these error simulations, the future trend process is projected over time horizon k, i.e.:

$$\hat{\kappa}^{(m)}_{g,t_n+k} = \kappa_{g,t_n} + k\hat{\theta}_g + \sum_{j=1}^{k} e^{(m)}_{g,j} \tag{4.14}$$

After plugging the simulated errors in, we get M simulations of future trends. For the prediction interval, the 5%-quantile and the 95%-quantile is taken from the simulations. In figure 4, the best estimate of the trend process is shown for both male and female. For the prediction interval, $M = 1000$ simulations were generated and the projection horizon is $k = 120$ years in the future.

Next, we will consider only taking parameter uncertainty into account. This is done by setting the errors to zero, i.e. $\epsilon_{g,t_n+k} = 0$, and the drift is assumed to be normally distributed with the drift estimate obtained from the SUR regression as the mean, and the standard error of the estimate as the variance, i.e.

$$\theta_g \sim N\left(\hat{\theta}_g, se(\hat{\theta}_g)\right). \tag{4.15}$$

By drawing the drift terms M times from this distribution, we generate future possible scenarios:

$$\hat{\kappa}^{(m)}_{g,t_n+k} = \kappa_{g,t_n} + k\theta^{(m)}_g \tag{4.16}$$

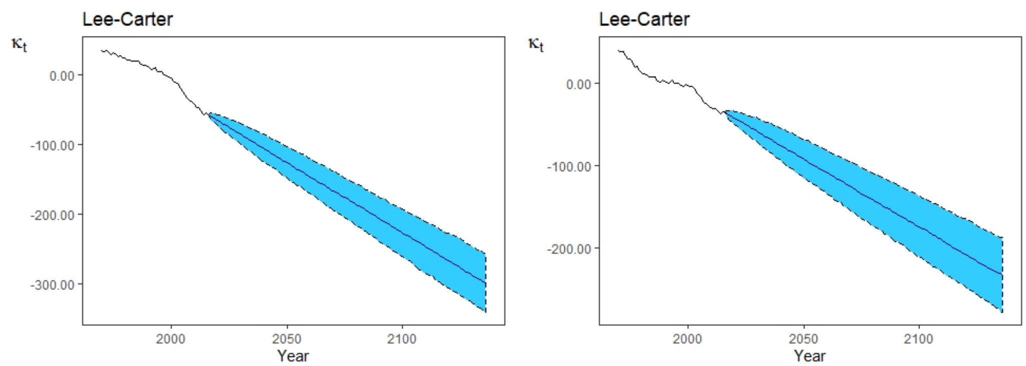

Figure 4: On the left: Forecasted best estimate male trend with prediction interval. On the right: Forecasted best estimate female trend with prediction interval. The number of simulations used is $M = 1000$. The prediction interval consists of taking the 5%-quantile and 95%-quantile of the simulations. The projection horizon is $k = 120$

By taking the 5%-quantile and 95%-quantile, we can again construct a prediction interval including only parameter uncertainty. In figure 5 the best estimate of the trend for both the Dutch male population and female population is shown, together with the prediction interval consisting only of parameter uncertainty.

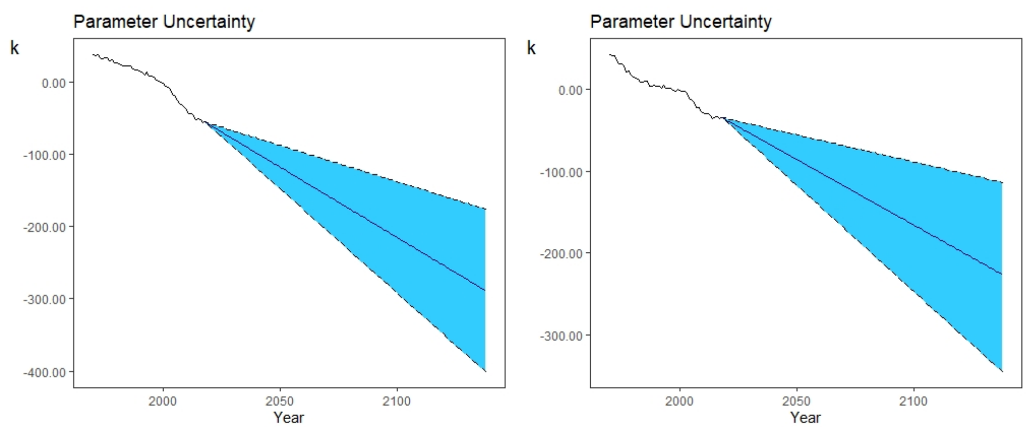

Figure 5: On the left: Forecasted best estimate male trend with prediction interval containing only parameter risk. On the right: Forecasted best estimate female trend with prediction interval containing only parameter risk. The number of simulations used is $M = 5000$. The prediction interval consists of taking the 5%-quantile and 95%-quantile of the simulations. The projection horizon is $k = 120$.

Now that both stochastic uncertainty and parameter uncertainty have been introduced, combining them both leads to:

$$\hat{\kappa}_{g,t_n+k}^{(m)} = \kappa_{g,t_n} + k\theta_g^{(m)} + \sum_{j=1}^{k} e_{g,j}^{(m)} \tag{4.17}$$

In figure 5, it can be observed that in the long run, parameter risk is the main cause of uncertainty, while in the short run, volatility risk is the main cause. In figure 6, the best estimate of the trend for both the Dutch male population and female population is shown, together with the prediction interval consisting of both stochastic and parameter uncertainty. For the prediction interval, the 5%-quantile and 95%-quantile of the simulations are taken.

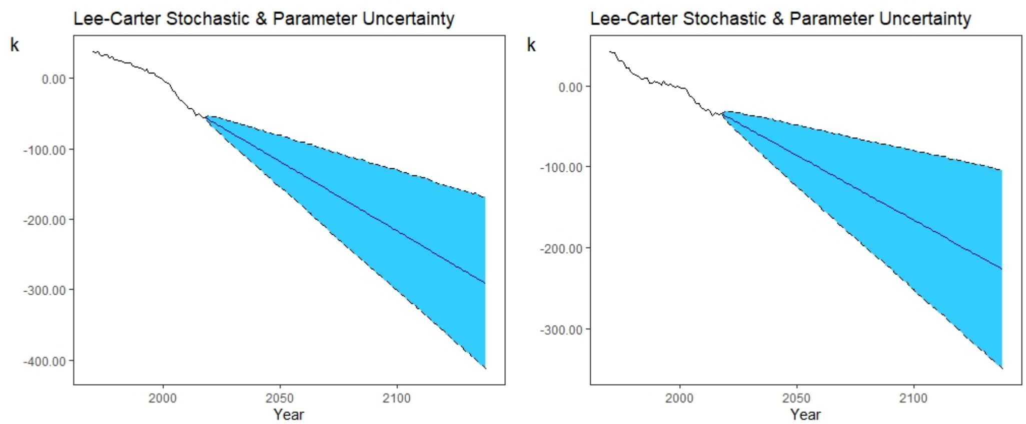

Figure 6: On the left: Forecasted best estimate male trend with prediction interval containing both volatility and parameter risk. On the right: Forecasted best estimate female trend with prediction interval containing both stochastic and parameter risk. The number of simulations used is $M = 5000$. The prediction interval consists of taking the 5%-quantile and 95%-quantile of the simulations. The projection horizon is $k = 120$

Now that we have obtained the best estimate trend process $\hat{\kappa}^{BE}_{g,t_n+k}$ for the Dutch population, it can be plugged in equation (4.1) to obtain the best estimate of the future log mortality rates, i.e.:

$$\log\, \hat{\mu}^{BE}_{g,x,t_n+k} = \hat{\alpha}_{g,x} + \hat{\beta}_{g,x}\hat{\kappa}^{BE}_{g,t_n+k}, \tag{4.18}$$

Next we apply Kannisto's (1992) method to $\log\, \hat{\mu}^{BE}_{g,x,t_n+k}$. When estimating the mortality probabilities of older individuals, i.e. individuals aged x in age group $\tilde{X} = \{91, ..., 120\}$, fluctuations may arise in the estimations. This is due to the lack of observations at high ages or if there aren't any observations available for specific high ages. In the case of the data obtained from HMD, there is data available for ages up until age $x = 110$. To fill the table or "close" the mortality table, an extrapolation method is used to determine the mortality probabilities of high ages, namely Kannistö's closing method. This method is also used in the Royal Dutch Actuarial Association's(AG) (2018), "Projection table AG2018".

The Kannistö's closing method is used for the aforementioned age group $x \in \tilde{X} = \{91, ..., 120\}$. This is done by using a logistic regression obtained from the table for ages $y \in X^{Kan} = \{80, 81, ..., 90\}$, which is similar to AG2018. Then counting the number of years underlying the regression y_k, we get $n = 11$. Then taking the mean of the ages and then the squared sum of deviations, we get $\bar{y} = \frac{1}{n}\sum_{k=1}^{n} y_k$ and $SSD = \sum_{k=1}^{n}(y_k - \bar{y})^2$ respectively. The extrapolation of the mortality rates according to Kannistö is thus as follows:

$$\mu_{x,t} = L\left(\sum_{k=1}^{n} w_k(x) L^{-1}(\mu_{y_k,t})\right), \tag{4.19}$$

for $x \in \tilde{X}$ and for evert t. $w_k(x)$ are the regression weigths and the functions $L(x)$ and its inverse $L^{-1}(x)$ are defined as

$$L(x) = \frac{1}{1+e^{-x}}, \quad L^{-1}(x) = -\log\left(\frac{1}{x} - 1\right). \tag{4.20}$$

As for the regression weights, they are defined as

$$w_k(x) = \frac{1}{n} + \frac{(y_k - \bar{y})(x - \bar{y})}{SSD} \tag{4.21}$$

4.2 The Li-Lee Model

Unlike the Lee-Carter model, which is a single-population model, the Li and Lee model is a multi-population model. It is a Lee-Carter extension that allows for the use of multiple countries. Currently, the Royal Dutch Actuarial Association (2018) is using it for the forecasting of the Dutch mortality rates, where 13 other countries, i.e. West-European countries, are included for fitting the model. This is because they observed a decrease in the differences in mortality probabilities between the selected European countries and they also have a similar upward trend in life expectancy. Because of these similarities, the choice was made to model Dutch mortality based on the development of these countries. By including these countries, the mortality forecast won't have to rely on Dutch data alone. This is a major advantage, since in the past there may have been specific fluctuations that occurred in Dutch data that is not useful for future developments. By including the selected countries, it greatly increases the amount of observations, which in turn leads to a more robust model and more stable projections.

4.2.1 Data

As Li and Lee (2005) mentioned, the population, i.e. in this case countries, should have similar socioeconomic conditions and close connection which are also expected to continue in the future. The Royal Dutch Actuarial Association (2018) made the choice based on Gross Domestic Product (GDP). The reasoning for this is because there is a positive correlation prosperity and aging. So European countries with similar GDP as the Netherlands. Since the Netherlands has an above-average GDP, similar countries with above-average GDP were chosen. These countries are Belgium, Denmark, Germany, Finland, France, Ireland, Iceland, Luxembourg, Norway, Austria, United Kingdom, Sweden and Switzerland. The data of the Netherlands is also included. The mortality data of these countries can all be found on HMD. The data range is from 1970 to 2016 for the European countries. This is because 2016 is the most recent data available on the HMD website for all the selected countries. When referring to European data, we mean the observed total number of deaths, $D^{EU}_{g,x,t}$ and exposures $E^{EU}_{g,x,t}$ of all these European countries.

4.2.2 Model and Calibration Model Parameters

The Li-Lee model specifies the logarithm of the Dutch mortality rates as follows:

$$\log \mu^{NL}_{x,g,t} = \log \mu^{EU}_{x,g,t} + \log \tilde{\mu}^{NL}_{x,g,t}, \tag{4.22}$$

with

$$\log \mu^{EU}_{x,g,t} = \alpha^{EU}_{x,g} + \beta^{EU}_{x,g} \kappa^{EU}_{g,t} \tag{4.23}$$

$$\log \tilde{\mu}^{NL}_{x,g,t} = \alpha^{NL}_{x,g} + \beta^{NL}_{x,g} \kappa^{NL}_{g,t}, \tag{4.24}$$

where $\mu^{NL}_{x,g,t}$ denotes the Dutch mortality rate for an individual aged $x \in \{0, 1, \ldots, 90\}$ in calender year $t \in \{1970, \ldots, 2016\}$ and gender $g \in \{male, female\}$. The European mortality rate is denoted as $\mu^{EU}_{x,g,t}$ and $\tilde{\mu}^{NL}_{x,g,t}$ denotes the Dutch deviation relative to the European mortality rate. $\beta^{EU}_{x,g} \kappa^{EU}_{g,t}$ is the common factor that models the long term trend and age specific movement that all the the selected European countries share. $\beta^{NL}_{x,g} \kappa^{NL}_{g,t}$ describes the short term deviations from the common trend, which is specific to the Netherlands. The $\kappa^{EU}_{g,t}$ describes the change of mortality over time and $\beta^{EU}_{x,g}$ is the sensitivity to $\kappa^{EU}_{g,t}$ for all the European countries. The $\beta^{NL}_{x,g}$ describes the differences between the patterns of change by age in mortality for the Dutch population and for the selected European countries. Originally, Li and Lee (2005) use a two step approach for the calibration of the model parameters by using singular value decomposition. However, the Poisson maximum likelihood method is used for the parameter calibration in this thesis.

The Poisson maximum likelihood method is also used in the Projection table AG2018 to calibrate the model parameters. The assumption that the number of deaths of the European population,

$D_{g,x,t}^{EU}$, is modeled as a Poisson distribution. First the maximization of the likelihood function is done for European data, so that we can determine the estimates of $\alpha_{g,x}^{EU}$, $\beta_{g,x}^{EU}$ and $\kappa_{g,t}^{EU}$:

$$\max_{\{\alpha_{g,x}^{EU},\beta_{g,x}^{EU},\kappa_{g,t}^{EU}\}} \prod_{x=x_1}^{x_m} \prod_{t=t_1}^{t_n} \frac{(E_{g,x,t}^{EU}\mu_{g,x,t}^{EU})^{D_{g,x,t}^{EU}} \exp(-E_{g,x,t}^{EU}\mu_{g,x,t}^{EU})}{D_{g,x,t}^{EU}!}. \tag{4.25}$$

To ensure a unique solution, we set the constraints:

$$\sum_{t=t_1}^{t_n} \kappa_{g,t}^{EU} = 0 \text{ and } \sum_{x=x_1}^{x_m} \beta_{g,x}^{EU} = 1. \tag{4.26}$$

The same method that was introduced in section 4.1.2, namely the Newton-Raphson algorithm, is implemented to obtain the estimates. Since the data of Europe is up to and including 2016 for the chosen countries, a linear extrapolation method is used to compute $\kappa_{g,2017}^{EU}$ and $\kappa_{g,2018}^{EU}$, i.e.

$$\kappa_{g,2017}^{EU} = \kappa_{g,2016}^{EU} + \frac{\kappa_{g,2016}^{EU} - \kappa_{g,1970}^{EU}}{2016-1970} \text{ and } \kappa_{g,2018}^{EU} = \kappa_{g,2017}^{EU} + \frac{\kappa_{g,2017}^{EU} - \kappa_{g,1970}^{EU}}{2017-1970} \tag{4.27}$$

So after obtaining $\kappa_{g,2017}^{EU}$, we use it to calculate $\kappa_{g,2018}^{EU}$. After obtaining the estimates of the EU data, the next step is to model for the Netherlands. The number of deaths in the Netherlands has the following Poisson distribution:

$$D_{g,x,t}^{NL} | \tilde{\mu}_{g,x,t}^{NL}, \mu_{g,x,t}^{EU} \sim \text{Poisson}\left(E_{g,x,t}^{NL}\mu_{g,x,t}^{EU}\tilde{\mu}_{g,x,t}^{NL}\right) \tag{4.28}$$

Next, the maximum log-likelihood method is applied to the Dutch data:

$$\max_{\{\alpha_{g,x}^{NL},\beta_{g,x}^{NL},\kappa_{g,t}^{NL}\}} \prod_{x=x_1}^{x_m} \prod_{t=t_1}^{t_n} \frac{(E_{g,x,t}^{NL}\mu_{g,x,t}^{NL})^{D_{g,x,t}^{NL}} \exp(-E_{g,x,t}^{NL}\mu_{g,x,t}^{NL})}{D_{g,x,t}^{NL}!}, \tag{4.29}$$

where $\mu_{g,x,t}^{NL} = \mu_{g,x,t}^{EU} \cdot \tilde{\mu}_{g,x,t}^{NL} = \mu_{g,x,t}^{EU} \cdot \exp(\alpha_{g,x}^{NL} + \beta_{x,g}^{NL}\kappa_{g,t}^{NL})$. Again we set the constraints $\sum_{t=t_1}^{t_n} \kappa_{g,t}^{NL} = 0$ and $\sum_{x=x_1}^{x_m} \beta_{g,x}^{NL} = 1$ and we use the Newton-Raphson algorithm for the estimation of the parameters.

In figure 7, the Li-Lee estimates of $\alpha_{g,x}^{NL}$ and $\alpha_{g,x}^{EU}$ are shown. In the case of $\alpha_{g,x}^{EU}$, which has a similar shape to the previous Lee-Carter estimate, the interpretation is the same as well as in the case of the Lee-Carter model. An example of this is the high mortality for new borns or the accident hump that occurs for individuals in their early twenties. This is also the case with the $\beta_{g,x}^{EU}$ estimate as shown in figure 8, i.e. it has the same shape as with the Lee-Carter estimates. However, the estimates of $\alpha_{g,x}^{NL}$ and $\beta_{g,x}^{NL}$ represents the deviation from the estimates of the European countries and the Dutch population, hence it has a different interpretation. In the case of the estimate of $\alpha_{g,x}^{NL}$, in the early ages, it fluctuates around zero. Then for the most of the ages it is below zero, so the general pattern of mortality in the Netherlands lie for the most part below that of the European countries. For the older ages, the difference tends to zero, which means the mortality pattern for the older ages in the Netherlands are similar to the selected European countries. As for the trend process $\kappa_{g,t}^{NL}$, which can be seen in figure 9, tends to not be equal to zero, which is especially clear for females. This means that there is definitely deviation from the European trend.

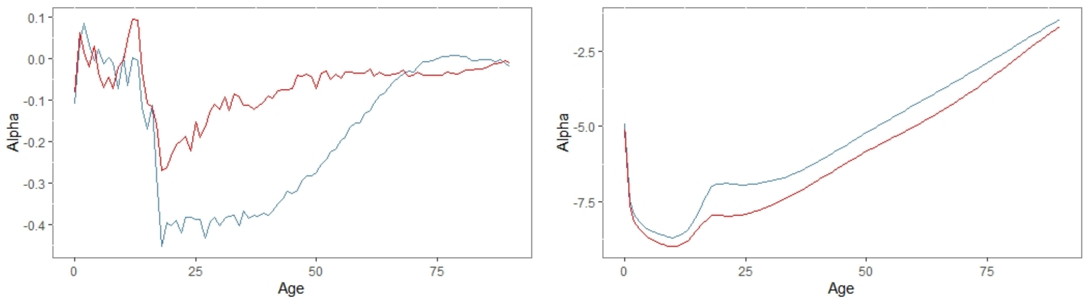

Figure 7: On the left: The Li-Lee estimates of α_x^{NL} for both male (blue) and female (red) of the Dutch population from 1970 up to and including 2018. On the right: he Li-Lee estimates of α_x^{EU} for both male (blue) and female (red) of the European countries from 1970 up to and including 2018.

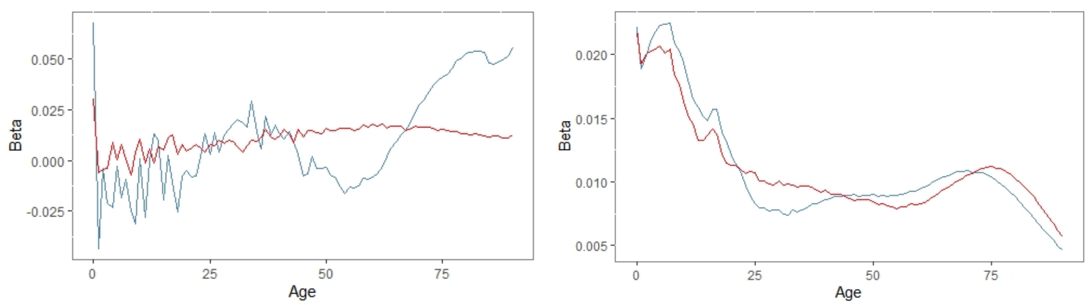

Figure 8: On the left: The Li-Lee estimates of β_x^{NL} for both male (blue) and female (red) of the Dutch population from 1970 up to and including 2018. On the right: he Li-Lee estimates of β_x^{EU} for both male (blue) and female (red) of the European countries from 1970 up to and including 2018.

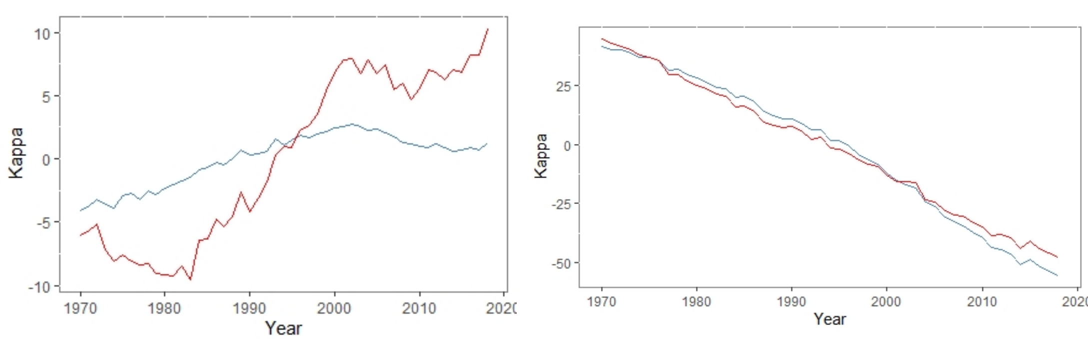

Figure 9: On the left: The Li-Lee estimates of κ_x^{NL} for both male (blue) and female (red) of the Dutch population from 1970 up to and including 2018. On the right: The Li-Lee estimates of κ_x^{EU} for both male (blue) and female (red) of the European countries from 1970 up to and including 2018.

4.2.3 Forecasting Mortality

As mentioned before, $\beta_{x,g}^{EU}\kappa_{g,t}^{EU}$ is the common long term trend that all countries share. This means that the country specific trend, i.e. Dutch trend, should converge to this common trend in the long run. Li and Lee (2005) chose that the random walk with drift should be used for the forecasting of the whole group, i.e. all of the selected EU countries, since it is known for its simplicity and straightforwardness when interpreting. Hence we have the following for the European trend:

$$\kappa_{g,t}^{EU} = \kappa_{g,t-1}^{EU} + \theta_g + \epsilon_{g,t}, \tag{4.30}$$

where θ_g is the drift and $\epsilon_{g,t}$ the error term.

As for $\kappa_{g,t}^{NL}$, according to Li and Lee (2005), it should converge to some constant level over time. This way, this approach can be successful. If it doesn't converge, and thus the differences remain in the long run, it will lead to divergence in the forecasts. Now for this approach to be successful, the random walk with drift cannot be assumed for $\kappa_{g,t}^{NL}$, since it will generate a trending long-term mean. To get it to tend to a constant, the $\kappa_{g,t}^{NL}$ are assumed to follow a first-order autoregressive AR(1) model without the constant term:

$$\kappa_{g,t}^{NL} = \rho_g \kappa_{g,t-1}^{NL} + \delta_{g,t}, \tag{4.31}$$

where ρ_g is a coefficient and $\delta_{g,t}$ refers to the error term.

Both time series process, i.e. for EU and NL, are jointly estimated for both genders using the SUR regression, which in turn generates the correlated error terms as follows:

$$\begin{pmatrix} \epsilon_{M,t} \\ \epsilon_{F,t} \\ \delta_{M,t} \\ \delta_{F,t} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tilde{C} \right), \tag{4.32}$$

where $\tilde{C}$ is the covariance matrix. The error terms $(\epsilon_{M,t}, \epsilon_{F,t}, \delta_{M,t}, \delta_{F,t})$ are assumed to be independent and identically distributed following the multi dimensional distribution given in equation (4.32). When simulating the trend processes in the future, error terms are drawn from the covariance matrix $\tilde{C}$. The future trend process for the European countries is estimated in the same way as equation (4.12), i.e.

$$\hat{\kappa}_{g,t_n+k}^{EU} = \kappa_{g,t_n}^{EU} + k\hat{\theta}_g + \sum_{j=1}^{k} \epsilon_{g,k}. \tag{4.33}$$

The future trend process for the Dutch deviation on the other hand, is estimated using:

$$\kappa_{g,t_n+k}^{NL} = \rho_g^k \kappa_{g,t_n}^{NL} + \sum_{j=1}^{k} \delta_{g,k}, \tag{4.34}$$

To obtain for the point forecast for the trend parameters, i.e. the best estimate of the future trend processes, on the horizon k, we again take the conditional expectation of the estimates of the future trend processes:

$$\hat{\kappa}_{g,t_n+k}^{EU,BE} = \mathbb{E}[\hat{\kappa}_{g,t_n+k}^{EU} | \kappa_{g,t_1}^{EU}, \kappa_{g,t_2}^{EU}, \ldots, \kappa_{g,t_n}^{EU}] = \kappa_{g,t_n}^{EU} + k\hat{\theta}_g, \tag{4.35}$$

$$\hat{\kappa}_{g,t_n+k}^{NL,BE} = \mathbb{E}[\hat{\kappa}_{g,t_n+k}^{NL} | \kappa_{g,t_1}^{NL}, \kappa_{g,t_2}^{NL}, \ldots, \kappa_{g,t_n}^{NL}] = \hat{\rho}_g^k \kappa_{g,t_n}^{NL}. \tag{4.36}$$

Next, the prediction interval for both trend processes is constructed. Again, it is constructed in a way to include both stochastic uncertainty and parameter uncertainty. For the European trend process, it is similar to equation (4.17), i.e.

$$\hat{\kappa}_{g,t_n+k}^{EU(m)} = \kappa_{g,t_n}^{EU} + k\theta_g^{(m)} + \sum_{j=1}^{k} \epsilon_{g,j}^{(m)}, \tag{4.37}$$

where the drift is assumed to have a normal distribution with the drift estimate as the mean and standard error of the drift estimate as the volatility, i.e. $\theta_g \sim N\left(\hat{\theta}_g, se(\hat{\theta}_g)\right)$ and $m \in \{1, \ldots, M\}$, where M is the number of simulations. As for the prediction interval for the Dutch deviation, we have:

$$\hat{\kappa}^{NL(m)}_{g,t_n+k} = \rho^k_{g,(m)}\kappa^{NL}_{g,t_n} + \sum_{j=1}^{k} \delta^{(m)}_{g,j}, \tag{4.38}$$

where the coefficient ρ_g is assumed to have a normal distribution with the coefficient estimate as mean and the standard error of the estimate of the coefficient, which is also obtained in the SUR regression, as the volatility. In other words, we have $\rho_g \sim N\left(\hat{\rho}_g, se(\hat{\rho}_g)\right)$. One problem that occurs when simulating parameter uncertainty this way is some of the simulated coefficients will be larger or equal to one, i.e. $\rho \geq 1$, which in turn leads to divergence. To avoid divergence, every simulated coefficient that is larger or equal to one will be changed to the coefficient estimate $\hat{\rho}_g$. In figure 10, the best estimate together with a 95% prediction interval is shown for both $\hat{\kappa}^{NL}$ and $\hat{\kappa}^{EU}$. The prediction interval includes both stochastic errors and parameter uncertainty.

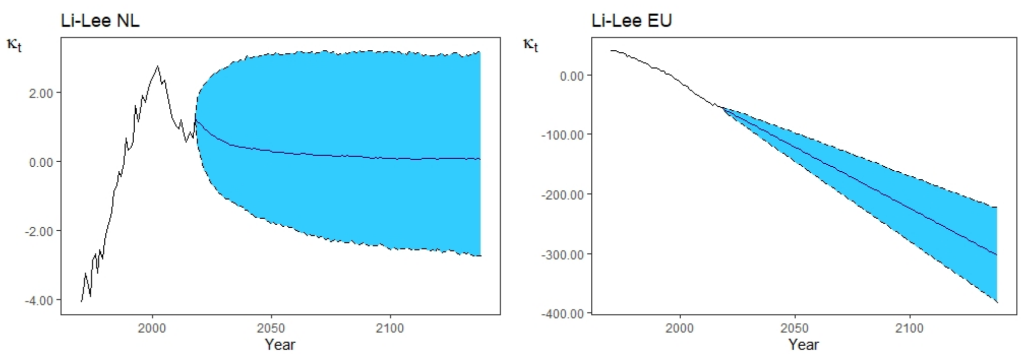

Figure 10: On the left: The Li-Lee simulations of κ^{NL}_x for the male population from 2018 up to and including 2138, i.e. N=120. The best estimate with a 95% prediction interval is shown. On the right: The Li-Lee simulations of κ^{EU}_x for the European male population. The best estimate and a 95% prediction interval can be seen. The number of simulations used is $M = 1000$. The prediction interval consists of parameter uncertainty and stochastic errors.

Now, the next step is to obtain the best estimate future Dutch log mortality rates. This is done as follows:

$$\log \hat{\mu}^{EU,BE}_{x,g,t_n+k} = \hat{\alpha}^{EU}_{x,g} + \hat{\beta}^{EU}_{x,g}\hat{\kappa}^{EU,BE}_{g,t_n+k} \tag{4.39}$$

$$\log \hat{\tilde{\mu}}^{NL,BE}_{x,g,t_n+k} = \hat{\alpha}^{NL}_{x,g} + \hat{\beta}^{NL}_{x,g}\hat{\kappa}^{NL,BE}_{g,t_n+k}, \tag{4.40}$$

which ultimately gives us

$$\log\ \hat{\mu}^{NL,BE}_{x,g,t_n+k} = \log\ \hat{\mu}^{EU,BE}_{x,g,t_n+k} + \log\ \hat{\tilde{\mu}}^{NL,BE}_{x,g,t_n+k}. \tag{4.41}$$

The Kannistö's closing method is also applied in this case to $\log\ \hat{\mu}^{NL,BE}_{x,g,t_n+k}$.

5 Longevity Risk

In the previous chapter, both the Lee-Carter model and the Li-Lee model had been introduced. Future mortality rates can be simulated by both models, and we have also shown the uncertainty that dwells around the best estimate. This uncertainty is referred to as longevity trend risk. Longevity risk in itself refers the uncertainty in mortality projections which is caused by long term deviations from the deterministic mortality projection. These deviations can be due to changes in the level, trend or volatility in mortality rates. As was introduced previously, longevity risk can be distinguished into micro-longevity risk and macro-longevity risk. Micro-longevity risk is the idiosyncratic risk that can be fully diversified by increasing the number of participants in a pension fund. Macro-longevity risk is the non-diversifiable systematic risk. According to Broeders et al. (2018) there are three sources of macro-longevity risk. These consists of stochastic variation, which is the random variation in aggregate observed number of deaths, parameter risk, which refers to the uncertainty around the true value of the model parameters, and lastly, model risk, which refers to the uncertainty that the chosen stochastic model is actually correct. On the other hand, Richards et al. (2013) considers several components more of longevity risk, which are also non-diversifiable, i.e. basis risk and trend risk. Basis risk arises because models are usually calibrated to population data and not the data of the portfolio that is considered. So, there is the risk that the mortality trend of the participants in the pension fund is different from that of the population used for the model calibration. Trend risk arises due to the fact that the trend estimate that we would have for, say next year, could differ from the current estimated trend. Longevity trend risk will be the main focus of the analysis done in this chapter.

Now that the simulated mortality rates have been acquired from the previous chapter, we can compute the amount of capital needed for insurers to cover this risk. In section 5.1, the Solvency II framework will be discussed and how the solvency capital requirement has to be computed under Solvency II. In section 5.2, the three shock methodologies will be discussed and compared. And lastly, the results of the three shock methodologies are discussed in section 5.3.

5.1 Solvency II

Solvency II is a regulatory framework for insurance companies in Europe. It consists of three pillars that need to be satisfied. The first pillar covers capital requirements, the second covers corporate governance and the third one covers disclosure and transparency. For the calculation of capital requirements, there are two levels at which capital is measured: The Minimum Capital Requirements (MCR) and the Solvency Capital Requirements (SCR). The SCR is the target capital (in excess of their total capital) which an insurance company should have to cover risk. It is used to determine if a company has enough capital to cover all losses over the course of one year with probability 99.5%. The MCR on the other hand represents the threshold below which a national regulatory agency would have to intervene.

It is preferable for insurance companies to implement internal models for their risk assessment, since it values risks that are company specific. So this leads to better accuracy in their risk assessment. However, as mentioned before, it is quite costly if an insurance company chooses to implement an internal model. For this reason, the European Commission with support of the Committee of Insurance and Occupational Pension Supervisors (CEIOPS) has allowed another way for the approximation of the capital requirements, namely the standard formula. The standard formula is a scenario based standard model, where the overall risk split into several categories or modules. These consist of are market risk, credit risk, non-life underwriting risk, life underwriting risk and operational risk. For each of these risks, the SCR needs to be computed individually. Then, the SCRs are aggregated. Mostly small to medium sized companies make use of the standard formula. Some larger companies also make use of the standard formula for a couple of the modules and thus implementing a partial internal model. Longevity risk falls under life underwriting risk. For each of the modules, it is required that an insurance company is able to cover their liabilities with a certainty of 99.5% over the course of one year. For longevity risk, the SCR is computed as the change in liabilities by assuming a shock that permanently reduces the mortality rates by 20%. In CEIOPS(2008), it has been argued by QIS4 participants who have not incorporated the standard formula that the reason they chose to do so is because the way the longevity stress is calculated with the standard formula failed to appropriately reflect the actual longevity risk. To be more specific, the standard formula does not allow for the risk of increases in the future mortality developments, i.e. longevity trend risk. This means that if a company chooses to implement the standard formula, it could lead to unnecessarily high capitalization of insurance companies if the current longevity shock overstates longevity risk. If the opposite happens, i.e. in case of underestimation of longevity risk, it could lead to a higher default probability than 0.5%. Since the standard formula is structured under solvency II to cover more than just longevity risk, it is expected to produce higher capital requirement. In the next section, the standard formula will be compared to two stochastic models using VaR frameworks as possible internal models. First, we will provide the SCR definition following Börger (2010).

First, let AC_t be the company's available capital at the end of year t, which is the difference between the market value of the company's assets, A_t, and the market value of the company's liabilities, L_t, at time t, i.e.

$$AC_t = A_t - L_t \tag{5.1}$$

This amount of capital could be used to cover future losses. Unlike the market value of the assets which is easily determined, the market value of the liabilities is much more complex to obtain. This is due to the fact that there is no liquid market for liabilities and also due to options and guarantees that are in the insurance contracts. Solvency II proposes to estimate the value of liabilities by the so-called Technical Provisions, which is composed of the Best Estimate Liabilities (BEL) and a Risk Margin (RM). At the end of year t, we have:

$$L_t = BEL_t + RM_t. \tag{5.2}$$

The (BEL) is the expected present value of the liabilities of the portfolio conditional on all survival rates up to time t and the value of these liabilities depend on stochastic future payments i.e.

$$BEL_t = \sum_{s\geq 1} \mathbb{E}[\tilde{L}_{t+s}|\mathcal{F}_t] \cdot P_t^{(s)}, \tag{5.3}$$

where $\tilde{L}_{t+s}$ is the liability payment at time $t+s$, $P_t^{(s)}$ denotes the discount factor for payments at time $t+s$ discounted to time t and $\mathcal{F}_t$ denotes the information set which contains the survival rates up to time t, i.e. $\mathcal{F}_t = \{p_{x+s,t+s}|\ \forall s\}$.

The risk margin needs to be added to the best estimate since it represents the non-hedgeable risks which the insurance company bears. These are risks that can not be fully hedged with instruments traded in an active market. To value these non-hedgeable risks, Solvency II prescribes that the risk margin should be calculated using the cost of capital approach. More specifically the capital base in Solvency II is the Solvency Capital Requirement (SCR) regarding the non-hedgeable risks. The cost of capital rate (CoC), which is an additional rate above the risk-free interest rate, reflects the return which an investor should receive if he were to purchase the liabilities over. The cost of capital rate is set according to the Solvency II standard model calibration, which is a fixed rate of 6%. The risk margin (RM) is then calculated as follows:

$$RM_t = \sum_{s\geq 1} \frac{CoC * SCR_{t+s}}{(1+r_{t+s}^f)^{t+s}}, \tag{5.4}$$

where CoC is the cost of capital rate, r_{t+s}^f is the risk-free rate at time t with maturity $t+s$ and SCR_t is the SCR at time t. As was introduced before, the SCR is the 99.5% VaR of the available capital over a one year time span. So, under Solvency II a insurance company is considered solvent if at time $t = 1$ the present value of the available capital is positive with a probability of at least 99.5 %. In other words, the smallest amount of x for which the following condition holds:

$$\mathbb{P}(AC_1 > 0 | AC_0 = x) \geq 99.5\%, \tag{5.5}$$

where AC_0 is the available capital at time $t = 0$ and AC_1 is the available capital after one year, i.e. at time $t = 1$. However, in practice Bauer et al.(2010) shows that a simpler but roughly equivalent notion of the SCR is used:

$$SCR = min\left\{x|P\left(AC_0 - \frac{AC_1}{1+r_1^f} \leq x\right) \geq 0.995\right\}. \tag{5.6}$$

Let the difference between the current available capital ($t = 0$) and the available capital one year from now discounted to $t = 0$, i.e. $AC_0 - \frac{AC_1}{1+r_1^f}$, be the one-year loss function observed at time $t = 0$. The probability that this loss over one year is less than or equal to the SCR has to be greater or equal to 99.5 %. As one can observe from the definitions, the SCR and the available capital have a mutual dependence. This leads to a loop in calculating the SCR and the available capital, since the SCR is calculated as the VaR of the available capital and in turn the available capital depends on the SCR through the risk margin. To obtain a solution for this, CEIOPS (2009) suggests that a risk margin should not be taken into account in the liabilities when calculating the SCR. This applies when a life underwriting risk stress is based on the change in value of assets minus liabilities. In conclusion, the assumption that the risk margin does not change in stress scenarios holds and thus the the AC can be estimated by the change in Net Asset Value (NAV):

$$NAV_t = A_t - BEL_t. \tag{5.7}$$

Hence, the SCR for longevity risk is

$$SCR^{VaR} = min\left\{x|\ P\left(NAV_0 - \frac{NAV_1}{1+r_1^f} \leq x\right) \geq 0.995\right\}. \tag{5.8}$$

In the case of the Solvency II standard formula, the SCR of longevity risk is calculated in a simplified way that is an approximation of SCR^{VaR}, i.e.

$$SCR^{SF} = NAV_0 - (NAV_0|\ longevity\ shock). \tag{5.9}$$

Now that we have a general idea of how the SCR should be calculated under Solvency II, we can compute the $SCRs$ for the different shock methodologies.

5.2 Shock Methodologies: VaR Approach

In the previous section, both the standard formula and the internal model are discussed and how the SCR is defined under Solvency II. If an insurance company develops an internal model, they should to base their shock model on a one-year 99.5% value-at-risk. Richards et al. (2013) developed a one-year VaR methodology that can be used for the construction of an internal model. However, he still argues that it is not appropriate to measure longevity risk over a one year horizon, since longevity risk lies in the long-term trend taken by mortality rates. So in this section, we will also introduce a terminal VaR methodology. In this section we will discuss and compare the three shock methodologies, i.e. the one-year VaR approach, the standard formula and the terminal VaR approach, for both the Lee-Carter model and the Li-Lee model. In section 5.2.1, the data of the pension fund as well as the portfolio setup is discussed. In section 5.2.2, an expression of the best estimate of the liabilities is derived for the chosen portfolio. In section section 5.2.3, the one-year VaR approach is discussed and in 5.2.5, the terminal VaR approach is discussed.

5.2.1 Data and Portfolio Setup

In order to measure the effects of the shocks due to longevity risk, we consider a model portfolio, which includes two annuity funds. The first fund, fund Green consist of mostly younger individuals that have yet to retire, while the second fund, fund Grey, consists of mostly older individuals that have already retired. This can be observed in figure 11. Both funds consist of 45% male and 55% female. In both funds, there are no new entrants into the funds. Also, no premiums are paid after time $t = 0$. All individuals enter at the age of $x = 20$ and the maximum age in the fund is $x = 100$. The term structure of interest rate is assumed to be flat and deterministic in this case. The retirement age is set to age 67.

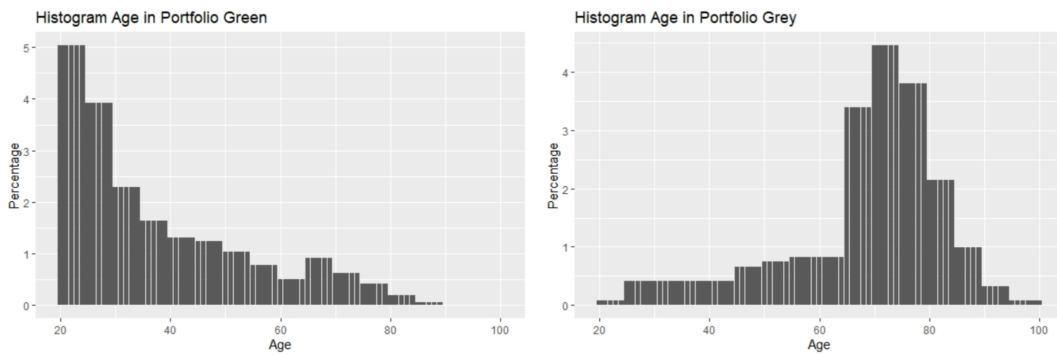

Figure 11: Fund Green has a high percentage of people in their twenties and thirties, while fund Grey has a high percentage of people that are over the age of 60.

The salary W_x of the individuals with the youngest age, i.e. $x = 20$, is $W_{20} = 20,000$ per year, while the salary of the individuals aged $x = 66$, i.e. individuals that are almost at the retirement age, is $W_{66} = 41,957$ per year. Throughout the working lifetime, which is assumed to be from age 20 to 66, of these individuals, a certain amount of pension rights will be build up. Then, when an individual reaches the retirement age, he or she will receive a benefit as a percentage of their accumulated salary of their working years. The annual rate of accrual of pension rights is 1.75%. Assuming that the salary increases with age, an individual aged x has built up

$$B_x = \sum_{i=20}^{\min\{x,66\}} 1.75\% \cdot W_i. \tag{5.10}$$

As one can observe, the salary and the pension benefits are gender neutral, i.e. it is the same for both male and female. Note that for individuals that have reached the retirement age, i.e. individuals aged $x \geq 67$, the benefit B_x will remain the same until death. In other words, for individuals older that age 66, we can denote B_x as B. The contracts which are considered are pension annuities, where annual payments are done while the pensioner is still alive. Now that we have a general idea of the model portfolio, we can continue with the estimation of the future liabilities.

5.2.2 Best Estimate Liabilities

The best estimate of the liabilities (BEL) introduced in section 5.1 is the expected present value of the liabilities given the survival rates up until time t, i.e.

$$BEL_t = \sum_{s \geq 1} \mathbb{E}[\tilde{L}_{t+s} | \mathcal{F}_t] \cdot P_t^{(s)}, \tag{5.11}$$

where $P_t^{(s)}$ denotes the discount factor for payments at time $t+s$ discounted to time t, which is a function of the interest rate i:

$$P_t^{(s)} = \left(\frac{1}{1+i}\right)^{t+s} \tag{5.12}$$

As mentioned before in the previous section, we consider pension annuity contracts, which pay a specific amount B for an individual aged $x \geq 67$ at time t on an annual basis. The first payment to the insured happens at the beginning of the following year after the individual has reached the retirement age, $x = 67$. The last payment is at the beginning of the year in which the insured dies. So in other words, the amount of future payments depends on the remaining lifetime of an individual.

Let $T_{x,t}^{(g)}$ be the remaining lifetime of an individual aged x at time t with gender g and let ${}_\tau p_{x,t}^{(g)}$ be the probability that a person aged x at time t and gender g will survive another τ years, i.e.:

$${}_\tau p_{x,t}^{(g)} = p_{x,t}^{(g)} \cdot p_{x+1,t+1}^{(g)} \cdot \dots \cdot p_{x+\tau-1,t+\tau-1}^{(g)}, \tag{5.13}$$

where the one-year survival probability $p_{x,t}^{(g)}$ is obtained using equation (3.1), i.e. $p_{x,t}^{(g)} = 1 - q_{x,t}^{(g)}$. The one-year death probability $q_{x,t}^{(g)}$ is assumed to have the following relationship with the force of mortality:

$$q_{x,t}^{(g)} = 1 - \exp(-\mu_{x,t}^{(g)}). \tag{5.14}$$

Let $\tilde{\mathcal{F}}_t$ be the information set containing all future survival probabilities as of time t, i.e. $\tilde{\mathcal{F}}_t = \{p_{x+1,t+1}^{(g)}, p_{x+2,t+2}^{(g)}, ..., p_{x+\tau,t+\tau}^{(g)}\} = \{p_{x+\tau,t+\tau}^{(g)} | \forall \tau\}$. The expected remaining lifetime at time t of a person aged x and gender g at time t given all future survival probabilities as of time t is

$$\mathbb{E}[T_{x,t}^{(g)} | \tilde{\mathcal{F}}_t] = \sum_{\tau=0}^{MaxAge-x} {}_\tau p_{x,t}^{(g)} \tag{5.15}$$

Next, let $V_{x\geq 67,t}^{(g)}$ be the present value of the payments at time t for a person who is aged 67 or older at time t with gender g, i.e.

$$V_{x\geq 67,t}^{(g)} = B \cdot P_t + B \cdot P_t^{(1)} + B \cdot P_t^{(2)} + ... + B \cdot P_t^{(T_{x,t}^{(g)})}, \tag{5.16}$$

where $P_t^{(s)}$ is the aforementioned discount factor for payments at time $t+s$ discounted to time t. The expected present value of the future payments at time t conditional on the survival rates up until time $t+\tau$ for a person who is aged 67 or older that will survive another τ years with gender g is :

$$\mathbb{E}[V_{x\geq 67,t}^{(g)} | \tilde{\mathcal{F}}_t] = \sum_{\tau=0}^{MaxAge-x} {}_\tau p_{x,t}^{(g)} \cdot B \cdot P_t^{(\tau)}. \tag{5.17}$$

In the case where a person is younger than the age 67, the payments start in $67 - x$ years. So the present value will depend on whether the person reaches the retirement age or not. The expected present value conditional on the survival rates up until $t + \tau$ for a person younger than age 67 at time t that will survive another τ years with gender g is

$$\mathbb{E}[V_{x<67,t}^{(g)} | \tilde{\mathcal{F}}_t] = \sum_{\tau=67-x}^{MaxAge-x} {}_\tau p_{x,t}^{(g)} \cdot B \cdot P_t^{(\tau)}. \tag{5.18}$$

The expected present value of the payments for a person aged x at time t can be written as

$$\mathbb{E}[V_{x,t}^{(g)} | \tilde{\mathcal{F}}_t] = \sum_{\tau=\max\{67-x,0\}}^{MaxAge-x} {}_\tau p_{x,t}^{(g)} \cdot B_x \cdot P_t^{(\tau)}. \tag{5.19}$$

With this we have defined the age-specific expected present value of the future payments for both genders, or in other words, the expected present value of the liability payments per individual aged x with gender g. So for the valuation of the expected present value of liabilities in each fund, we need to take the sum over the total amount of participants in each fund. Let $I_{x,t}^{(g,j)}$ be the number of participants with gender g, aged x at time t in fund $j \in \{green, grey\}$. Then we have the following for the expected present value of the liabilities per fund:

$$\mathbb{E}[L_{t+\tau}^{(j)}|\tilde{\mathcal{F}}_t] = \sum_{g,x} \mathbb{E}[V_{x,t}^{(g)}|\tilde{\mathcal{F}}_t] \cdot I_{x,t}^{(g,j)}. \tag{5.20}$$

Then the total expected present value of the portfolio is obtained by adding the expected present value of both funds.

$$\mathbb{E}[L_{t+\tau}|\tilde{\mathcal{F}}_t] = \sum_{j,g,x} \mathbb{E}[V_{x,t}^{(g)}|\tilde{\mathcal{F}}_t] \cdot I_{x,t}^{(g,j)}. \tag{5.21}$$

For the survival rates, which are used to determine the best estimate of liabilities, we use the best estimate of the force of mortality, which we obtained in the previous sections for the Dutch population through simulation using the Lee-Carter model and the Li-Lee model. So the best estimate force of mortality of the Dutch population $\hat{\mu}_{x,g,t_n+k}^{NL,BE}$ is plugged in equation (5.14), such that we obtain the best estimate one-year death probabilities for person aged x and gender g at time t_n+k, i.e. $\hat{q}_{g,x,t_n+k}^{BE}$, for $\forall k$. Plugging this in equation (3.1), we get the best estimate one-year survival probabilities $\hat{p}_{g,x,t_n+k}^{BE}$, which in turn we use to get the k-year survival rates ${}_k\hat{p}_{g,x,t}^{BE}$ with $t = t_n$ by using equation (5.13). Hence the best estimate of the liabilities is calculated by

$$BEL_t = \sum_{j,g,x} \sum_{k=\max(67-x,0)}^{100-x} {}_k\hat{p}_{g,x,t}^{BE} \cdot B \cdot P_t^{(k)} \cdot I_{x,t}^{(g,j)}. \tag{5.22}$$

The results of the best estimate of the liabilities for each fund is shown in table 1. For both funds, the Li-Lee model gives a larger BEL compared to the Lee-Carter model. In other words, the Lee-Carter underestimates the present value of the liabilities given that the Li-Lee model models the Dutch mortality rates more accurately.

Table 1: Best Estimate of the Liabilities of fund Green and fund Grey.

Present Value Liabilities	Fund Green	Fund Grey
BEL LC	1,277,280,106	1,489,449,331
BEL LL	1,339,527,204	1,529,623,646

5.2.3 One-Year VaR Approach

In this section, longevity risk will be modeled using a one-year VaR framework. Richards et al. (2013) introduces a method for the one-year VaR approach, where the main idea is to simulate a single scenario of the stochastic mortality rates in the first projected year. This is then considered a new data point, which in turn is used to refit the model. Then, the new best estimate of mortality rates is estimated for the rest of the projected years. In other words, after including the new data point, the best estimate is observed to see how it is affected. After obtaining the new best estimate, the present value of the liabilities for this single simulation is calibrated. This process is repeated M times, where M is the number of simulations. The VaR is then the 99.5th quantile of the simulated liabilities.

This method is applicable to a wide range of stochastic projection models. First, the method will be applied to the Lee-Carter model, where the trend process follows a random walk with drift. After fitting the model, one simulation of the trend process is projected in the following year. The simulation contains both stochastic and parameter uncertainty, i.e.

$$\hat{\kappa}^{LC}_{g,t_n+1} = \kappa^{LC}_{g,t_n} + \theta_g + e_{g,t_n+1}, \tag{5.23}$$

with θ_g drawn from $N(\hat{\theta}_g, se(\hat{\theta}_g))$ and $e_{g,1}$ drawn from $N(0, \hat{C})$, as in the previous section. With $\hat{\kappa}^{LC}_{g,t_n+1}$, the stochastic mortality rates $\hat{\mu}^{LC}_{g,x,t_n+1}$ are computed for the first projected year. After obtaining the simulated mortality rates of the first projected year, the corresponding one-year death probabilities $\hat{q}^{(g)}_{x,t_n+1}$ are computed. Next, the number of deaths $D^{(g)}_{x,tn+1}$ is simulated as a binomial random variable, i.e.

$$D^{(g)}_{x,tn+1} \sim \text{Bin}(E^{(g)}_{x,t_n}, \hat{q}^{(g)}_{x,t_n+1}),$$

for each age x. For simplicity, the subscript NL is removed from the notation for the number of deaths and exposure. To simulate the number of deaths for the projected year, it is assumed that the population exposure is the same in year t_n+1 as in year t_n. If the simulated number of deaths equates to zero, then it will be changed to the expected number of deaths, i.e. $E^{(g)}_{x,t_n} \cdot \hat{q}^{(g)}_{x,t_n+1}$, since this is essential for the Newton-Raphson algorithm to work. The next step is to add the simulated data to the real current data set, creating a simulated data set we would have one year from now. Then the model is refitted to the simulated data set. After the refit, we obtain the new common parameter estimates $\hat{\hat{\alpha}}_{g,x}$, $\hat{\hat{\beta}}_{g,x}$ and $\hat{\hat{\kappa}}_{g,t_n+1}$. Next, the SUR regression is applied to the trend processes, which estimates the new drift terms $\hat{\hat{\theta}}_g$. Then when forecasting the trend process at time t_n+1, we have

$$\hat{\tilde{\kappa}}_{g,t_n+1+k} = \hat{\tilde{\kappa}}_{g,t_n+1} + k\tilde{\theta}_g + \sum_{j=1}^{k} \tilde{e}_{g,t_n+1+j}, \tag{5.24}$$

where $\tilde{e}_{g,j} \sim N(0, \hat{\hat{C}})$ and $\tilde{\theta}_g \sim N(\hat{\hat{\theta}}_g, se(\hat{\hat{\theta}}_g))$, with $\hat{\hat{C}}$ the new co-variance matrix estimate obtained from the SUR regression and $\hat{\hat{\theta}}_g$ the new drift estimate also obtained from the same SUR regression. However, the rest of the projection is based on the best estimate, i.e.

$$\hat{\hat{\kappa}}^{BE}_{g,t_n+1+k} = \hat{\hat{\kappa}}_{g,t_n+1} + k\hat{\hat{\theta}}_g. \tag{5.25}$$

Finally, the new best estimate mortality rates are simulated as of year t_n+1, i.e.

$$\log \hat{\hat{\mu}}^{BE}_{g,x,t_n+N} = \hat{\hat{\alpha}}_{g,x} + \hat{\hat{\beta}}_{g,x}\hat{\hat{\kappa}}^{BE}_{g,t_n+N}. \tag{5.26}$$

With this, the one-year survival probabilities and thus the τ-year survival probabilities ${}_\tau\hat{\hat{p}}^{BE}_{g,x,t}$ can be computed. Using this we can compute the best estimate of the liabilities as of $t+1$, with $t = t_n$:

$$BEL_{t+1} = \sum_{j,g,x} \sum_{\tau=\max(67-x-1,0)}^{100-x} {}_\tau\hat{\hat{p}}^{BE}_{g,x+1,t+1} \cdot B \cdot P^{(\tau)}_{t+1} \cdot I^{(g,j)}_{x,t+1}, \tag{5.27}$$

where the number of participants aged x in the fund at time $t+1$ is binomially distributed conditional on the survival probabilities in year t, i.e. $I^{(g,j)}_{x,t+1}|\hat{p}^{(g)}_{x,t} \sim \text{Bin}(\hat{p}^{(g)}_{x,t}, I^{(g,j)}_{x,t})$. In other words, in one year not everyone in the fund will survive. Hence, $I^{(g,j)}_{x,t+1} \sim \mathbb{E}[I^{(g,j)}_{x,t+1}|\hat{p}^{(g)}_{x,t}] = \hat{p}^{(g)}_{x,t_n} \cdot I^{(g,j)}_{x,t}$.

Next, the present value of the liabilities need to be computed. In this case, there are two factors that play a role in the computation of the present value of the liabilities, namely the cash outflow due to payments of death in year $t+1$ and the best estimate of the liabilities at time $t+1$. There is payment only if the individual is retired and alive, hence the cash outflow $CF_{t,t+1}$ at time $t+1$ discounted to time t is

$$CF_{t,t+1} = \mathbb{1}_{\{x\geq 67\}} \sum_{g,j,x} \hat{p}^{(g)}_{x,t} \cdot B_{x,t} \cdot P^{(1)}_t \cdot I^{(g,j)}_{x,t}. \tag{5.28}$$

The present value of the liabilities at time t is as follows:

$$L_t^{shock} = \frac{1}{1+r}\left(CF_{t,t+1} + BEL_{t+1}\right). \tag{5.29}$$

This whole process is counted as one simulation. To obtain the one-year VaR, this whole process is repeated M times. After repeating this process M times and obtaining M simulations of the present value of the liabilities, the 99.5th quantile, i.e. $Q_{0.995}(\vec{L}_t)$, where $\vec{L}_t$ is the vector with the simulated present value of the liabilities, is taken to get the 99.5% one-year value-at-risk.

Next, the SCR needs to be calculated. As stated in section 5.1, the SCR of the one-year VaR is computed according to equation 5.8, i.e.

$$SCR^{VaR} = min\left\{x|\ P\left(NAV_0 - \frac{NAV_1}{1+r} \leq x\right) \geq 0.995\right\},$$

where $NAV_t = A_t - BEL_t$. Assuming assets pay out a return r, we have the following at time $t+1$,

$$A_{t+1} = A_t(1+r) - CF_{t,t+1}. \tag{5.30}$$

In this case we are only considering the cash outflow, hence why the cash flow is subtracted. Now this implies

$$\begin{aligned} NAV_0 - \frac{NAV_1}{1+r} &= A_0 - BEL_0 - \frac{A_1 - BEL_1}{1+r} \\ &= A_0 - BEL_0 - \frac{A_0(1+r) - CF_{0,1} - BEL_1}{1+r} \\ &= \frac{BEL_1 + CF_{0,1}}{1+r} - BEL_0 \\ &= L_0^{shock} - BEL_0 \end{aligned} \tag{5.31}$$

which in turn leads to the SCR formula:

$$SCR^{VaR} = min\left\{x|\ P\left(L_0^{shock} - BEL_0 \leq x\right) \geq 0.995\right\}. \tag{5.32}$$

Hence, SCR can be obtained by first taking the differences between the M simulated present value of the liabilities and the best estimate of the liabilities at time $t=0$, i.e. $L_0^{(m)} - BEL_0$, and then taking the 99.5th quantile of the differences between the simulated present value of the liabilities and the best estimate of the liabilities at time $t=0$. This is equivalent to subtracting the best estimate of the liabilities at time $t=0$ from the 99.5 % VaR, i.e. $Q_{99.5}(L_t) - BEL_0$, since BEL_0 is deterministic. In algorithm 1, an overview of the one-year VaR methodology is given.

In figure 12, the simulations of the trend processes of the Dutch male population are shown when applying the one-year VaR framework to the Lee-Carter model. One can observe how the best estimate is affected after a new data point is simulated and added to the data set and then refitted.

Algorithm 1 One-year VaR algorithm: Lee-Carter

- For $j = 1, ..., M$
  - Load/select the observed deaths $D_{x,t}^{(g)}$ and population exposures $E_{x,t}^{(g)}$ for $t = t_1, ..., t_n$.
  - Fit data to the Lee-Carter model, obtaining estimates for $\hat{\mu}_{x,g,t}^{LC}$.
  - Run SUR regression to obtain estimates of θ_g and the co-variance matrix.
  - Simulate $\hat{\kappa}_{g,t_n+1}^{LC(j)} = \kappa_{g,t_n}^{LC} + \theta_g^{(j)} + e_{g,t_n+1}^{(j)}$.
  - Compute log $\hat{\mu}_{x,g,t_n+1}^{LC(j)} = \hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}_{g,t_n+1}^{LC(j)}$.
  - Compute $\hat{q}_{x,t_n+1}^{(g,j)} = 1 - \exp(-\hat{\mu}_{x,g,t_n+1}^{LC(j)})$.
  - Simulate $D_{x,t_n+1}^{(g,j)} \sim \text{Bin}(E_{x,t_n}^{(g)}, \hat{q}_{x,t_n+1}^{(g,j)})$.
  - Append the simulated mortality rates to the current data set.
  - Refit the model to the newly appended data set, obtaining $\hat{\hat{\alpha}}_{g,x}$,$\hat{\hat{\beta}}_{g,x}$ and $\hat{\hat{\kappa}}_{g,t_n+1}$.
  - Compute the best estimate of the future trend process $\hat{\hat{\kappa}}_{g,t_n+1+k}^{BE} = \hat{\hat{\kappa}}_{g,t_n+1} + k\hat{\hat{\theta}}_g$ for $k = 1, ..., N$.
  - Compute log $\hat{\hat{\mu}}_{g,x,t_n+1+k}^{BE} = \hat{\hat{\alpha}}_{g,x} + \hat{\hat{\beta}}_{g,x} \hat{\hat{\kappa}}_{g,t_n+1+k}^{BE}$ for $k = 1, ..., N$.
  - Compute $\hat{\hat{p}}_{g,x+1,t_n+1+k}^{BE} = \exp(-\hat{\hat{\mu}}_{g,x,t_n+1+k}^{BE})$ for $k = 1, .., N$.
  - Forecast the $BEL_{t+1}^{(j)}$.
  - Compute the present value of the liabilities $L_t^{(j)} = \frac{1}{1+r}(BEL_{t+1}^{(j)} + CF_{t,t+1}^{(j)})$.
- Take 99.5th quantile of the M present values of the liabilities, i.e. $L_t^{shock} = Q_{99.5}(\vec{L_t'})$.
- Compute $SCR = L_t^{shock} - BEL_t$

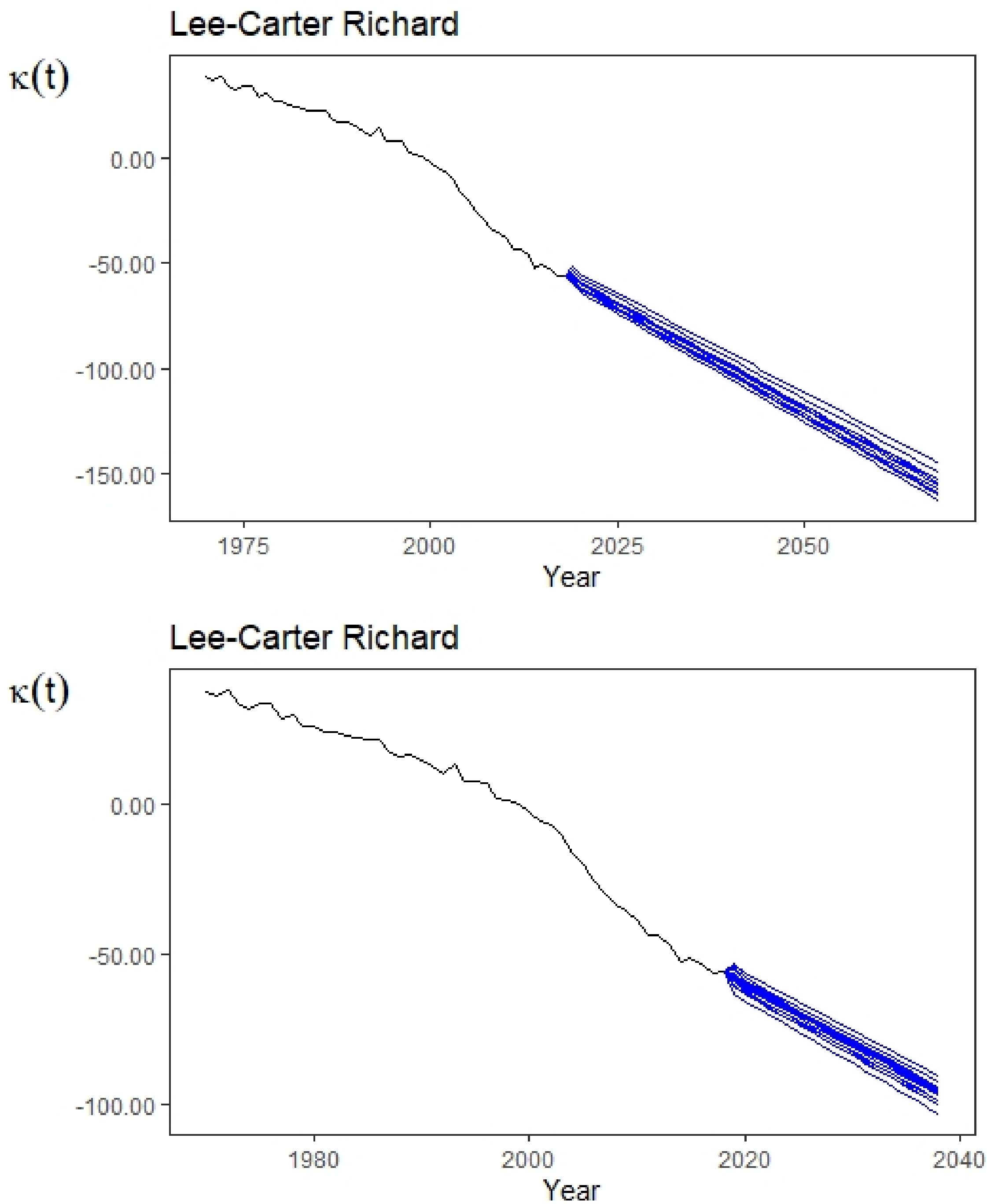

Figure 12: Simulations of the trend processes for the Dutch Male population using the one-year approach. For both figure, $M = 20$ simulations are shown. In the first projected year, i.e. $t = 2019$, a data point is generated. The model is then refitted every time there is a simulated data point in year $t = 2019$. For each simulation, the change in the best estimate can be seen and how it can be affected over the course of one year. The top figure shows us the newly simulated best estimate 50 years in the future, while the bottom figure has a horizon of 20 years in the future. This way the one-year simulations are better observed.

Next, we consider the Li-Lee model. In this case we need to simulate a sample path for both the common (European) trend $\kappa_{g,t}^{EU}$ and the country-specific (Dutch) trend $\kappa_{g,t}^{NL}$. The sample paths are projected one year into the future, i.e.

$$\hat{\kappa}_{g,t_n+1}^{EU} = \kappa_{g,t_n}^{EU} + \theta_g^{EU} + \epsilon_{g,t_n+1}, \tag{5.33}$$

where $\theta_g^{EU} \sim N\left(\hat{\theta}_g^{EU}, se(\hat{\theta}_g^{EU})\right)$ and

$$\hat{\kappa}_{g,t_n+1}^{NL} = \rho_g \kappa_{g,t_n+1}^{NL} + \delta_{g,t_n+1}, \tag{5.34}$$

with $\rho_g \sim N\left(\hat{\rho}_g, se(\hat{\rho}_g)\right)$ and both error terms follow the same distribution, i.e. $N(0, \tilde{C})$. As shown previously, this allows us to get the simulated mortality rates for Europe $\hat{\mu}_{x,g,t_n+1}^{EU}$ and the Dutch deviation $\hat{\tilde{\mu}}_{g,x,t_n+1}^{NL}$ and thus allowing us to compute the simulated Dutch mortality rates $\hat{\mu}_{g,x,t_n+1}^{LL}$.

One thing to take note of, is when parameter risk is included in the ARIMA model, it can lead to divergence. This is because the trend simulations of the Dutch deviation can exceed one, which in turn leads to divergence. To avoid this problem, the condition is set such that if the coefficient exceeds one, it is set to $\rho_g = 0.9999$. The next step is to compute the corresponding one-year death probabilities for the Dutch population $\hat{q}_{x,t_n+1}^{LL(g)} = 1 - \exp(-\hat{\mu}_{g,x,t_n+1}^{LL})$ and then simulate the number of deaths in year $t_n + 1$ by

$$D_{x,t_n+1}^{NL(g)} \sim \text{Bin}(E_{x,t_n}^{NL(g)}, \hat{q}_{x,t_n+1}^{LL(g)}).$$

Of course this also needs to be done for the European number of deaths, i.e. simulate

$$D_{x,tn+1}^{EU(g)} \sim \text{Bin}(E_{x,t_n}^{EU(g)}, \hat{q}_{x,t_n+1}^{EU(g)}),$$

where $\hat{q}_{x,t_n+1}^{EU(g)} = 1 - \exp(-\hat{\mu}_{x,t_n+1}^{EU})$. With this we can append the simulated observations to the European and Dutch data set and refit the model. After the refit, the best estimate of the mortality rates are computed as was shown in section 4.2.3 at time $t_n + 1$, where we obtain the best estimate Dutch mortality rates $\hat{\tilde{\mu}}_{x,t_n+1+N}^{LL,BE}$.

The following steps are similar to the Lee-Carter model, i.e. the computation of the best estimate survival rates and inserting them in equation (5.27) and thus computing the best estimate of the liabilities as of time $t + 1$, BEL_{t+1}, with $t = t_n$. The present value of the liabilities are also then easily obtained as discussed before using equation (5.28) and (5.29) using the estimated survival probabilities of the Li-Lee model in year t, $\hat{p}_{x,t}^{LL(g)}$, prior to the refit. Again, this whole process is repeated M times, which gives us M present values of the liabilities. The 99.5th quantile is taken of these M liabilities, which is then the 99.5% VaR, giving us the stressed present value of the liabilities. The one-year VaR methodology for the Li-Lee model is summarized in algorithm 2.

Algorithm 2 One-year VaR algorithm: Li-Lee

- For $j = 1, ..., M$
  - Load/select $D_{x,t}^{NL(g)}$, $E_{x,t}^{NL(g)}$, $D_{x,t}^{EU(g)}$ and $E_{x,t}^{EU(g)}$ for $t = t_1, ..., t_n$.
  - Fit data to the Li-Lee model, obtaining estimates for $\hat{\mu}_{x,g,t}^{LL}$.
  - Simulate $\hat{\kappa}_{g,t_n+1}^{EU(j)} = \kappa_{g,t_n}^{EU} + \theta_g^{EU(j)} + \epsilon_{g,t_n+1}^{(j)}$.
  - Simulate $\hat{\kappa}_{g,t_n+1}^{NL(j)} = \rho_g^{(j)} \kappa_{g,t_n}^{NL(j)} + \delta_{g,t_n+1}^{(j)}$
  - If $\rho_g^{(j)} \geq 1$, then the coefficient is changed to $\rho_g^{(j)} = 0.9999$
  - Compute $\log\ \hat{\mu}_{g,x,t_n+1}^{EU(j)} = \hat{\alpha}_{g,x}^{EU} + \hat{\beta}_{g,x}^{EU} \hat{\kappa}_{g,t_n+1}^{EU(j)}$ and $\log\ \hat{\tilde{\mu}}_{g,x,t_n+1}^{NL(j)} = \hat{\alpha}_{g,x}^{NL} + \hat{\beta}_{g,x}^{NL} \hat{\kappa}_{g,t_n+1}^{NL(j)}$.
  - Then, $\log\ \hat{\mu}_{g,x,t_n+1}^{LL(j)} = \log\ \hat{\mu}_{g,x,t_n+1}^{EU(j)} + \log\ \hat{\tilde{\mu}}_{g,x,t_n+1}^{NL(j)}$
  - Compute $\hat{q}_{x,t_n+1}^{LL(g)} = 1 - \exp(-\hat{\mu}_{g,x,t_n+1}^{LL})$ and $\hat{q}_{x,t_n+1}^{EU(g)} = 1 - \exp(-\hat{\mu}_{g,x,t_n+1}^{EU})$.
  - Simulate $D_{x,t_n+1}^{NL(g)} \sim \text{Bin}(E_{x,t_n}^{NL(g)}, \hat{q}_{x,t_n+1}^{LL(g)})$ and $D_{x,t_n+1}^{EU(g)} \sim \text{Bin}(E_{x,t_n}^{EU(g)}, \hat{q}_{x,t_n+1}^{EU(g)})$.
  - Append the simulated mortality rates to the current data set.
  - Refit the model to the newly appended data set, obtaining $\hat{\hat{\alpha}}_{g,x}^{EU}, \hat{\hat{\beta}}_{g,x}^{EU}$, $\hat{\hat{\kappa}}_{g,t}^{EU}$ and $\hat{\hat{\alpha}}_{g,x}^{NL}, \hat{\hat{\beta}}_{g,x}^{NL}$, $\hat{\hat{\kappa}}_{g,t}^{NL}$.
  - Estimate the new drift $\hat{\hat{\theta}}_g^{EU(j)}$ and time coefficient $\hat{\hat{\rho}}_{g(j)}^k$ through the SUR regression.
  - Calculate the best estimate of the trend processes $\hat{\hat{\kappa}}_{g,t_n+1+k}^{BE,EU(j)} = \hat{\hat{\kappa}}_{g,t_n+1}^{BE,EU(j)} + k\hat{\hat{\theta}}_g^{EU(j)}$ and $\hat{\hat{\kappa}}_{g,t_n+1+k}^{BE,NL} = \hat{\hat{\rho}}_{g(j)}^k \hat{\hat{\kappa}}_{g,t_n+1}^{NL}$ for $k = 1, ..., N$.
  - Compute $\log\ \hat{\hat{\mu}}_{g,x,t_n+1+k}^{LL,BE} = \log\ \hat{\hat{\mu}}_{g,x,t_n+1+k}^{EU,BE} + \log\ \hat{\hat{\mu}}_{g,x,t_n+1+k}^{NL,BE}$ for $k = 1, ..., N$.
  - Compute $\hat{\hat{p}}_{g,x+1,t_n+1+k}^{LL,BE} = \exp(-\hat{\hat{\mu}}_{g,x,t_n+1+k}^{LL,BE})$ for $k = 1, .., N$.
  - Forecast the $BEL_{t+1}^{(j)}$ using $\hat{\hat{p}}_{g,x+1,t_n+1+k}^{LL,BE}$ for $k = 1, .., N$.
  - Compute the present value of the liabilities $L_t^{(j)} = \frac{1}{1+r}(BEL_{t+1}^{(j)} + CF_{t,t+1}^{(j)})$.
- Take 99.5th quantile of the M present values of the liabilities, i.e. $L_t^{shock} = Q_{99.5}(\vec{L_t})$.
- Compute $SCR = L_t^{shock} - BEL_0$

In table 3, the results of the SCR as a percentage of the best estimate of the liabilities are shown for the one-year VaR for both the Lee-Carter and the Li-Lee model. The SCRs for the other approaches will also be presented in this way, i.e.

$$SCR\ percentage = \frac{L_t^{shock} - BEL_t}{BEL_t}. \tag{5.35}$$

As it can be observed, the Lee-Carter model generates a smaller SCR for both funds compared to the Li-Lee model. Also both SCRs of fund Green are smaller than that of fund Grey in this case.

Table 2: SCR percentage of one-year VaR for both the Lee-Carter model and the Li-Lee model.

	Fund Green	Fund Grey	Total
SCR LC	2.30%	5.21%	7.50%
SCR LL	4.355%	7.57%	11.93%

5.2.4 Standard Formula

For the sake of completeness and for comparison, this subsection is dedicated to the shock calculation according to the aforementioned standard formula. The standard formula approach assumes an immediate fall of $f = 20\%$ in the current and future best estimate of mortality rates, i.e.

$$\mu_{x,t}^{SF} = (1-f)\mu_{x,t}^{BE}. \tag{5.36}$$

As mentioned in section 5.1, the SCR formula for the standard formula is

$$SCR^{SF} = NAV_0 - (NAV_0|\ \textit{longevity shock}),$$

with $NAV_0 = A_0 - BEL_0$. This leads to

$$\begin{aligned} SCR^{SF} &= A_0 - BEL_0 - (A_0 - BEL_0|\ \textit{longevity shock}) \\ &= (BEL_0|\ \textit{longevity shock}) - BEL_0. \end{aligned}$$

The way in which $(BEL_0|\ \textit{longevity shock})$ is computed, is by obtaining $\mu_{x,t}^{SF}$ and transforming it into k-year survival probabilities ${}_kp_{x,t}^{(SF)(g)}$ with $k = 1, ..., N$. Plugging these survival probabilities into equation 5.22 leads to obtaining $(BEL_0|\ \textit{longevity shock})$. In table 3, the results of the SCR^{SF} are shown.

Table 3: SCR results of Standard Formula for Lee-Carter and Li-Lee model

	Fund Green	Fund Grey	Total
SCR^{SF} LC	6.59%	8.21%	14.79%
SCR^{SF} LL	6.17%	8.09%	14.26%

5.2.5 Run-off Approach: Terminal VaR

The run-off approach, also referred to the multi-year approach or the stressed trend approach, is according to Richards et al. (2013) the most appropriate way to investigate longevity trend risk, since the risk lies in the long term trend taken by the mortality rates. The main idea behind this approach is to consider the fluctuations in mortality rates over the potential lifetime of an annuitant/pensioner or until the maturity date of a policy. Basically, in the run-off approach, the stress scenario is the prediction interval for the best estimate of liabilities. Recall that in section 4.1.3 and 4.2.3, we discussed the prediction interval for the Lee-Carter model and the Li-Lee model. Using these prediction intervals, we can calculate the run-off VaR or in other words terminal VaR.

First we consider the Lee-Carter model. The first step is to simulate trajectories of the trend processes, $\hat{\kappa}_{g,t}^{LC}$, into the future. As discussed before, both parameter uncertainty and stochastic uncertainty is taken into account when simulating the trajectories. Recall that the trend process is modeled as a random walk with drift, i.e.

$$\kappa_{g,t}^{LC} = \kappa_{g,t-1}^{LC} + \theta_g + e_{g,t}, \tag{5.37}$$

with normally distributed error terms:

$$e_{g,t} = \begin{pmatrix} e_{M,t} \\ e_{F,t} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{M,e}^2 & r\sigma_{M,e}\sigma_{F,e} \\ r\sigma_{M,e}\sigma_{F,e} & \sigma_{F,e}^2 \end{pmatrix}\right) \tag{5.38}$$

The SUR regression is used to obtain the drift estimates $\hat{\theta}_g$ and the estimate of the co-variance matrix $\hat{C}$. The subscript LC refers to the Lee-Carter model, $g \in \{M, F\}$ which refers to male and female, and $t = t_2, ..., t_n$ as defined in the previous chapter.

To include stochastic uncertainty, the errors are drawn from the normal distribution with mean zero and co-variance matrix $\hat{C}$ and to include parameter uncertainty the drifts are drawn from the normal distribution with mean $\hat{\theta}_g$ and variance $se(\hat{\theta}_g)$. Both the error terms and the drift are simulated M times. The drift is constant over time for each sample path, but the errors are drawn over each future year.

$$\hat{\kappa}^{LC(m)}_{g,t_n+k} = \kappa^{LC}_{g,t_n} + k\theta^{(m)}_g + \sum_{j=1}^{k} e^{(m)}_{g,j} \quad m = 1, ..., M \text{ and } k = 1, ..., N \tag{5.39}$$

where N is the terminal year. In this case, the terminal year refers to the year that the youngest individual at time t dies with certainty or in other words, the year in which the youngest individual at time t reaches the maximum age. With this we get an NxM matrix of simulated future trend processes. Plugging the trend processes in equation (4.1), we obtain a NxM matrix of log mortality rates and thus obtaining the mortality rates. After applying Kannistö's extrapolation to obtain mortality rates for older people, we insert the mortality rates in equation (5.14), which gives us the simulated one-year death probabilities and thus the one-year survival probabilities as well, i.e. $p^{(m)(g)}_{x+k,t+k}, k \in \{1, ..., N\}$.

Using these simulated survival probabilities, the k-year survival probabilities, ${}_kp^{(m)(g)}_{x,t}$, are calculated and thus we can simulate the corresponding present value of the liabilities. This is done by calculating the present value of the liabilities for each trajectory. Let $L^{(m)}_t$ be the present value of the liabilities at time t for trajectory m. Then we have

$$L^{(m)}_t = \sum_{j,g,x} \sum_{k=\max(67-x,0)}^{N-x} {}_kp^{(m)(g)}_{x,t} \cdot B \cdot P^{(k)}_t \cdot I^{(g,j)}_{x,t}, \tag{5.40}$$

for $m = 1, ..., M$. So this gives us M simulations of the present value of the liabilities. Then the terminal VaR is obtained by simply taking the 99.5th quantile of the simulated liabilities that the compandy would have in year $t = N$:

$$L^{shock}_t = Q_{0.995}(\vec{L}_t), \tag{5.41}$$

where $\vec{L}$ is the vector with the M simulated liabilities. An overview of the terminal VaR calculation is shown in Algorithm (3). For the calculation of the SCR, we take the difference between the present value of the stressed liabilities and the present value of the best estimate of the liabilities:

$$SCR_t = L^{shock}_t - BEL_t \tag{5.42}$$

Algorithm 3 Terminal VaR algorithm: Lee-Carter

- Simulate M drift terms $\{\theta^{(1)}, \theta^{(2)}, ..., \theta^{(M)}\}$
- For $k = 1, ..., N$
  - Simulate M stochastic errors, i.e. $\{e_k^{(1)}, e_k^{(2)}, ..., e_k^{(M)}\}$
  - Forecast $\hat{\kappa}_{t_n+k}^{(m)} = \kappa_{t_n} + k\theta^{(m)} + \sum_{j=1}^{k} e_j^{(m)}$ for all M trajectories.
- Obtain the simulated log mortality rates; log $\hat{\mu}_{x,t_n+k}^{LC(m)} = \hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}_{t_n+k}^{LC(m)}$ for $k = 1, ..., N$
- Calculate the one-year survival probabilities $p_{x+k,t_n+k}^{(m)} = \exp(-\hat{\mu}_{x,t_n+k}^{LC,(m)})$ and thus obtaining the k-year survival probabilities ${}_kp_{x,t}^{(m)}$ for $k = 1, ..., N$.
- Calculate the present value of the liabilities of each trajectory $L_t^{(m)}$ using equation (5.40)
- Take the 99.5th quantile of the M liabilities in year N, i.e. $L_t^{shock} = Q_{0.995}(\vec{L}_t)$
- Compute the $SCR_t^{shock} = L_t^{shock} - BEL_t$

Next, we will consider the Li-Lee model. In this case, trajectories for both the common (European) trend, i.e. $\kappa_{g,t}^{EU}$, and the country-specific (Dutch) trend, i.e. $\kappa_{g,t}^{NL}$ need to be simulated. We will keep using this notation for the Li-Lee model. As was discussed in the previous chapter, the common trend is modeled as a random walk with drift and the country-specific trend is modeled as an AR(1) model. For the simulation of the trajectories of the European trend processes including both parameter and stochastic uncertainty, we have

$$\hat{\kappa}_{g,t_n+k}^{EU(m)} = \kappa_{g,t_n}^{EU} + k\theta_g^{EU(m)} + \sum_{j=1}^{k} \epsilon_{g,j}^{(m)} \quad m = 1, ..., M \text{ and } k = 1, ..., N \tag{5.43}$$

where the simulated drift terms are drawn from the normal distribution, i.e. $\theta_g^{EU} \sim N\left(\hat{\theta}_g^{EU}, se(\hat{\theta}_g^{EU})\right)$ and the simulated error terms are drawn from the normal distribution with mean zero and co-variance matrix $\tilde{C}$ as discussed in section 4.2.3, i.e.

$$\begin{pmatrix} \epsilon_{M,t} \\ \epsilon_{F,t} \\ \delta_{M,t} \\ \delta_{F,t} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tilde{C}\right). \tag{5.44}$$

Next, the trajectories of the country specific (Dutch) trend are simulated. The coefficients $\rho_{g,(m)}$ are drawn from a normal distribution, i.e. $\rho_g \sim N(\hat{\rho}_g, se(\hat{\rho}_g))$, which generates a coefficient per trajectory. The error terms are drawn from the same normal distribution with mean zero and co-variance matrix $\tilde{C}$.

$$\hat{\kappa}_{g,t_n+k}^{NL(m)} = \rho_{g,(m)}^{k} \kappa_{g,t_n}^{NL} + \sum_{j=1}^{k} \delta_{g,j}^{(m)} \quad m = 1, ..., M \text{ and } k = 1, ..., N. \tag{5.45}$$

After obtaining the simulations of both trend processes, the future mortality rates for the Dutch population are then calculated. And with this, the one-year survival probabilities are also obtained. Let $\tilde{p}_{x+k,t+k}^{(m)(g)}$ denote the simulated one-year survival probabilities obtained from the Li-Lee model. With this the k-year survival probabilities are calculated and inserted in equation (5.40) for $k =$

$1, ..., N$:

$$\tilde{L}_t^{(m)} = \sum_{j,g,x} \sum_{k=\max(67-x,0)}^{N-x} {}_k\tilde{p}_{x,t}^{(m)(g)} \cdot B \cdot P_t^{(N)} \cdot I_{x,t}^{(g,j)}. \quad (5.46)$$

Then the terminal value at risk and SCR are calculated using equation (5.41) and (5.42). In Algorithm 4, an overview of how the approach is done for the Li-Lee model is shown.

Algorithm 4 Terminal VaR algorithm: Li-Lee

- Simulate M drift terms $\{\theta^{EU(1)}, \theta^{EU(2)}, ..., \theta^{EU(M)}\}$
- Simulate M coefficients $\{\rho_{(1)}, \rho_{(2)}, ..., \rho_{(M)}\}$
- For $k = 1, ..., N$
  - Simulate M stochastic errors for the European trend, i.e. $\{\epsilon_k^{(1)}, \epsilon_k^{(2)}, ..., \epsilon_k^{(M)}\}$
  - Simulate M stochastic errors for the Dutch trend, i.e. $\{\delta_k^{(1)}, \delta_k^{(2)}, ..., \delta_k^{(M)}\}$
  - Forecast $\hat{\kappa}_{t_n+k}^{EU(m)} = \kappa_{t_n}^{EU} + k\theta^{EU(m)} + \sum_{j=1}^{k} \epsilon_j^{(m)}$ for all M trajectories.
  - Forecast $\hat{\kappa}_{t_n+k}^{NL(m)} = \rho_{(m)}^k \kappa_{t_n}^{NL} + \sum_{j=1}^{k} \delta_j^{(m)}$ for all M trajectories.
- Obtain simulated mortality rates through $\log\ \hat{\mu}_{x,t_n+k}^{LL(m)} = \log\ \hat{\mu}_{x,t_n+k}^{EU(m)} + \log\ \hat{\tilde{\mu}}_{x,t_n+k}^{NL(m)}$, with $\log\ \hat{\mu}_{x,t_n+k}^{EU(m)} = \hat{\alpha}_x^{EU} + \hat{\beta}_x^{EU} \hat{\kappa}_{t_n+k}^{EU(m)}$ and $\log\ \hat{\tilde{\mu}}_{x,t_n+k}^{NL} = \hat{\alpha}_x^{NL} + \hat{\beta}_x^{NL} \hat{\kappa}_{t_n+k}^{NL,(m)}$ for $k = 1, ..., N$ and $m = 1, ..., M$
- Calculate the one-year survival probabilities $\tilde{p}_{x+k,t_n+k}^{(m)} = \exp(-\hat{\mu}_{x,t_n+k}^{LL})$ and thus obtaining the k-year survival probabilities ${}_k\tilde{p}_{x,t}^{(m)}$ for $k = 1, ..., N$ and $m = 1, ..., M$.
- Calculate the present value of the liabilities of each trajectory $\tilde{L}_t^{(m)}$ using equation (5.40)
- Take the 99.5th quantile of the M liabilities in year N, i.e. $L_t^{shock} = Q_{0.995}(\vec{L}_t)$
- Compute the $SCR_t = L_t^{shock} - BEL_t$

In table 4, the resulting SCRs as percentages of the best estimate (BEL) of the multi-year approach can be seen. The SCR generated by the Lee-Carter model is larger in fund Green compared to the Li-Lee model, however in fund Grey, it is the opposite, i.e. the Li-Lee model generates a larger SCR. The total SCR of both funds of the Li-Lee model is larger than the Lee-Carter. So, the Lee-Carter model indeed underestimates the total SCR in this case.

Table 4: SCR of 99.5% Terminal VaR for both the Lee-Carter model and the Li-Lee model.

	Fund Green	Fund Grey	Total
SCR LC	8.11%	4.84%	12.95%
SCR LL	8.01%	5.96%	13.97%

5.3 Discussion

In this section, we will compare and discuss the results that were obtained and shown in the previous section, namely the SCRs for longevity risk of the three shock methodologies.

5.3.1 Comparison of SCR

As was previously shown, for each model, the SCRs for longevity trend risk were calculated through the one-year VaR approach, the terminal VaR approach and the standard formula. This was done for the two different funds, i.e. fund Green and fund Grey. The same portfolio assumptions from the previous section holds, i.e. the same number of participants as well as the same male to female ratio. In table 5, table 6, and table 7 the three SCRs from the shock methodologies for each fund are summarized as well as the total SCR, i.e. the portfolio SCR containing both funds. Again, the SCRs shown in percentages of the best estimate of the liabilities.

When comparing the results of fund Green in table 5, i.e. the SCRs of the one-year VaR, the terminal VaR, and the standard formula, it is apparent that the terminal VaR is much larger compared to the one-year VaR. As for the standard formula, it generates SCRs that are larger than the one-year VaR but smaller than the Terminal VaR. However, in table 6 it can be seen that it is the other way around for fund Grey. The one-year VaR framework generates larger SCRs compared to the terminal VaR framework and the standard formula generates the largest SCRs of them all. If we look at the total SCRs in table 7, the one-year VaR generates the smallest SCR, then it is followed by the terminal VaR, leaving the largest generating SCRs to the standard formula. This is the result for both the Lee-Carter model and the Li-Lee model. Note that since the Li-Lee model is assumed to be the ”correct” model, we can conclude that the Lee-Carter model very much underestimates the SCRs, especially in the case of the one-year VaR.

The results of table 7 have come out as expected when looking at the totals of the three methodologies, i.e. the one-year VaR framework has the smallest SCR of the three and the standard formula has the largest SCR. In other words, the standard formula requires more capital than the one-year VaR and the terminal VaR. This does not come off as a surprise, since every one-year death rate decreases by the 20% in this case, and also considering the fact that the older individuals have a relatively high death rate, the mortality rates in return decrease significantly. And considering that it is specifically the older population, where the liabilities need to be paid and thus is the main age group in the portfolio concerning the results, the SCR increases at a much faster rate over time and is thus larger to begin with. Fernando et al. (2017) confirms that the shock implemented by the standard formula does not adequately reflect longevity risk that life annuity portfolios face, stating that the shock will tend to overestimate or underestimate the current longevity risk for all ages. However, one should also keep in mind that the standard formula considers other risks, for example the risk of using the ”wrong” model. It also considers deviations in the level of the mortality rates, or in other words, the probability that the mortality rates of the people considered are not the same as those of the entire population. In the case of the one-year VaR or terminal VaR framework, only the trend risk is considered.

Table 5: *SCRs* from fund Green of all three shock methodologies for both the Lee-Carter model and the Li-Lee model.

Fund Green	One-year 99.5% VaR	Terminal 99.5% VaR	Standard Formula
Model LC SCR	2.30%	8.11%	6.59%
Model LL SCR	4.36%	8.01%	6.17%

Table 6: *SCRs* from fund Grey of all three shock methodologies for both the Lee-Carter model and the Li-Lee model.

Fund Grey	One-year 99.5% VaR	Terminal 99.5% VaR	Standard Formula
Model LC SCR	5.21%	4.84%	8.21%
Model LL SCR	7.57%	5.96%	8.09%

Table 7: Total SCR of all three shock methodologies for both the Lee-Carter model and the Li-Lee model.

	One-year 99.5% VaR	Terminal 99.5% VaR	Standard Formula
Model LC SCR	7.50%	12.95%	14.79%
Model LL SCR	11.93%	13.97%	14.26%

In fund Green, the population consists mostly of younger people who have yet to reach the retirement age. In funds consisting of mostly younger individuals, it is expected that the capital requirements are much larger compared to a fund consisting of mostly older individuals. This is because they suffer from a larger longevity risk caused by the fact that the death rates of these young individuals are on a longer time horizon, and thus have more time to deviate from the current estimations in the long-run. This is especially apparent when comparing the terminal VaR of the two funds, where the SCR of the terminal VaR of fund Green is much larger for both models than that of fund Grey. In general, it is expected that a portfolio consisting of older people will have lower SCRs since there is less uncertainty concerning the future development of mortality rates.

In case of the one-year VaR, the SCR of fund Grey is larger than that of fund Green. A possible explanation for this is due to the fact that the shock happens only in the first projected year, where it will only have an effect on the older population, since they need to receive their benefits that year. On the other hand the cash outflow to the younger generations, i.e. non-retired people, is zero. So the main source in this case of the high SCR in the older population, is the cash outflow of the future projected year.

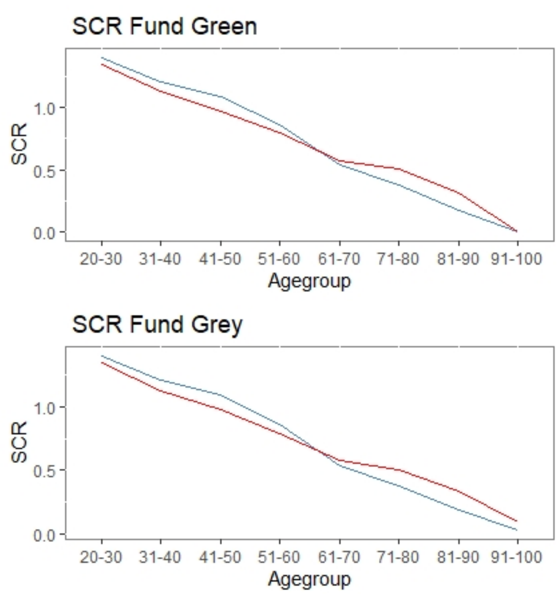

Figure 13: SCR of the 99.5% terminal VaR framework as a function of age using the Lee-Carter model. The red line represents the female population in the fund and the blue line represents the male population. Ages are distributed in age groups of 10. The figure on top shows the SCR of the age groups of fund Green and at the bottom, the figure shows the SCR of fund Grey.

According to Richards et al. (2013), the capital requirement for the terminal VaR approach is dependent on the model, outset age and the discount rate. In figure 13, the SCR of the 99.5% terminal VaR for different age groups in each fund can be observed for the Lee-Carter model. This is done for both male and female. In both funds, the female population generates a lower SCR from ages 20 to 70 compared the the male population of the same ages. After that, it gives a larger SCR for the older age groups compared to the males, i.e. ages 70 to 100. In general, the terminal VaR approach gives a higher SCR for the younger age groups compared to the older age groups for both male and female and the SCR gradually decreases as the age increases.

In figure 14, the SCRs from the one-year 99.5% VaR are shown as a function of age groups for both funds. For both fund Green and Grey, the SCR is larger for the older age groups compared to the younger age groups. In fund Green, it can be seen that there are no people who are in the age group 91 to 100. The SCR percentage in fund green is in general slightly higher than that of fund Grey. Another thing that can be observed in fund Green, is the SCR for females are smaller for all age groups compared to that of the males. This is not the case in fund Grey, as the SCR for females becomes larger than that of males as of age group 71 to 80. Similar figures containing the relationship between the SCRs and age can be seen for the Li-Lee model in the appendix, i.e. figure 15 and figure 16. In the case of the terminal VaR approach of the Li-Lee model, the SCRs for females are higher than that of males in both funds. And this is the case for all age groups. As for the one-year VaR, this is also the case for fund Grey, but for fund Green the SCR for females is higher than that of males up until age group 61 to 70. After that it lies slightly below the SCR of males.

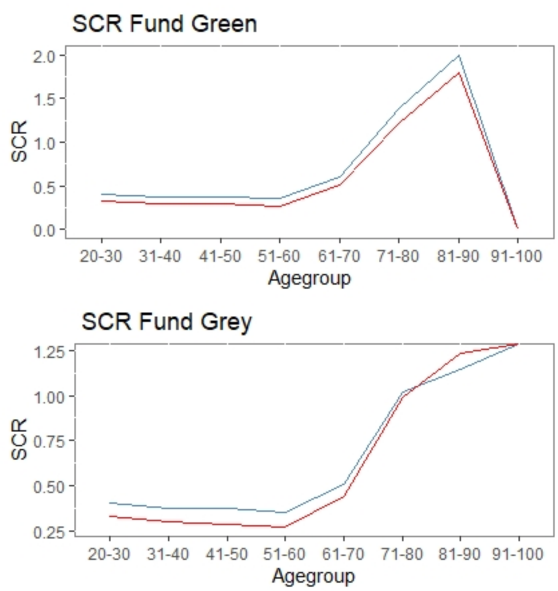

Figure 14: SCR of the one-year 99.5% VaR framework as a function of age using the Lee-Carter model. The red line represents the female population in the fund and the blue line represents the male population. Ages are distributed in agegroups of 10. The figure on top shows the SCR of the agegroups of fund Green and at the bottom, the figure shows the SCR of fund Grey.

After analyzing from the age group perspective, we consider another important factor that also plays a crucial role in how the SCR is affected, namely the interest rate. In table 8, the results of the SCRs are shown for the Lee-Carter model with three different interest rates. Up until now, we have assumed the interest rate to be $r = 2\%$, hence we consider a lower interest rate and a higher interest rate, i.e. $r = 1\%$ and $r = 3\%$ to see how the SCRs are affected. From the results we can conclude that a lower interest rate generates a higher SCR, while a higher interest rate generates lower SCRs. According to Börger (2010), this is indeed the case since he notes that capital requirements increase when the interest rate decreases.

	$r = 1\%$		$r = 2\%$		$r = 3\%$	
Lee-Carter	green	grey	green	grey	green	grey
1-year VaR	2.83%	5.42%	2.30%	5.21%	1.46%	4.64%
Terminal VaR	8.52%	5.28%	8.11%	4.84%	5.56%	3.40%
SF	7.03%	8.68%	6.59%	8.21%	6.20%	7.76%

Table 8: Calculated SCRs of the one-year 99.5% VaR, the 99.5% terminal VaR and the standard formula for different interest rates using the Lee-Carter model.

6 Summary and Conclusion

This thesis investigates and compares the three shock methodologies for longevity risk, namely the one-year VaR framework, the terminal VaR framework and the standard formula. The one-year VaR framework and the standard formula are both shock methodologies prescribed by Solvency II for longevity risk, since Solvency II states that every insurance company should have a risk based capital that covers 99.5% of all events over the time span of one year. The main issue with the Solvency II framework is that it measures longevity risk over a one year horizon. Since longevity risk lies in the long-term trend of the mortality rates, it is more fitting to measure it with a multi-year framework. However, Solvency II has already been implemented and hence, we will investigate if the one-year VaR framework meets a multi-year requirement. Now, to implement these methodologies, mortality rates need to be modeled stochastically. Since we have chosen to model the Dutch mortality rates, the Li-Lee model has been chosen together with the Lee-Carter model. The Lee-Carter model is chosen due to its simplicity, and has all around a great performance. Li-Lee on the other hand is chosen, because it is the chosen model of the Royal Dutch Actuarial Association, which is assumed to best represent the Dutch mortality data. For each of these model, the best estimate of the liabilities is calculated and the three shock methodologies are applied. First the one-year 99.5% VaR approach was considered, where the VaR framework of Richards et al. (2013) was implemented. This methodology forecasts one simulation of the mortality rates one year in the future, appends these forecasted mortality rates to the current data set, and refits the model. Then the best estimate of the liabilities is recalculated using this refitted model to obtain the present value of the liabilities. This whole process is repeated M times and then the 99.5th-quantile of the liabilities is taken, which becomes the 99.5% VaR. Then the SCR is calculated as the difference between the 99.5% VaR and the best estimate of the liabilities. Next, the 99.5% terminal VaR framework is considered. For this shock methodology, we simulate M sample paths of future mortality rates and let them develope over time. At the terminal year, which is the year that the youngest individual reaches the maximum age, the present value of the liabilities are calculated. The 99.5% terminal VaR is then the 99.5th-quantile of the liabilities. This is followed by the corresponding SCR calculation. The last and the simplest shock methodology is that standard formula, which assumes a fall of 20% in the current best estimate of mortality rates. Then the present value of the liabilities is calculated and thus the SCR as well.

From intuition one expects the one-year VaR to deliver the smallest SCR compared to the other two shock methodologies. The results show that this is indeed the case when calculating the total SCRs of our portfolio. The standard formula on the other hand generates the highest SCR out of the three methodologies. And lastly the terminal VaR lies between these two. This holds true for both the Lee-Carter model and the Li-Lee model. Comparing the Lee-Carter and the Li-Lee model, the Lee-Carter underestimates the SCRs for the one-year VaR and the terminal VaR. When comparing the one-year VaR and the terminal VaR of the Li-Lee model, the difference is around 2%. This means that the one-year 99.5% VaR does not meet the multi-year capital requirements. In this case, it is then recommended to make the one-year shock methodology a bit more conservative. This can be done by, for example adding more assumptions and conditions to the one-year VaR shock model. The standard formula on the other hand, is too conservative and overestimates the necessary capital requirements.

Next to comparing the total SCRs of the portfolio, we also dived deeper to compare the SCRs of the three shock models for each fund in the portfolio. One fund consisted of mostly younger individuals who have yet to reach retirement age, while the other fund consisted of mostly older people who have reached or passed retirement age. To our finding, the SCR of terminal VaR is smaller than the SCR of the one-year VaR when considering the fund with the older population. This is the case for both the Li-Lee and the Li-Carter model. This is contrary to the results of the SCRs of the fund consisting of younger individuals. In their case, the SCR of the one-year VaR is smaller than the SCR of the terminal VaR. So it would seem that it could be better to consider different shock methodologies for different age groups. Especially because older age groups are sensitive to short-term shocks, while the younger age groups are more sensitive to long-term shocks. Next to analyzing the two funds and different age groups, we also took a look to see how the total portfolio SCRs would react to different interest rates. We considered an interest rate which is lower and one that is higher than the current interest rate. From the results, we conclude that a lower interest rate generates a higher SCR. This is the case for all three shock methodologies.

For future research, we would like to further investigate our finding of the SCR of the terminal VaR for the older population, since it generated quite small numbers for the SCR. Also, it would be quite interesting to see what other conditions and assumptions could be implemented such that the one-year VaR does meet the multi-year requirement. And lastly, research on considering different shock models for different age groups would also be interesting. An idea for this is to consider specific trend processes for different age groups.

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7 Appendix

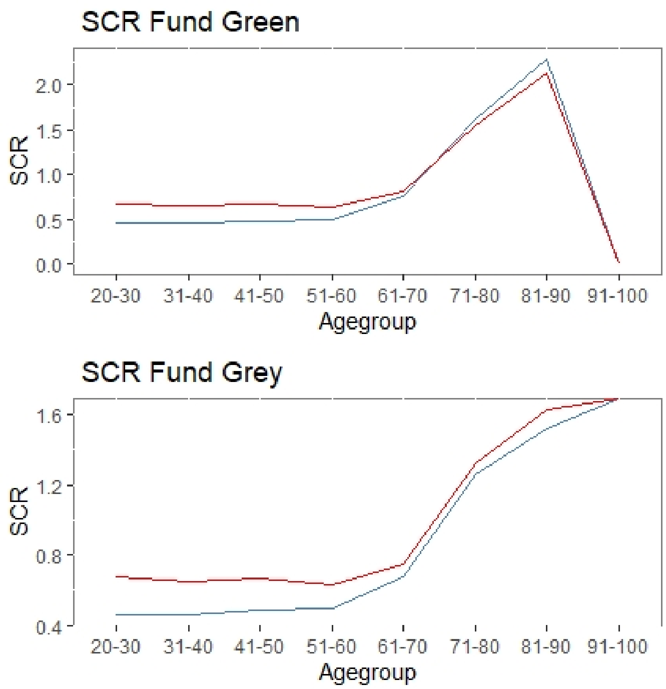
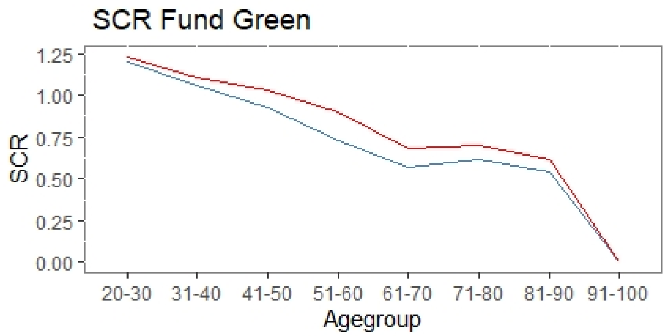


Figure 15: SCR of the one-year 99.5% VaR framework as a function of age using the Li-Lee model. The red line represents the female population in the fund and the blue line represents the male population. Ages are distributed in age groups of 10. The figure on top shows the SCR of the age groups of fund Green and at the bottom, the figure shows the SCR of fund Grey.



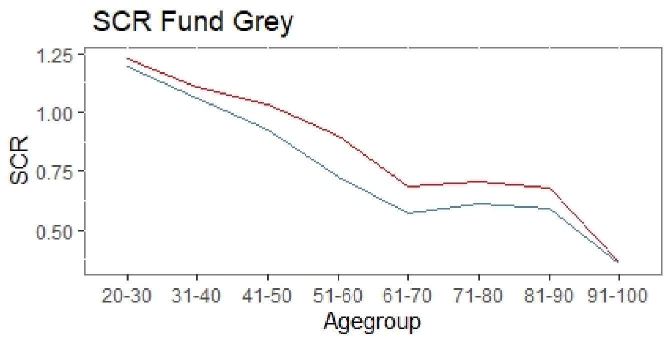


Figure 16: SCR of the 99.5% terminal VaR framework as a function of age using the Li-Lee model. The red line represents the female population in the fund and the blue line represents the male population. Ages are distributed in age groups of 10. The figure on top shows the SCR of the age groups of fund Green and at the bottom, the figure shows the SCR of fund Grey.