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Life-Cycle Investment Design

Strategies for Dutch top-up plans

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Life-Cycle Investment Design
Strategies for Dutch top-up plans

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"In the long run, there's just another short run."

— Abba P. Lerner

Abstract

Top-up pension plans were introduced in 2015 as a consequence of the implemented maximum pensionable salary. In this thesis, the intertemporal investment problem for a net pension top-up plan participant is studied. A stylized framework is developed that captures the main features of the top-up plan. These main features include the conversion of accrued financial wealth into a life annuity subject to a particular funding ratio surcharge at retirement. Additionally, the implicit presence of guaranteed pension benefits as well as classic human capital are taken into account. Within this framework, a life-cycle investment strategy is determined that best serves the expected utility maximizing participant. In case the surcharge equals the actual funding ratio at retirement, closed-form solutions are provided under a certain set of assumptions.

Keywords: stochastic processes, Vasicek model, individual defined contribution schemes, top-up plans, expected utility, lifetime portfolio selection, human capital, life-cycle investing, hedging, funding ratio, life annuity

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[1]The views expressed in this thesis do not necessarily represent the views of APG Asset Management.

Management Summary

Research motive

Top-up pension plans were introduced in 2015 as a consequence of implementing a previously non-existent maximum with regard to the pensionable salary. APG is currently managing top-up plans for the following pension funds: ABP, bpfBouw, PPF APG and SPW. A prominent and widely utilized top-up plan is the net pension plan. For the accrued capital in the net pension scheme, APG currently applies a uniform life-cycle strategy for the aforementioned pension funds[1].

Recent changes in legislation[2] have prompted a revision of the existing investment policies of these plans. In particular, the net pension plan no longer carries the requirement that accrued capital be converted upon retirement into pension payments at a relatively unfavorable annuity rate that includes a surcharge for the pension fund's required solvency (VEV). Instead, the surcharge is replaced by the larger of the fund's actual funding ratio at the time of retirement, and the fund's minimum required solvency (MVEV). The change implies a change in the participant's risk exposures over the life cycle, and may well necessitate an alteration in the net pension plan's investment policy.

Model development

The life-cycle investment strategy applied in the net pension plan should adequately cover the participant for both financial market risks and risks associated with the net pension plan design. Risks associated with the plan design are for example the risk factors induced by the nature of the mandatory conversion regime at retirement. Moreover, the fact that participation in a top-up plan implies the presence of additional retirement saving components, affects the desired life-cycle strategy.

In this thesis, we develop a theoretical framework in which the (new) investment problem for a net pension top-up plan participant can be analyzed. Within this framework, we can evaluate which portfolio composition best serves a net pension participant over the life cycle. This stylized framework needs to capture the main features of the top-up plan. These main features include the conversion of accrued financial wealth into a life annuity subject to the asymmetric funding ratio surcharge at retirement. Additionally, the implicit presence of both accumulated pension rights in the underlying occupational pension fund and the state pension guarantee are taken into account. Moreover, classic human capital, which consists of fiscally maximized pension premiums for the top-up plan, is also incorporated. In this model, a (power) utility maximizing participant is considered, for which his financial wealth is invested in an economy that exhibits both interest rate and equity risk.

There are three stages in which the model framework is developed. We start out with an unrestricted problem setting and build up in model complexity from there. An increasing complexity corresponds with accounting for an increasing number of 'restrictions' related to the net pension top-up plan. First of all, a general intertemporal consumption problem with exogenous pension contributions is considered. This problem represents an investment-linked drawdown account. Thereafter, we consider a problem specification in which the individual is subject to the mandatory conversion regime at retirement, abstracting away from the minimum level of the funding ratio surcharge. This second phase answers

[1] *Pensioenreglement 2018.* ABP. (2018). p 127-130

[2] *Besluit van 9 januari 2018 tot wijziging van het besluit uitvoering pensioenwet en wet verplichte beroepspensioenregeling vanwege wijziging van het inkooptarief voor nettopensioen*

the question of how the analytically optimal pension portfolio is composed in case of this simplified conversion regime. The third phase incorporates the minimum funding ratio surcharge. A straightforward numerical approach is applied in order to determine the optimal asset allocation subject to this condition.

Main results & policy revisions

Within the proposed framework, we identified a particular desired portfolio composition for the net pension participant. Closed-form solutions for the optimal asset allocation strategies are provided in case the surcharge equals the actual funding ratio at retirement[3]. This optimal investment strategy consists of a well-known mean-variance (speculative) portfolio and a particular minimum risk (or hedge) portfolio that corresponds to different features of the top-up plan and their respective risks.

Based on the resulting investment strategies, the corresponding analyses and a well-considered benchmark parameter set, the following main recommendations have been made regarding the revised composition of a uniform and deterministic life-cycle investment strategy for the net pension plan and similar top-up plans:

- A general increase in equity holdings over the lifetime is desirable.
- The duration hedge position with respect to the annuity purchase at retirement should be built up more gradually over the entire life cycle.
- Accurate calibration of level of risk aversion of the participants pool it of the utmost importance.

Based on these recommendations and the theoretical insights created by this thesis, the revised life-cycle strategy was constructed. This revised strategy is going live in January 2019 and can be found in e.g. *Pensioenreglement 2019* (p. 125-127), ABP (2019). One can immediately notice that the first recommendation of increasing the equity exposure over the life-cycle is directly incorporated. In order to facilitate the second recommendation in a practical manner, the asset class *long-term treasuries* is introduced in the revised life-cycle mix. Furthermore, the asset classes *index-linked bonds* and *long-term treasuries* have a hump-shaped glide-path pattern over the last 35 years before retirement. This pattern facilities a more gradual build-up of the duration hedge regarding the annuity purchase at retirement. Additionally, the asset class *treasuries* exhibits a sharp inclining glide path starting at a 10-year horizon, partly to complement the earlier initiated duration hedge for the upcoming annuity purchase.

As a final remark, this research strongly urges to explore the options for applying a stochastic life-cycle product in top-up plans. In this thesis, we provide a stochastic investment strategy that is individually-tailored, which could be implemented with a limited amount of basic information. This basic information consists of the individual's accrued financial wealth in both the top-up plan and the occupational pension plan, age and time of entry in the top-up plan. Additionally, a simple labor income prognosis completes the information that is required for utilizing a stochastic life-cycle regime. This information provision is generally of low cost and the resulting individual strategies can easily be incorporated in the monthly rebalancing of the life-cycle assets that already takes place. By applying a more tailored strategy, a significant increase in the participant's utility could be established.

[3]Note that the derived optimal portfolio compositions are only valid under the particular set of stated assumptions.

Contents

Chapter 1

Introduction

Nowadays, concerning a Dutch pension system in which people gradually acquire more and more responsibility to fund their own retirement, it is of the utmost importance for individuals to be able to make complex financial decisions associated with retirement savings and spending. This decision process should be guided by adequate and clear advice provided by pension funds, employers and the government. Such advice should be reflected by and incorporated in the assortment of pension contract options. This shift to a more individualized approach to retirement saving and spending is also emphasized by the Sociaal-Economische Raad (SER) (2016), who evaluated the current pension system and performed an exploration on possible adjustments. In this exploration, individually tailored retirement saving combined with collective risk sharing came forth as an attractive option. Recent developments in the Netherlands with regard to this option are discussed by e.g. Bovenberg and Nijman (2017).

The net pension top-up plan is such a combination of individual retirement saving and traditional, defined benefit (DB) schemes of a collective nature. Amongst others, the net pension plan is facilitated by the largest pension fund of the Netherlands, namely the Algemeen Burgelijk Pensioenfonds (ABP) (ABP, 2018b). This top-up pension plan was introduced in 2015 as a result of the pensionable salary for the second pillar being capped at an upper limit. The net pension is a voluntary retirement savings plan with respect to the part of the gross salary above this upper limit. The plan's objective is to restore the total accumulation of benefits in the underlying collective pension fund best as possible. One could label this top-up plan to be an individual defined contribution (IDC) plan, as the participant solely bears the investment risk. However, at the retirement date, the accrued financial wealth is mandatorily converted to a pension right in the underlying occupational pension fund. A retirement or dissaving phase in this pension plan is therefore absent. This mandatory conversion regime is subject to a number of risk factors. In contrast to voluntary savings in the third pillar, the premiums in the top-up plan are taxed, whereas the returns on investment and benefits are exempt from wealth tax. Existing excedentregelingen of e.g. Stichting Bedrijfstakpensioenfonds voor de Bouwnijverheid (bpfBouw) can also be labeled as top-up pension plans amongst others, similar to the net pension (bpfBouw, 2018). Nonetheless, we focus on the net pension in this thesis.

The current investment strategy for the accumulated wealth in top-up plans managed by the All Pensions Group (APG), consists of a uniform life-cycle strategy in which the proportion invested in fixed income securities is gradually increased as one approaches retirement. This life-cycle design was determined based on popular strategy in the market place. Utilizing a life-cycle strategy for top-up plans serves as part of the pension fund's fulfillment on legislation with respect to the prudent person principle (*Pensioenwet*, 2006, Article 135). The prudent person principle emphasizes the pension fund's duty to look after the customer best as possible when it comes to voluntary saving schemes. Recent changes in legislation have prompted a revision of the existing investment policies of these top-up plans[1]. Therefore, a study on optimal life-cycle investment design with regard to the investment policy of these top-up pension plans was needed to formulate an up-to-date investment strategy that best captures the participant's needs.

Once in retirement, one is dependent on the state pension, the second pillar pension benefits and possible voluntary savings such as top-up plans to provide an income stream for the remaining years. An optimal composition of the pension portfolio and allocation of financial means is therefore important to strive for the ultimate goal of maintaining the standard of living best as possible within retirement. The seminal work of Merton (1969, 1971) and Samuelson (1969) in the area of optimal lifetime portfolio selection and consumption choice can be considered as the start of a wide variety of studies regarding life-cycle investing. They derive closed-form solutions for the optimal asset allocation over the lifetime under a certain set of assumptions and constant equity risk exposure. However, an important risk faced by a long-term, finite horizon investor is interest rate risk, which leads to a stochastic investment opportunity set. This lifetime asset allocation problem was first researched by Sørensen (1999). Thereafter, one of the leading works developed on modeling lifetime portfolio and consumption choice accounting for both interest rate risk and inflation risk, is created by Brennan and Xia (2002).

In this thesis, we develop a theoretical framework in which the investment problem for a net pension top-up plan participant can be analyzed. This stylized framework needs to capture the main features of a general top-up plan, and in particular the net pension. A top-up plan implicates the presence of an underlying occupational pension scheme. Therefore, it is important to take a holistic approach towards retirement saving and spending, such that other retirement savings components are taken into account as well. These other retirement savings components are e.g. the state pension and accrued benefits in the underlying pension fund. The main features of the net pension top-up plan itself comprise of the pension premiums and the conversion regime at retirement. The pension premiums for the net pension are fiscally maximized, dependent on age. The conversion regime can be considered as purchasing a fixed life annuity subject to a particular funding ratio surcharge. This surcharge exhibits a certain guaranteed minimum surcharge level. The conversion regime plays an important role in the asset allocation problem. Within this framework, the individual maximizes the expected utility of pension wealth subject to exposure of interest rate risk and equity risk. These included risk factors are conform the recent leading publications on lifetime portfolio and consumption choice. The classical life-cycle investment design comes into play by incorporating human capital as a personal asset that is closely related to fixed income securities. Human capital generally depletes as one's career progresses.

Then, within this problem setup, the main research question of how to invest in the available asset menu over the lifetime to best serve the net pension participant, is addressed. There are three stages in which this research question will be answered. We start out with an unrestricted problem setting and build up in the amount of restrictions related to the net pension top-up plan from there. First of all, a general intertemporal consumption problem with exogenous pension contributions is taken into account, for which the model of Brennan and Xia (2002) serves

[1] *Besluit van 9 januari 2018 tot wijziging van het Besluit uitvoering Pensioenwet en Wet verplichte beroepspensioenregeling vanwege wijziging van het inkooptarief voor nettopensioen* (2018, January 9)

as a stepping stone. This problem represents an individual defined contribution scheme: one's pension consists of an investment-linked drawdown account. We could consider this investment-linked drawdown account to represent a variable life annuity. Thereafter, we consider a problem specification in which the individual is subject to the mandatory net pension conversion regime at retirement, abstracting away from the minimum level of the funding ratio surcharge. Part of the model proposition of Cairns, Blake, and Dowd (2006) is used to appropriately deal with the fixed annuity purchase at retirement in an analytical setting. This second phase answers the question of how the analytically optimal pension portfolio is composed in case of this simplified conversion regime. The third phase incorporates the minimum funding ratio surcharge. A straightforward numerical approach is applied in order to determine the optimal asset allocation subject to this condition. Note that the derived optimal portfolio compositions are only valid under the particular set of stated assumptions.

As for every model framework choice, the following trade-off is made: the main sources of risk, behavior and saving components concerning long-term investing need to be accounted for, without losing model tractability. The model framework needs to be as realistic as possible and still yield results that are well-interpretable. The latter is of particular importance, as this model framework and the resulting desired strategies also serve as a tool to provide policy makers with both insight and economic intuition regarding life-cycle investing for top-up plan participants.

The main results of this thesis are as follows. In the proposed framework, we identified a speculative and minimum risk (or hedge) portfolio. The hedge demand is caused by the desire to (partly) eliminate annuity risk at retirement, as well as adverse changes in both the interest rate in general and the funding ratio surcharge. The weight of the minimum risk portfolio with respect to the components held to hedge against adverse changes in the interest rate and annuity risk, is determined by the level of risk aversion. On the contrary, the weight in the minimum risk portfolio with respect to the components held to immunize for adverse changes in the funding ratio, is mainly determined by both the level of risk aversion and the minimum funding ratio surcharge. Furthermore, in order to hedge against adverse changes in the funding ratio surcharge, a certain demand for equity (additional to the speculative demand) is identified. In general, the demand for the available assets, especially equity, increases as pension guarantees are taken into account. Moreover, the human capital definition and therefore the time of entry in the top-up plan have a significant impact on the desired life-cycle strategy for financial wealth in the top-up plan. Based on this portfolio composition and a reasonably chosen benchmark parameter set, the following main recommendations have been made. A general increase in equity is desirable, partly to account for the changes in legislation of the net pension top-up plan. Secondly, the duration hedge position with respect to the annuity purchase should be built up more gradually over the life cycle.

We contribute to the existing literature on lifetime portfolio selection problems in the following ways. First of all, we explore the implications of accounting for fixed pension premiums, a variable life annuity and human capital in an intertemporal consumption problem with equity risk and interest rate risk. Secondly, we show how to appropriately deal with the top-up plan's stochastic conversion regime in a terminal wealth problem, for which the martingale method is applied to derive the optimal investment strategy. Additionally, we provide a method to incorporate human capital with respect to the top-up plan, including a guaranteed pension benefit, in this asset allocation problem. Thirdly, we propose a straightforward methodology to handle a minimum level with respect to the surcharge on the annuity purchase at retirement.

The practical relevance of this thesis speaks for itself, since it serves as a theoretical backbone to support decisions regarding the policy revision of the life-cycle investment strategy for top-up plans managed by APG. Adapted versions of such plans can potentially become a tool to increase freedom of choice regarding second pillar pension accrual in the future. With this life-cycle model, we also provide support for pension providers with regard to a possible implementation of a stochastic life-cycle strategy for top-up plans. This stochastic strategy can be considered as a highly tailored investment strategy that can differ for each participant. We argue that only a basic information provision and a small number of assumptions are needed to implement such a low-cost strategy.

Note that deviating from the preferred strategies would result in a theoretical welfare loss for top-up plan participants. However, a number of practicalities prevented the theoretically optimal strategies from being implemented directly. These practicalities consist of e.g. the absence of short positions and the application of a uniform strategy. A certain mapping of these theoretical strategies was undergone to eventually yield a practical and 'feasible' life-cycle strategy. In the subsequent thesis *'From Theoretical Life-Cycle Strategies to Best Practices in Dutch Top-Up Plans'* by Temme (2019)[2], the aim is to quantify the resulting welfare losses that originate from common, practical implementation choices with regard to the theoretically optimal (dynamic) life-cycle strategies for Dutch top-up plans. This research identified the highest overall welfare losses to be instigated by the strategies that serve as a simplification of an optimal, dynamic strategy. Consequently, this simulation study indicates amongst others that a dynamic life-cycle strategy would significantly improve the current welfare of participants in top-up plans.

The remainder of this thesis is organized as follows. In Chapter 2, the theoretical foundation for life-cycle investing is discussed and an overview of recent extensions to the literature on lifetime portfolio selection is provided. Some choices regarding the model framework in this thesis are discussed as well. Moreover, we shed light upon life-cycle strategies in practice and the net pension case. Secondly, in Chapter 3, the tailored intertemporal consumption problem that represents an individual defined contribution scheme is studied. The financial market, wealth processes and preferences of the individual, all of which generally apply to the remaining chapters, are introduced. Thereafter, Chapter 4 explores a tractable specification for the fixed life annuity price at retirement. Subsequently, Chapter 5 discusses the terminal wealth problem that incorporates this specification into the conversion regime of the net pension top-up plan. Then, Chapter 6 incorporates the minimum level of the funding ratio surcharge on the annuity purchase at retirement, which was previously abstracted away from. Lastly, Chapter 7 concludes this thesis and discusses both the resulting recommendations and suggestions for further research.

[2] Available at `https://papers.ssrn.com/abstract=3460242`

Chapter 2

Life-cycle investing and best practices

In this chapter, a brief overview is presented of the rather fragmented academic literature with the subject of individual portfolio and consumption choice over the lifetime. First of all in Section 2.1, the economic theory that lays the foundation for consumption and investing over the lifetime, is briefly discussed. Secondly, one of the first fundamental theoretical works in this area by Robert Merton (1969, 1971) is touched upon and recent additions to the literature in various directions are reviewed. Since the problem of lifetime investing is broadly researched over the years, this review probably does not to paint a complete picture. One can however form a general idea of the recent, relevant extensions made to this research area. The discussed extensions form a basis for the framework development in this thesis. Lastly, in Section 2.2, we make the transition to life-cycle investing in practice with respect to the investment strategies for the top-up plans in Dutch pension schemes, particularly the net pension. The net pension in general, as well as recent changes in legislation are briefly reviewed. Additionally, the first elements for the development of a mathematical model for this problem are discussed.

2.1 Literature on lifetime consumption and portfolio choice

Economic theory on lifetime savings and consumption

Concerning the development of the theory of saving and consumption, the following fundamental hypotheses regarding the determinants of consumption and saving patterns have shaped the ideas on this topic nowadays. Firstly, Keynes (1936) stated that the absolute level of current disposable income mainly determines the individual's present saving and consumption behavior. An increase in the absolute level of income would lead to a relatively larger part of the income being saved. This is known as the absolute income hypothesis. Contrary to Keynes, Duesenberry (1949) followed up on this idea by stating that relative rather than absolute income would be the main determinant of the individual's present saving and consumption. One would evaluate his current income with respect to previous income and the income of peers within the same socio-economic class, which would influence the attitude towards consumption and saving. This is also known as the relative income hypothesis, which is often associated with the expression: 'Keeping up with the Joneses'.

A few years later, Modigliani (1966) brought to life the life-cycle hypothesis of consumption and savings. This hypothesis states that the savings and consumption pattern over the life cycle is mainly determined by the expected lifetime income and the age of the individual, of which the latter determines the position within the life cycle. The goal of the individual is to even out consumption over his lifetime despite of substantial fluctuations in income, mainly depending on age. This consumption smoothing and resulting stable lifestyle would maximize the expected utility of welfare over the lifetime. Saving and investing can be seen as mechanisms to move income between periods, which enable the individual to accomplish this goal. According to this life-cycle theory, people build up financial assets throughout the career and deplete these means within retirement (even when other incentives to save are taken away, e.g. an economic environment with negative interest rates). In general, the largest component of an individual's aggregate savings will be for the purpose of financing retirement. This will lead to the well-known hump-shaped pattern of savings over age, as for example found and addressed by Cocco, Gomes, and Maenhout (2005).

The life-cycle hypothesis is still one the most famous and broadly supported theories regarding lifetime saving and consumption. The theory lays the foundation for the notion that wealth plays a crucial role in the consumption and therefore savings decision. Additionally, it describes that this wealth does not only constitute of tangible financial assets, but also incorporates the expected lifetime income. This expected lifetime income is often labeled as 'human capital', which in the simplest case equals the present value of future labor earnings. This basic notion can imply a certain investment strategy over the life cycle, which is nowadays implemented in the so-called life-cycle pension funds and target-date funds, as generally described by Viceira (2007). These funds, based on the earlier discussed principles, are popular investment tools in defined contribution schemes in many countries (Bovenberg & Nijman, 2017). We discuss the implementation of life-cycle investing in more detail in Section 2.2. First, the theoretical findings regarding lifetime consumption and portfolio choice are discussed. These findings form the theoretical foundation for the life-cycle investing principle.

Merton model

As previously mentioned, one of the first theoretical findings in the area of lifetime portfolio and consumption choice was achieved by Merton (1969, 1971) and Samuelson (1969). They derived a closed-form expression for the optimal asset allocation and consumption decision over the lifetime under a certain set of assumptions regarding the financial market, labor income, and the individual himself. This 'Merton model' is now briefly discussed.

First of all, the financial market of the Merton model consist of only one source of uncertainty, equity risk. Therefore, the Merton model embodies an asset menu of a risky stock and a risk-free asset, namely the bank account. The interest rate and inflation are assumed to be deterministic and constant over time, as well as the equity risk premium and the volatility of equity. Prices are therefore treated as given. The dynamics of the stock price are modelled in such a way that the risky asset price is stationary and independently, identically, log-normally distributed. The financial market is assumed to be dynamically complete, which implicates that individuals can participate in continuous trading and rebalancing of their portfolio at all times without having to take into account short selling constraints. The market is also assumed to be frictionless, which implies the absence of transaction costs. The described market in this section is often referred to in the literature as the Black and Scholes financial market.

Secondly, we consider the assumed preferences in the Merton model. An individual is presumed to be maximizing the expected lifetime utility of consumption, as was previously mentioned as part of the life-cycle hypothesis. The individual's preferences are represented by a utility function that belongs to the class of constant relative risk aversion (CRRA). This implies that a rational individual prefers a stable lifestyle, as was also discussed within the life-cycle hypothesis. Section 3.1 elaborates on this class of utility functions and its properties, as this type of utility function is also used as the primary representation of preferences regarding the model development in this thesis. Mainly the assumption on the type of utility function enabled Merton (1969) to find a closed-form solution to the intertemporal investment and consumption problem for CRRA utility (as well as constant absolute risk aversion (CARA) utility).

Thirdly, the assumptions regarding the individual himself are as follows in the Merton model. The lifespan of the individual is assumed to be known ex ante and therefore the time of death is predictable. Consequently, the individual is able to take into account the deterministic investment horizon of his pension wealth. The labor income component is risk-free and constant over time, exogenous with respect to the model. Labor supply is also assumed to be fixed and constant over time.

Under this set of assumptions, both Merton (1969) and Samuelson (1969) derived the theoretical result that the optimal fraction of wealth allocated to the risky stock is constant over the lifetime of the individual, independent of age and thus the investment horizon. However, the absolute level of consumption is both horizon-dependent and dependent on the state variable wealth. Whereas the rate of consumption as fraction of the current wealth (depletion speed of wealth) is only horizon-dependent. The optimal fraction is affected by the market price of equity risk, the volatility of equity and a parameter that indicates the level of risk aversion of the individual, which is further discussed in Section 3.1. This theoretical result implies that a long-term investor (youngster) is just as vulnerable to a relative change in wealth as a short-term investor (senior) regarding relative changes in the rate of consumption. They should therefore hold the same portfolio fraction in risk-bearing assets.

Regarding the solution technique, Merton and Samuelson applied a dynamic programming technique for stochastic optimal control problems to find a closed-form expression for their formulated problem. This technique is discussed in more detail in Chapter 4, though the problem setting is different. As an alternative to this method, Karatzas, Lehoczky, and Shreve (1987) as well as Cox and Huang (1989) developed the so-called martingale method, which maps the dynamic portfolio problem in a static variational problem. Although, this method is only applicable in a complete markets setting, a great variety of papers apply this method to lifetime portfolio and consumption choice problems. In this thesis, the martingale method is primarily used. This method is further discussed for an intertemporal problem setting in Chapter 3, and for a terminal wealth problem setting in Chapter 5.

Intuitively, this result of a constant fraction of wealth invested in risk-bearing assets, implies the following in case human wealth is accounted for. Since the life-cycle hypothesis poses that an individual at the start of his working life possesses relatively more non-tradable human capital than an individual who is about to retire, the tangible financial wealth should be adjusted over the lifetime in order to maintain this optimal constant relative exposure of wealth to the risky asset. When the illiquid human capital is assumed to be deterministic and riskless, it is basically equivalent to a position in the risk-free asset, and the entire risk exposure should come from financial wealth, as described by Bovenberg, Koijen, Nijman, and Teulings (2007). Human capital in this setting is often labeled in the literature as bond-like human capital and therefore attributes bond-like qualities to future labor income. Since human capital declines with age and financial wealth tends to increase over one's career path because part of the human wealth is utilized for saving, the optimal fraction of financial wealth invested in the risk-bearing asset tends to decrease over the lifetime. At the start of the working life, the individual possesses a relatively large alternative income source, namely the expected lifetime income. Therefore, he does rely less on the performance of financial wealth to finance future consumption. This economic intuition is formally proven by Bodie, Merton, and Samuelson (1992), who incorporate a labor supply decision variable in the model, therefore taking into account human wealth. The resulting investment strategy additionally depends on the market value of human capital and state variable financial wealth. Note that this investment strategy is both time-varying and stochastic, as it takes into account the performance of the investments. They evaluate human capital as if it were a certain financial asset that can be replicated, which makes it possible to correctly price human capital. This enables the individual to borrow against this non-tradable asset to obtain the desired optimal exposure, especially in the early stages of the life cycle. Bodie et al. (1992, p. 439) report that under certain sets of benchmark parameters, an individual that finds himself at the start of the working life desires to take on a short position in the risk-free asset between 2.6 and 6.5 times the annual labor income.

Now that the original theory and intuition that form the foundation of the classical life-cycle investment pattern of a decreasing portfolio fraction invested in risky assets is discussed, we address recent relevant extensions of this fundamental basic model. These extensions often imply a relaxation of the very restricting set of assumptions needed in the Merton model in order for the previously reviewed theoretical findings to be valid. These relaxations could serve the purpose of investigating certain effects previously outside of the model scope, or simply reconcile theory with real world observations and attain a model which embodies a more realistic reflection of reality. In recent publications, often multiple combinations of restriction relaxations are proposed and investigated.

Human capital extensions

The assumption of labor income to be completely risk-free and deterministic is very restrictive, even more so in a constant interest rate setting. Economic intuition suggests that for certain groups of individuals, like stock brokers and entrepreneurs, human capital might act more like an equity holding regarding its intrinsic risk factors. Some studies have investigated the correlation of human capital and certain assets, for example Campbell (1996) finds a high correlation between human capital and market returns. However, e.g. Cocco et al. (2005) conclude an insignificant correlation between labor income and stock market returns in their setting. Bodie et al. (1992) also acknowledged the possibility of risky human capital and investigated the impact of assuming stochastic wage income subject to equity risk instead of deterministic wage income. Nonetheless, they assume perfectly hedgeable wage income, therefore keeping intact the assumption of a dynamically complete financial market where all contingent claims can be replicated. Over the years, a number of studies have developed frameworks for determining optimal consumption and portfolio choice over the lifetime incorporating stochastic human capital that exhibits exposure to a number of risk factors in the economy. These studies generally derive closed-form solutions in case of fully spanned human capital risk. The resulting impact on the composition of the life-cycle portfolio varies substantially over the course of these studies, dependent on the investigated model changes. For example, Benzoni, Collin-Dufresne, and Goldstein

(2007) develop a model in which human capital and equity dividends are cointegrated and find a resulting hump-shaped life-cycle portfolio fraction in equity.

Regarding the framework adjustments, a clear split within the literature can be observed when it comes to the assumption of complete versus incomplete markets. In case the stochastic human capital is defined as exhibiting risk exposure that cannot be replicated by a combination of available assets (like uninsurable unemployment), human capital risk is considered to be unspanned. The market consequently becomes incomplete, which implies the absence of a closed-form solution. One has to resort to numerical optimization methods in this case to derive optimal consumption decisions and investment strategies over the lifespan. Viceira (2001) incorporates uninsurable labor income risk in a dynamic model of optimal consumption and portfolio choice and therefore has to rely on approximated solutions to investigate the impact of this assumption and possibilities to partly hedge this risk. Also Cocco et al. (2005) and Cairns et al. (2006) defined part of the labor income risk as non-tradable and perform numerical analysis using simulation to investigate the impact of the portfolio allocation and consumption decision. Munk and Sørensen (2010) acknowledge the fact that previous studies which underline labor income to be partly a risk-free asset, such as Cocco et al. (2005), do not distinguish between short-term risk-free assets and long-term risk-free assets. This might affect portfolio allocations. They develop a more comprehensive model that incorporates stochastic interest rates and in which labor income represents (a combination of) cash and bonds. Labor income also exhibits risk exposure to equity, changes in the term structure of interest rates and unhedgeable labor income risk. Contributing to this specification of labor income, is defining the expected labor income growth as an affine function of the short rate. They consider the impact on lifetime portfolio choice of both the specification of unspanned and fully spanned labor income risk and perform a calibration exercise for the labor income process, which reports no significant effect of the short rate on the expected income growth rate.

This last finding is explicitly mentioned because it is used for validating certain choices regarding the model development later on (Section 4.3). In this thesis, we assume labor income growth not to dependent on the short rate and therefore not to be cash-like. However, we do attend to the theoretical implications of different exposures to the available risk factors under the assumption of fully spanned human capital in Chapter 5.

Alternative utility functions

Correctly capturing a person's preferences has been subject of study for behavioral economists for a long time. As mentioned earlier, the Merton model assumes preferences that exhibit constant relative risk aversion in an expected utility maximizing setup. However, empirical research over the years (partly) rejects this assumption. As a counterpart to the expected utility theory, Kahneman and Tversky (1979) developed the so-called prospect theory. Contrary to expected utility theory, prospect theory takes into account the phenomenon of loss aversion, which states that individuals perceive losses and gains differently; an individual is more sensitive to losses than gains of equal size regarding choice outcomes. This would lead to kinked utility function, of which the impact on lifetime consumption and investment decisions is investigated by e.g. van Bilsen (2015). However, since most of the studies on the subject of lifetime portfolio allocation relevant to the problem description in this thesis apply expected utility theory, the choice is made to retain the expected utility theory framework.

Another feature of preferences observed in practice is habit formation, for which Sundaresan (1989) and Constantinides (1990) provided the model foundation. Habit formation entails the behavior that individuals steer and evaluate consumption relative to a certain reference level. Note that this behavioral feature was also addressed in the relative income hypothesis, posed by Duesenberry (1949). In general, a distinction is made between two kinds of habit formation: internal habit formation, for which the individual's past consumption acts as a reference level, and external habit formation, for which a certain exogenous reference level (e.g. mean consumption of peers) acts as a reference level. The effect of internal habit formation on intertemporal consumption and investment decisions, which among others causes time-inseperable utility of consumption, is extensively studied by van Bilsen (2015). Habit formation can be incorporated in the CRRA utility function, which is applied by e.g. Gomes and Michaelides (2003). In this thesis, we briefly touch upon this subject of habit formation as part of the model development in Chapter 4.

Also within the expected utility theory, there is a variety of different function forms of utility functions that can be implemented to represent preferences. However, the CRRA utility function possesses certain desirable qualities, both with respect to preferences and mathematical representation, which are further discussed in Section 3.1. Since this functional form of utility is applied in the vast majority of studies on lifetime portfolio allocation and consumption choice over the recent years, we elect to implement this class of utility functions in this thesis.

Stochastic interest rates

An important risk faced by a long-term, finite horizon investor is interest rate risk. For example, changes in the interest rate over a typical 40-year career path could materialize in a significant impact on the financial wealth of the individual, as well as the market value of human capital. The term structure of interest rates also impacts the price of an annuity in case pension wealth is converted. This interest rate risk is not acknowledged in the simple Merton model, since interest rates are assumed to be constant over time. Together with the assumption of constant risk premia, this leads to a constant investment opportunity set over the lifetime. Merton (1971) first recognized that changes in the investment opportunity set over time creates a hedge demand besides the standard myopic demand that represents the optimal portfolio allocation in a single-period mean-variance framework, as created by Markowitz (1952). When this changing opportunity set is caused by stochastic variation in interest rates, instruments such as long-term bonds can provide a hedge against adverse shifts in the future investment opportunity set.

With regard to the modeling of interest rate processes, a general distinction can be made between equilibrium models and no-arbitrage models (Hull, 2012). Equilibrium models do not provide an exact fit to the current term structure of interest rates, since this is endogenously determined by the chosen model parameter values. On the contrary, no-arbitrage models are designed to be consistent with the current term structure, which is important in case of derivative valuation. However, since we are interested in scenario analysis over long horizons regarding lifetime portfolio selection, in which today's term structure shape exerts only little influence on the lifetime risks, equilibrium models generally prevail in the literature with respect to this topic.

Several papers have studied the optimal portfolio and consumption choice for long-lived investors in case of a stochastic term structure of interest rates and therefore a stochastic investment opportunity set. The functional form of the term structure of interest rates seems to have a significant impact on the resulting lifetime portfolio allocation, which is researched by Munk and Sørensen (2003). As was mentioned in the previous subsection, Munk and Sørensen (2010) later developed a model with stochastic interest rates and investigate the effect of incorporating risk-bearing labor income on lifetime consumption and portfolio allocation. They specify the stochastic term structure of interest rates by stating that the instantaneous short rate follows a Vasicek (1977) one-factor process, as do Omberg (1999) and Sørensen (1999) when determining optimal dynamic asset allocation under certain sets of assumptions, among which constant market prices of risk. Brennan and Xia (2000) deviate from this specification and study the effects of a term structure governed by a Hull-White (1994) two-factor model. Differences aside, the papers mentioned above all find a certain demand of the individual, depending on the defined framework, to hedge possible changes in the term structure of interest rates. The Vasicek model and its implications are further discussed in Section 3.2.

One of the leading works on modeling lifetime portfolio and consumption choice accounting for both interest rate risk and inflation risk, is developed by Brennan and Xia (2002). They model both the inflation and the real interest rate as a mean-reverting process and again, the risk premia are assumed to be constant. They consider an investor with a finite horizon that can invest in nominal bonds, cash and stocks and find an optimal portfolio allocation that is both horizon-dependent and state-dependent in case of the intertemporal consumption problem. When abstracting away from inflation risk, the real interest rate is also governed by a Vasicek one-factor model. This paper will be used as a stepping stone in this thesis and therefore we adopt this model to represent the stochastic short rate, further described in Section 3.2. The advantage of applying the Vasicek one-factor model is that it captures in a clear and intuitive way the distinctions in interest rate risk exposure of fixed income securities with different underlying maturities. Although the single factor may not capture the entire term structure shifts correctly, in general a large part is explained. The Vasicek one-factor model also possesses the desirable feature that the level of risk aversion of the individual, in the case of CRRA utility, is positively correlated with the bond-stock ratio for all possible investment horizons and bond maturities. On the contrary, this is not always true for its multi-factor counterpart (Brennan & Xia, 2000, p. 203).

Annuities and longevity risk

The part of the individual's life cycle in which research on lifetime portfolio allocation and consumption decisions takes interest in, can generally be divided into the following categories: the accumulation phase, the conversion phase upon retirement, the decumulation phase or a combination of these different phases. The original Merton model does not account for a stochastic time of death or a consumption guarantee, established by an annuity purchase, in any way. Research regarding the conversion phase usually involves investigating the effects of different strategies to obtain certain annuity products. The annuitization of pension wealth in a lifecycle investment framework, is a broadly researched topic with respect to pension contract design. For example, Horneff, Maurer, and Stamos (2008) discuss optimal portfolio choice over the lifespan, facing a number of risks and incorporating a life annuity into the investment opportunity set to hedge against possible longevity risk. Horneff, Maurer, Mitchell, and Dus (2008) also perform a simulation study to evaluate and compare different portfolios of investment-linked, phased withdrawal schemes and life annuity purchases on different time points near the retirement date. They report a utility gain between 25% and 50% for strategies that partly integrate annuity purchases over time, compared to full annuitization at retirement. Note that these works both implement numerical methods. In case of analytical solutions, such as provided by Milevsky and Young (2007) (discussed below), often a constant interest rate is assumed, therefore not accounting for a significant risk factor associated with annuity purchases.

The annuitization at retirement may yield a disappointing retirement benefit stream if the economic conditions are unfavorable at the time of conversion upon retirement. One of the main determinants is the governing interest rate at time of retirement: a relatively high interest rate results in a relatively favorable annuity factor. Therefore, interest rate risk plays a leading role in this annuitization risk and can be appropriately hedged by purchasing long-term nominal and inflation-linked bonds to establish the desired interest rate (and inflation) immunization. Koijen, Nijman, and Werker (2011) analyse this portfolio choice problem to optimally anticipate the annuity risk before retirement, as well as decision making between different types of annuities at the retirement date. They assume a financial market closely related to the framework of Brennan and Xia (2002), where the interest rate model consists of a Vasicek two-factor model, although time-varying risk premia are included. They derive closed-form solutions for optimal annuity purchase at retirement and anticipating on this annuitization decision before retirement by investing appropriately in the bond and equity markets. A more stylized, well-rounded model is proposed by Cairns et al. (2006), who develop a framework in which an investor hedges a minimum guarantee subject to interest rate risk that results from a single annuity product purchase at retirement. The interest rate model in this paper is described as a general one-factor model. They argue a significant welfare gain for DC plan members when instead of a uniform deterministic life-cycle strategy, a stochastic dynamic strategy is adapted based on the economic state and individual characteristics. This framework also incorporates human capital in the form of pension premiums of a stochastic labor income process and addresses a form of habit formation as well. Therefore, the model of Cairns et al. (2006) is used as a basis for the model development regarding the annuity purchase in this thesis, which is further discussed in Chapter 4.

Annuities also provide a natural hedge against longevity risk for the individual retiree. Also further investment risk is decreased in case the retiree decides to keep investing after retirement. Milevsky and Young (2007) focus on the impact of aging and mortality rates on the life annuity purchases of individuals for both annuitization at retirement and annuitizing parts of financial wealth before retirement. Bommier (2010) also analyzes the impact of different stochastic mortality rate models on optimal financial strategies in a life-cycle framework. However, we choose to abstract away from stochastic mortality rate models. In case we take into account mortality, deterministic and predetermined mortality rates for the Dutch population are applied, since the main focus of this thesis is the portfolio allocation strategy.

2.2 Life-cycle strategies in practice for top-up plans

In the previous section, we addressed the scientific foundation from a modern portfolio theory standpoint for age-based and risk-based investing. Within the defined contribution landscape, the mutual funds available for pension contract investments that rest on this foundation, can generally be divided in two main categories: life-cycle funds and life-style funds (Viceira, 2007). Within a life-cycle fund, age (or horizon) is the main determinant for the asset mix based on human capital considerations according to the previously discussed life-cycle hypothesis, whereas in life-style funds horizon effects are neglected and the presumably constant risk appetite of the individual is the main determinant of a constant asset mix. Regarding a typical life-cycle fund, the so-called glide path that represents the change over time of the (crudely defined) stock-bond ratio of the fund, is decreasing and therefore generally resulting in a low-risk portfolio near retirement, as discussed by Viceira (2007, p. 4). This glide path can be interpreted as the manner in which rebalancing of assets takes place in order to materialize a predetermined, horizon-dependent target mix. Hybrids of these life-cycle and life-style funds have also been put into practice over the years in a number of countries, for example a life-cycle fund which reduces the stock-bond ratio over time relatively more for a more risk averse individual. This is often labeled as a 'conservative' life-cycle, which is an option out out of an assortment of available life-cycles differentiated based on the presumed risk aversion of the individual.

In the Netherlands, the third pillar of voluntary pension savings has shown an increasing variety of DC life-cycle products over the last couple of years. According to market research on this topic, conducted by Lane, Clark & Peacock Netherlands (2018), a substantial spread exists on target asset mixes and therefore risk profiles of available life-cycle funds. This is also the case with respect to interest rate risk immunization over the lifetime of the products, for which the reported bandwidth of the investigated funds is 65% on a 10-year horizon. Part of the variety in these asset allocation strategies can be explained by the fact that these funds may strive for different goals. Individuals have been recently granted the right to convert their third pillar accumulated pension wealth in a variable, investment-linked annuity rather than a fixed annuity (*Wet verbeterde premieregeling*, 2016, June 23). Also, the possibility exists to convert only part of the pension wealth into a fixed annuity, or one might want to keep the option of switching between annuity types within retirement. All these specifications result in different exposures to certain risks such as interest rate risk. Then, a particular life-cycle can be designed to hedge against the relevant risks, for which the level of risk coverage is in tune with the risk appetite of the individual in question (Lane, Clark & Peacock Netherlands, 2018, p. 19). Despite of this great variety in products, all the available products exhibit a deterministic glide path over the lifespan. There is no mention of dynamic life-cycles, which adjust to the individual's situation, economic situation and previous returns.

2.2.1 The net pension case

Not belonging to the third pillar within the Dutch pension system, but obliged to be invested in a life-cycle product, is the net pension. As described earlier, the net pension was introduced in 2015 as result of capping the pensionable salary at €100,000 [1] (*Besluit van 11 december 2014 tot wijziging van het Besluit uitvoering Pensioenwet en Wet verplichte beroepspensioenregeling in verband met uitvoering van het nettopensioen en de waarborg voor fiscale hygiëne van het nettopensioen*, 2014, December 11). This net pension scheme is a voluntary retirement benefit scheme regarding the gross salary above the upper limit, with the objective to restore the total accumulation of benefits within the underlying second pillar occupational pension fund best as possible. Similar to DC schemes in the third pillar, the premium contributions to this top-up plan are fiscally limited, dependent on age. However, the fiscal maximum contributions, expressed as a percentage of the pension base, are generally lower for all age categories. The current maximum allowed contributions for the net pension, based on a 3% actuarial rate, can be found in Appendix

[1] In 2018, the upper limit of the pensionable salary is equal to €105,075.

B. In general, participants first pays labor income tax and the remaining net salary above the upper limit acts as a pension base for the net pension. According to the previously mentioned legislation, the net pension needs to be treated like a pure defined contribution scheme, in which the participant carries both the longevity and the investment risk within the accumulation phase (De Nederlandsche Bank, 2017a). However, note that there exist fiscal differences regarding the contributions and benefits.

Since the net pension is lawfully a defined contribution scheme in the accumulation phase, the financial service providers are obliged to take into account the prudent person principle regarding the investment policy for this plan (*Pensioenwet*, 2006, Article 135). The prudent person principle states that the financial well-being of the individuals over the lifetime needs to be the pension provider's main priority, which with regard to defined contribution schemes implies an age-based investment strategy according to the life-cycle principle. This life-cycle investment strategy may be adjusted if the pension provider can prove that the strategy deviation provides an effective cover to certain risks, such as equity risk and interest rate risk (De Nederlandsche Bank, 2016).

Now that the investment strategy obligations in the accumulation phase of the net pension are discussed, the conversion phase of the net pension is considered. Upon retirement, the accrued net pension wealth is converted into a certain pension right in the underlying occupational pension scheme. This buying into the underlying pension scheme can be interpreted as purchasing a fixed life annuity. A net pension participant neither has freedom of choice on which pension fund to buy into, nor is the option present to convert the accrued pension wealth into a variable annuity. The participant is obliged to buy into the specific underlying occupational pension fund (De Nederlandsche Bank, 2017c). Up until 2017, the participant was required to convert the accrued capital into pension payments at a relatively unfavorable annuity rate that includes a surcharge for the underlying pension fund's required solvency. A recent change in legislation replaced this surcharge by the larger of the fund's actual funding ratio at the time of retirement, and the fund's minimum required solvency (*Besluit van 9 januari 2018 tot wijziging van het Besluit uitvoering Pensioenwet en Wet verplichte beroepspensioenregeling vanwege wijziging van het inkooptarief voor nettopensioen*, 2018, January 9). The minimum of this asymmetric funding ratio surcharge for e.g. ABP equalled 104.2% in 2017 (ABP, 2018a, p. 39). This change implies a change in the participant's risk exposures over the life cycle.

The focus of the net pension life-cycle investment strategy in the accumulation phase should therefore be a 'most favorable' conversion of accrued wealth into a particular fixed life annuity at retirement, subject to the described funding ratio surcharge. The life-cycle strategy needs to adequately protect the participant against risks associated with the plan's setup in order to satisfy the prudent person principle requirement. These risks include, amongst others, interest rate risk and equity risk. Also taking into account the nature of the net pension: a top-up plan, affects the desired investment strategy for the participants.

2.2.2 Current net pension life cycle operated by APG

The following pension funds for which APG is the financial service provider, offer a net pension scheme: ABP, bpfBouw, Personeels pensioen fonds APG (PPF APG) and Stichting Pensioenfonds voor de Woningcorporaties (SPW). The currently implemented life-cycle for the net pension of the pension funds under management by APG, is displayed in Figure A.1a of Appendix A. The portfolio is composed of a total of seven asset classes: developed markets equity, emerging markets equity, tactical real estate, commodities, fixed income credits, treasuries and index-linked bonds. These asset pools facilitate market consistent valuation of monthly deposits and withdrawals regarding the net pension contracts. The investment strategy is deterministic and horizon-dependent, which implies the portfolio fractions of the individual's net pension financial wealth are dependent on age. Besides changes in the investment horizon, variation in returns over the asset pools also cause deviations from today's intended allocation and the actual portfolio composition. In order to realize the intended portfolio composition for each individual, monthly rebalancing of the net pension capital takes place. Note that the glide path is uniform for all net pension participants.

In order to facilitate a more straightforward view of the financial market, the original seven asset classes are bundled together into two main categories based on their characteristics, which can be labeled as the fixed income securities category and the equities category. The following division is made: asset classes fixed income credits, treasuries and index-linked bonds are bundled into the general class of fixed income securities, whereas the remaining four classes are labeled as equity. This simplified life-cycle is displayed in Figure A.1b of Appendix A, which shows that the portfolio fraction of equity initially equals 80% at 50 years from retirement. Starting from a 35-year horizon up until retirement, this fraction is then periodically reduced to 30%. Note that an accelerating glide path descent is chosen, whereas this could also have been e.g. linear, to enable profiting for a longer period from the relatively higher yield on equities compared to fixed income securities. This form and composition of the life-cycle was originally based on comparative analysis of similar products on the market.

Chapter 3

Individual defined contribution scheme

In this chapter, a general intertemporal consumption problem with exogenous pension premiums and human capital is discussed, based on the model proposed by Brennan and Xia (2002). Firstly, both the general assumptions and the financial market on which the model rests, are discussed. Then, the different wealth processes of the individual are stated and a set of benchmark parameter values is provided in Section 3.4. Lastly, in Section 3.5, we derive the optimal consumption and investment strategies with respect to both total and financial wealth.

3.1 Assumptions

In this section, the initial assumptions with regard to both the individual and the financial market are discussed. First of all, the assumptions regarding the considered timeline are discussed. We consider an individual that starts his working life at time $t = 0$ and retires at time $t = T_R$. The date of death, $t = T_D$, is fixed and known ex ante. A fixed date of death implies the individual does not face micro longevity risk, which could be considered as either the individual being fully insured or the micro longevity risk being completely pooled away. In this context, we also abstract away from macro longevity risk; the risk of increasing expected lifetimes. This is in line with the problem setup discussed by Brennan and Xia (2002).

Secondly, we consider the components of the individual's human capital. In this chapter, the assumption is made that human capital only consists of a labor income component. The individual's labor income, denoted by Y_t, is considered risk-free and equal to zero once in retirement. Therefore, we do not consider risks associated with labor income, such as unemployment risk and disability risk. In other words, labor demand is abundant and the individual stays in good health. There are no discontinuities in salary payments. Since the salary payments are guaranteed throughout the career path, the labor income is bond-like.

Regarding the preferences of the individual, the following assumptions are made. The individual maximizes the expected lifetime utility by choosing optimal consumption over the lifetime and the optimal investment strategy of wealth that finances this optimal consumption choice. The individual has no bequest motive and therefore depletes his wealth completely upon time of death. We assume that the individual's preferences exhibit (positive) constant relative risk aversion (CRRA). This is incorporated by the following time-separable isoelastic utility function:

$$u(c_t) = \begin{cases} \frac{1}{1-\gamma} c_t^{1-\gamma} & \text{if } \gamma \in (0, \infty) \backslash \{1\} \\ \log c_t & \text{if } \gamma = 1 \end{cases}, \tag{3.1.1}$$

where γ denotes the coefficient of relative risk aversion. This relative risk aversion parameter can be interpreted as an appreciation measure of a stable consumption stream over both economic states and time. For example, a risk loving individual values a stable consumption stream over time and economic states less than a highly risk averse individual. The inverse of γ can be seen as the elasticity of intertemporal substitution, since lifetime consumption problem is in essence an intertemporal choice problem. The coefficient of relative risk aversion corresponding to a certain utility function, is also known as the Arrow-Pratt measure of relative risk aversion

$$R(c_t) = c_t A(c_t) = -c_t \frac{u''(c_t)}{u'(c_t)}, \tag{3.1.2}$$

where $A(c_t)$ is the Arrow-Pratt measure of absolute risk aversion (1964), $u'(c) = \frac{\mathrm{d}u(c)}{\mathrm{d}c}$ and $u''(c) = \frac{\mathrm{d}^2 u(c)}{\mathrm{d}c^2}$. If we determine $R(c_t)$ for the isoelastic utility function stated in (3.1.1), we see that indeed $R(c_t) = \gamma$. Note that the utility function (3.1.1) has some desirable properties. First of all, positive marginal utility, which implies the trivial intuition that a higher consumption level yields a strictly higher utility. Secondly, diminishing marginal utility, which implies the function $u(c_t)$ is concave in c_t and hence embodies the principle that the individual prefers a fixed amount over a fair lottery. This is a desirable property because a less risky equivalent payoff will yield a higher utility, which implies the risk adversity of the individual.

The individual is also presumed to exhibit impatience regarding consumption. Time preference is generally displayed as a discount function with respect to utility of future consumption. Consequently, a consumption unit today yields a higher utility compared to the utility of the same consumption unit in the future. The rate of time preference is represented by δ and assumed to be positive. Now that the general assumptions on the individual's life and preferences have been discussed, we consider the risks to which the individual is exposed in the financial market.

3.2 Financial market description

In this section the financial market in which the individual operates, is described. Both the investment opportunity set; the asset menu over which pension wealth is allocated, and the risk factors driving the financial market are addressed. The considered market is assumed to be complete and arbitrage opportunities are absent.

We consider a financial market where both equity risk and interest rate risk are present. The uncertainty in the financial market that originates from those risks, is described by a two-dimensional Brownian motion $\boldsymbol{Z}^\top = (Z^S, Z^r)$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The standard filtration of $\boldsymbol{Z}$ is denoted by $\mathfrak{F} = \{\mathcal{F} : t \geq 0\}$, with respect to which all processes appearing in this thesis are assumed to be progressively measurable ($\mathfrak{F}_t$- adapted). The formal definition of the employed Brownian motion, which serves as the basic building block of other processes, is compliant with the standard convention. Inflation is considered deterministic and equal to zero, such that the nominal interest rate and the real interest rate coincide. The investment opportunity set therefore consists of three assets: an instantaneous risk-free bank account (B_t), a nominal zero-coupon bond (P_t^h) with time to maturity $t+h$ (constant bond horizon h) and a stock price index (S_t).

As was mentioned before, we model the interest rate risk as in the Vasicek (1977) one-factor model and therefore describing the instantaneous interest rate as a mean reverting Ornstein-Uhlenbeck process:

$$\mathrm{d}r_t = \kappa(\bar{r} - r_t)\,\mathrm{d}t + \sigma_r\,\mathrm{d}Z_t^r, \tag{3.2.1}$$

where κ is the speed of mean reversion, $\bar{r}$ the long-term (equilibrium) mean interest rate level and σ_r represents the volatility of the instantaneous interest rate. The term $\mathrm{d}Z_t^r$ represents a standard Brownian motion that drives the interest rate process, which implies $\mathrm{d}Z_t^r \sim \mathcal{N}(0, \mathrm{d}t)$. Note that because of the normally distributed Brownian motion, future instantaneous interest rates are also normally distributed. The associated conditional expectation of a future instantaneous interest rate can be expressed as

$$\begin{aligned}\mathbb{E}_t[r_{t+s}] &= \mathbb{E}_t\left[r_t + (\bar{r} - r_t)(1 - e^{-\kappa s}) + \sigma_r \int_t^{t+s} e^{-\kappa(t+s-u)}\,\mathrm{d}Z_u^r\right] \\ &= e^{-\kappa s} r_t + (1 - e^{-\kappa s})\bar{r}, \quad s \geq 0.\end{aligned} \tag{3.2.2}$$

We can observe from (3.2.2) that both the current interest rate level r_t and the long-run level $\bar{r}$ affect the future interest rate, where the influence of level r_t as well as the shock at time t decline over time. Using a similar methodology, the conditional variance of a future instantaneous interest rate can be expressed as

$$\begin{aligned}\mathbb{V}_t\left[r_{t+s}\right] &= \frac{1}{2\kappa}\sigma_r^2 e^{-2\kappa(t+s)}\left(e^{2\kappa(t+s)} - e^{2\kappa t}\right) \\ &= \frac{\sigma_r^2}{2\kappa}\left(1 - e^{-2\kappa s}\right), \quad s \geq 0.\end{aligned} \tag{3.2.3}$$

The derivation of (3.2.2) and (3.2.3) can be found in part C of the Appendix and rests on the well-known Itô's lemma as well as the properties of a stochastic integral, among which the Itô isometry property (Itô, 1944). Note that for $s \to \infty$, the conditional mean is equal to the long-run mean $\bar{r}$ and the conditional variance approaches $\frac{\sigma_r^2}{2\kappa}$. A similar observation could be made with regard to the speed of mean reversion: for the same horizon s, the conditional expectation becomes closer to the long-run level as κ increases. Also the conditional variance is decreasing in κ.

The stock price index process is modelled as a geometric Brownian motion:

$$\begin{aligned}\frac{\mathrm{d}S_t}{S_t} &= \mu_t\,\mathrm{d}t + \sigma_S\,\mathrm{d}Z_t^S \\ &= (r_t + \lambda_S\sigma_S)\,\mathrm{d}t + \sigma_S\,\mathrm{d}Z_t^S,\end{aligned} \tag{3.2.4}$$

where the drift μ_t is the expected return of the stock price index and the diffusion coefficient σ_S is the volatility of the stock price index. For the second equality in (3.2.4) the definition of the Sharpe ratio (Sharpe, 1966):

$$\lambda_S = \frac{\mu_t - r_t}{\sigma_S}, \tag{3.2.5}$$

is used. The Sharpe ratio, λ_S, represents the market price of equity risk as it denotes the instantaneous excess rate of return per unit of risk. Note that the expression for λ_S is related to the general arbitrage pricing theory, developed by Ross (1976), as well as risk neutral valuation, since it represents the change of drift in the underlying Brownian motion when changing from real-world measure $\mathbb{P}$ to risk-neutral measure $\mathbb{Q}$. It can be seen from (3.2.4) that we assume the volatility to be constant over time and there are no dividends on equity. Z_t^S denotes a standard Brownian motion that drives the stock price process. We assume $\mathrm{d}Z_t^r$ and $\mathrm{d}Z_t^S$ to be independent, which results in uncorrelated stock price and instantaneous interest rate processes in the above specifications. In practice, this assumption of uncorrelated processes can be easily relaxed by substituting $\mathrm{d}Z_t^S$ in (3.2.4) with

$$\mathrm{d}\hat{Z}_t^S = \rho_{rS}\,\mathrm{d}Z_t^r + \sqrt{1-\rho_{rS}^2}\,\mathrm{d}Z_t^S, \tag{3.2.6}$$

where $\rho_{rS} \in [-1, 1]$ now represents the correlation coefficient of the two processes. In this case $(\mathrm{d}Z_t^r, \mathrm{d}\hat{Z}_t^S)$ is a bivariate Brownian motion with unit variances and correlation coefficient ρ_{rS}. By using Itô's formula and the dynamics of S_t in (3.2.4) ,we are able to derive the closed-form expression for S_{t+s} at time t, which is stated in (3.2.7).

$$S_{t+s} = S_t \exp\left\{\int_t^{t+s} \nu_u\,\mathrm{d}u + \sigma_S \int_t^{t+s} \mathrm{d}Z_u^S\right\},\ \ s \geq 0, \tag{3.2.7}$$

where $\nu_u = \mu_u - \frac{1}{2}\sigma_S^2$. The derivation of (3.2.7) can be found in Appendix D.

Next, we consider the pricing kernel of the economy, also often known as the stochastic discount factor. The pricing kernel for an arbitrary horizon $s \geq 0$, can be explicitly expressed in the following way, in accordance with Brennan and Xia (2002):

$$\begin{aligned}\frac{M_{t+s}}{M_t} &= \exp\left\{-\int_t^{t+s}\left(r_u + \frac{1}{2}\|\boldsymbol{\lambda}\|^2\right)\mathrm{d}u - \int_t^{t+s}\boldsymbol{\lambda}'\,\mathrm{d}\mathbf{Z}_u\right\}, \quad \text{with}\\ \boldsymbol{\lambda} &= \begin{pmatrix}\lambda_S\\ \lambda_r\end{pmatrix}, \mathrm{d}\mathbf{Z}_t = \begin{pmatrix}\mathrm{d}Z_t^S\\ \mathrm{d}Z_t^r\end{pmatrix},\end{aligned} \tag{3.2.8}$$

where $\|\cdot\|$ represents the Euclidean vector norm. The market price of interest rate risk and the market price of equity risk are given by respectively λ_r and λ_S. One can interpret (3.2.8) as the financial deflator at time t for a cash flow at maturity date $t+s$. This expression follows from Girsanov's Theorem, as can be seen in the derivation of (3.2.8) in Appendix E, where we take a similar approach as Hainaut and Deelstra (2011). The pricing kernel is uniquely defined, since we assumed a complete market setting, where every contingent claim can be replicated. By using Itô's lemma, the dynamics of M_t, that satisfy the closed-form expression (3.2.8), can be derived:

$$\frac{\mathrm{d}M_t}{M_t} = -r_t\,\mathrm{d}t - \boldsymbol{\lambda}'\,\mathrm{d}\mathbf{Z}_t, \quad M_0 = 1. \tag{3.2.9}$$

We refer to the Appendix F for the explicit derivation of (3.2.9).

Now that the stochastic discount factor is defined, the bond price at time t can be expressed as the conditional expectation of this discount factor multiplied by the payout. Since the nominal bond in our asset menu is a zero-coupon bond with constant (remaining) maturity h, the payout solely equals one at the maturity date. Consequently, the bond price can be expressed as

$$P_t^h = \mathbb{E}_t\left[\frac{M_{t+h}}{M_t}\right], \tag{3.2.10}$$

where the pricing kernel M_t is given in (3.2.8). Note that the time t-governing term structure of interest rates is used to price the bond with maturity h at time t. In Appendix G, a general functional form of this conditional expectation is derived, which results in the following expression for the nominal bond price:

$$P_t^h = e^{-a(h)-D(h)r_t}, \quad \text{with} \tag{3.2.11}$$

$$a(h) \equiv \left(\bar{r} - \frac{\sigma_r \lambda_r}{\kappa}\right)(h - D(h)) - \frac{\sigma_r^2}{\kappa^2}\left(\frac{1}{2}h - D(h) + \frac{1}{4}D(2h)\right), \tag{3.2.12}$$

$$D(h) \equiv \frac{1}{\kappa}\left(1 - e^{-\kappa h}\right). \tag{3.2.13}$$

It can be observed from (3.2.11) that the current instantaneous interest rate directly affects the bond price P_t^h and letting h vary, enables us to define the complete yield curve at time t. We interpret $D(h)$ as the interest rate duration or sensitivity of the bond, since

$$\frac{1}{P_t^h}\frac{\partial P_t^h}{\partial r_t} = -D(h).$$

This duration definition does deviate from the classical definition of duration, since this would be equal to the maturity h for a zero-coupon bond (Luenberger, 2013, p. 57). Note that zero-coupon bonds with different maturities are all perfectly correlated to an interest rate shock. The interest rate duration $D(h)$ is declining in the speed of mean reversion κ and increases with bond horizon h, conform expectation. Using again Itô's formula, we can derive the following dynamics for the rolling bond price (for the complete derivation, see Appendix H):

$$\frac{\mathrm{d}P_t^h}{P_t^h} = (r_t - \sigma_r D(h)\lambda_r)\,\mathrm{d}t - \sigma_r D(h)\,\mathrm{d}Z_t^r, \tag{3.2.14}$$

where the bond price volatility equals $-\sigma_r D(h) < 0$. Consequently, the bond price decreases in case of an upward interest rate shock and vice versa, as expected. The bond risk premium equals $-\lambda_r \sigma_r D(h)$, based on the Sharpe ratio. Because one is exposed to interest rate risk when holding fixed income securities, the bond risk premium should be positive, which results in negative estimates for λ_r when the model is calibrated properly, as can be seen e.g. in the calibration of a more comprehensive model by Koijen et al. (2011) and the capital market model for the Netherlands estimated by Draper (2012). Furthermore, note that the bond risk premium increases with fixed maturity h, which implies relatively higher bond risk premia for long-term bonds compared to short-term bonds. A final remark on (3.2.14) is that these dynamics concern a rolling zero-coupon bond of constant maturity h, which represents the available zero-coupon bond continuously rebalanced over time to keep constant maturity h.

Lastly, the growth of the locally risk-free asset, namely the risk-free bank account (B_t), can be described as follows:

$$B_t = B_0 \exp\left\{\int_0^t r_u\,\mathrm{d}u\right\}, \tag{3.2.15}$$

since the bank account is continuously compounded and earns the instantaneous risk-free interest rate. Using the Leibniz integral rule, it can be easily shown that the explicit form of B_t in (3.2.15) satisfies the following ordinary differential equation:

$$\frac{\mathrm{d}B_t}{B_t} = r_t\,\mathrm{d}t. \tag{3.2.16}$$

Note that the dynamics stated in (3.2.16) also follow from $\lim_{h\downarrow 0}\frac{\mathrm{d}P_t^h}{P_t^h}$ and r_t can therefore also be interpreted as the short rate at time t.

3.3 Wealth processes

Now that the financial market, its risk factors and the available assets have been discussed, we are able to address the individual's wealth process. Firstly, the total wealth of the individual and its dynamics is discussed. Secondly, the components that are encapsulated by the total wealth are clarified. Since there exist three assets in our specified financial market, the value of wealth at time t can be expressed as:

$$W_t = \eta_t^S S_t + \eta_t^P P_t^h + \eta_t^B B_t, \tag{3.3.1}$$

where $\boldsymbol{\eta}_t^\top = (\eta_t^S, \eta_t^P, \eta_t^B)$ denote the number of the different assets held at time t. Consequently, using among others (3.2.4), (3.2.14) and (3.2.16), the dynamics of wealth can be stated as follows (see Appendix I for the full derivation):

$$\mathrm{d}W_t = \left(r_t + \boldsymbol{\omega}_t'\boldsymbol{\Sigma}\boldsymbol{\lambda}\right) W_t\,\mathrm{d}t + W_t\boldsymbol{\omega}_t'\boldsymbol{\Sigma}\,\mathrm{d}\boldsymbol{Z}_t - c_t\,\mathrm{d}t, \tag{3.3.2}$$

with

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_S & 0 \\ 0 & -\sigma_r D(h) \end{pmatrix}, \quad \boldsymbol{\omega}_t = \begin{pmatrix} \omega_t^S \\ \omega_t^P \end{pmatrix},$$

where ω_t^S (ω_t^P) denotes the fraction of wealth invested in the stock price index (nominal zero-coupon bond) at time t. Therefore, the fraction of wealth allocated to the risk-free bank account at time t (ω_t^B) is equal to $1 - \boldsymbol{\omega}_t^\top \boldsymbol{\iota}$. The rate of consumption at time t is denoted as c_t in (3.3.2) and we can see that consumption indeed depletes the individual's wealth. The dynamics of wealth (3.3.2) can be interpreted as the dynamic budget constraint regarding individual lifetime investment and consumption problems.

As was discussed before, the life-cycle principle is based on the fact that one possesses a certain amount of human capital besides financial wealth. Therefore, wealth W_t defined above can be interpreted as total wealth and constitutes of both human capital (H_t) and financial wealth (F_t), similar to e.g. Bodie et al. (1992), Cairns et al. (2006), and Munk and Sørensen (2010):

$$W_t = H_t + F_t. \tag{3.3.3}$$

At the start of the working life, financial wealth is assumed to be zero, $F_0 = 0$, which results in $W_0 = H_0$. Note that for now, we operate under the assumption that the individual's human capital solely consists of a labor income component (see Section 3.1). Then, human capital is equal to the sum of discounted future salaries. As the career of the individual progresses, his human capital is converted into financial wealth and is fully depleted at time of retirement, $H_{T_R} = 0$, which results in $\{W_t = F_t : t \geq T_R\}$. The individual was assumed to lack bequest motive and therefore we have $W_{T_D} = 0$. These human, financial and total wealth processes are graphically displayed in Figure 3.1a, for a set of benchmark parameter values chosen in Section 3.4 and the optimal consumption and investment strategy implemented.

However, in case of exogenous mandatory pension contributions in the accumulation phase, human capital can be seen as the present value of the future pension contributions rather than labor income (Gollier, 2005, 2008). Complementary, the part of the labor income which is not applied for pension accumulation, is interpreted as exogenous consumption. Note that the individual is not able to adjust the pension premiums (and therefore consumption) based on the actual investment returns and economic state of the market. We assumed labor income to be bond-like and therefore can be seen as a certain zero-coupon bond portfolio composed of bonds with different maturities. Consequently, human capital at time t is expressed as

$$H_t = \int_0^{T_D-t} \mathbb{E}_t \left[\frac{M_{t+h}}{M_t} K(t+h) \right] \mathrm{d}h = \int_0^{T_D-t} P_t^h K(t+h) \, \mathrm{d}h, \tag{3.3.4}$$

where M_t is the stochastic discount factor as stated in (3.2.8) and $K(u)$ equals the deterministic fixed pension contribution at time u, which can be expressed as

$$K(u) = \begin{cases} p_u Y_u & \text{for } u \in [0, T_R) \\ 0 & \text{for } u \in [T_R, T_D] \end{cases}. \tag{3.3.5}$$

Y_t is the labor income of the individual at time t and p_t the fixed premium as a percentage of labor income at time t. We discuss the choice of benchmark parameter values for these quantities in the next section.

3.4 Benchmark parameters and simulation method

In order to be able to simulate the previously defined continuous-time stochastic processes, we need to employ a method to discretize these processes. There are several different ways to establish a discretization of a continuous-time proces, of which we mainly apply the Euler approximation scheme. Although the Euler method is a rather crude approximation scheme, it seems sufficiently accurate for our model setup. So, suppose we have the following general stochastic differential equation for quantity X_t, with drift $\mu(t, X_t)$ and diffusion coefficient $\sigma(t, X_t)$:

$$\mathrm{d}X_t = \mu(t, X_t) \, \mathrm{d}t + \sigma(t, X_t) \, \mathrm{d}Z_t. \tag{3.4.1}$$

Then, for a small time step $\Delta t = \frac{T}{N}$, with T the terminal time and N the number of time steps to be taken, we can express (3.4.1) as follows:

$$X_{j+1} = X_j + \mu(t_j, X_j)\Delta t + \sigma(t_j, X_j)\sqrt{\Delta t} U_j, \qquad \text{for } j = 0, 1, ..., N-1, \tag{3.4.2}$$

with known X_0, $t_j = j\Delta t$ and $U_j \sim \mathcal{N}(0, 1)$. However, for a geometric Brownian motion, the Euler scheme produces an avoidable approximation error of X_t. This error can be asymptotically nullified by first simulating the log transformation of X_t and then transforming the log realizations back in order to obtain realizations of X_t. Note that also the wealth process (3.3.2) can be written as a Brownian motion, therefore W_t is simulated via a log transformation. The instantaneous short rate process (3.2.1) however is not specified as a Brownian motion. In order to obtain a numerical simulation of r_t that is valid for any Δt, the following exact formula, proposed by Gillespie (1996), is applied:

$$r_{j+1} = e^{-\kappa \Delta t} r_j + (1 - e^{-\kappa \Delta t})\bar{r} + \sigma_r \sqrt{\frac{(1 - e^{-2\kappa \Delta t})}{2\kappa}} U_j, \qquad \text{for } j = 0, 1, ..., N-1, \tag{3.4.3}$$

with known r_0. Note that this method is based on the closed-form expression of r_t derived in Appendix C. Regular, non-stochastic integrals are approximated as a corresponding left Riemann-Stieltjes sum for the same value of N, therefore evaluated at $\{t_j : j = 0, 1, ..., N-1\}$.

Now that the simulation method is described, the benchmark parameter values for the individual's life-cycle model are discussed. The figures and results displayed in this chapter will be based on these parameter values. We assume the individual starts working at age 25 ($t = 0$), up until time of retirement at age 68 ($T_R = 43$). The deterministic date of death is chosen at age 87, which implies $T_D = 62$. During his woking life, he receives labor income, which is assumed to be constant and normalized to one for convenience's sake. Hence, $Y_t \equiv 1$ for $0 < t < T_R$. The premiums p_t are assumed to be equal to the fiscal maximum contribution for regular defined contribution plans in the Netherlands, according to *Loonheffingen, inkomstenbelasting. Pensioenen; beschikbare-premieregelingen en premie en kapitaalovereenkomsten en nettopensioenregelingen* (2017, November 23), as discussed in Section 2.2.1. These age-dependent percentages of the pension base are displayed in the right column of Table B.1 in Appendix B. These percentages are based on an average yearly pension accrual rate of 1.875%, without taking into account spouse pension or disability insurance.

Regarding the parameters that represent behavioral trades and preferences of the individual, a constant relative risk aversion $\gamma = 5$ is chosen as a default parameter, since this value generally represents a moderate risk averse individual. Consequently, a number of related papers apply this value for the risk aversion parameter as a benchmark, amongst which Brennan and Xia (2002), Koijen et al. (2011) and Milevsky and Young (2007). The subjective rate of time preference is taken as $\delta = -\log 0.97$, in accordance with e.g. Horneff, Maurer, Mitchell, and Dus (2008).

When electing default parameter values for the financial market described in Section 3.2, the following line of thought is pursued. In case a uniform life-cycle design is applied for a defined contribution pension product in which the individual participant cannot exert any influence on this predetermined investment strategy, the risk appetite of the participating group needs to be coherent with the risk appetite implied by the implemented investment strategy (De Nederlandsche Bank, 2017b). This risk appetite can be interpreted as the level of risk aversion (γ) in our framework and forms a link between desired pension result and the investment strategy used to accomplish this goal. The manner in which the risk appetite is reported in practice, entails a maximum acceptable deviation between the pension result in the expected scenario and the pension result in a 'bad weather' economic scenario. To check whether the currently applied investment strategy is still coherent with the established participants' risk appetite, the investment strategy is evaluated based on a quarterly updated uniform scenario set, for which certain parameter regulations are in effect (*Besluit financieel toetsingskader pensioenfondsen*, 2018, July 1, Article 23a). The applied scenario set is based on the financial market framework as proposed by Koijen, Nijman, and Werker (2010) and further discussed an calibrated by Draper (2012). The financial market model developed by Koijen et al. (2010) contains both equities, inflation-linked and nominal bonds, for which the nominal instantaneous short rate is governed by a Vasicek two-factor model. Additionally, time-varying risk premia are included in the model. The last available scenario set is with respect to the fourth quarter of 2018 and consists of 2,000 scenario paths for the state variables over a 60-year horizon, subject to yearly intervals (De Nederlandsche Bank, 2018). We refer to the scenario set for a data description. Because this scenario set reflects the latest view on the financial market of policy makers and panels of academics and professionals when addressing pension related products, we elect to use this set to obtain default parameter values rather than perform a calibration ourselves. However, since the model used to generate the scenario set differs from our financial market model, we still need to perform an estimation. Note that the estimation is performed in a simplistic, naive manner.

For the equity price process S_t, the excess return is determined and using the observed volatility, the Sharpe ratio λ_S can be determined averaged over the scenarios, which fluctuates over the 60-year horizon due to time-varying risk premia. Since we assumed constant risk premia, the average over the 60 years is taken as a benchmark value. The volatility of the stock price is approximated by calculating the standard deviation per scenario and taking the average over the scenarios.

For the instantaneous short rate process r_t, benchmark parameter values are obtained in the following way. The Ornstein-Uhlenbeck dynamics can be approximated by an AR(1)-process resulting from applying the exact scheme (3.4.3), taking $\Delta t = 1$ to match the data frequency. We perform a simple ordinary least squares regression to fit an AR(1)-model to the data of R_t^0 in each scenario. R_t^0 represents the instantaneous nominal interest rate as described in the model by Koijen et al. (2010). The AR(1)-model is described as

$$R_t^0 = \alpha + \beta R_{t-1}^0 + \sigma_r \sqrt{\frac{1 - e^{-2\kappa}}{2\kappa}}\, \epsilon_t, \qquad \text{for } 1 < t < 61, \tag{3.4.4}$$

where ϵ_t is a standard normally distributed error term. After transforming the coefficients to obtain estimates for κ and $\bar{r}$ in each scenario, the average of these estimates over the scenarios is taken as benchmark parameter values. Straightforward calculation yields $\bar{r} = \alpha/(1-\beta)$ and $\kappa = -\log\beta$ in each scenario path. Also within in each scenario, the standard deviation of the residuals is determined and scaled by the factor $\sqrt{2\kappa/(1-e^{-2\kappa})}$. Then, the average over the scenarios provides an estimate for the interest rate volatility σ_r.

A zero-coupon bond maturity of $h = 10$ years is chosen as available asset in our financial market. This is an arbitrary choice, which does not influence the resulting optimal asset allocation in bonds or optimal consumption decision, as follows from Section 3.5. Only the duration of the bond $D(h)$ is of importance. In case a higher maturity is chosen, the optimal asset allocation will decrease, since a lower bond position already satisfies the required portfolio duration. In a similar fashion as the Sharpe ratio for stocks, the Sharpe ratio for this bond is determined as average over time over all scenario's. The resulting benchmark parameter values are displayed in Table 3.1.

The explicit starting values for the state variable processes are as follows. We assume the interest rate to start from an equilibrium state, therefore taking $r_0 = \bar{r}$. The equity price index is assumed to start at one, therefore $S_0 = 1$.

parameter	**symbol**	**value**
long-run mean interest rate	$\bar{r}$	0.025
speed of mean reversion	κ	0.160
interest rate volatility	σ_r	0.013
market price of interest rate risk	λ_r	-0.164
market price of equity risk	λ_S	0.250
stock price volatility	σ_S	0.168
zero-coupon bond maturity (years)	h	10

Table 3.1: Benchmark parameter values financial market

The benchmark parameters in Table 3.1 result in an excess return on the available bond of $\lambda_r \sigma_r D(h) \approx 0.011$ and an excess return on equity of $\lambda_S \sigma_S = 0.042$. The relatively low speed of mean reversion implies a half-life of approximately 4.3 years with respect to the interest rate process. Note that in general, it is considered quite difficult to obtain an accurate estimate for the speed of mean reversion.

3.5 Optimal consumption and investment decision

In this section, the optimal consumption and investment decision is derived for the assumptions and financial market stated in the previous sections in this chapter. In order to accomplish this, the martingale method is employed to find a solution to this intertemporal consumption problem, in accordance with Brennan and Xia (2002). This martingale method is originally developed by Cox and Huang (1989) as an alternative to the dynamic programming method traditionally used for stochastic optimal control problems. The martingale method is often perceived as more flexible to work with, although it can only be applied within a complete markets setting. This section is set up in the following way. Firstly, we state the actual maximization problem. Secondly, we derive the optimal consumption decision for every economic state and every time t. Then, we focus on the portfolio choices that need to be made in every state at every time to finance the earlier derived optimal consumption strategy. The investment strategy is a result from the principles of individual budget balance and stochastic replication strategies. Lastly, the human capital component is taken into account, which results in the optimal investment strategy of actual financial wealth of the individual. Additionally, we analyze the impact of including a guaranteed pension component in human capital besides the labor income component.

3.5.1 Maximization problem

Recall that in Section 3.1 the individual was assumed to be maximizing the expected power utility of consumption over the lifetime. As was also discussed, the consumption in the accumulation phase is exogenously determined, since it is the complement of the mandatory fixed pension contributions before retirement. Therefore, in the determination of the initial total wealth W_0 (discounted future fixed premiums) , the exogenous fixed consumption is implicitly included and can be seen as equal to zero within the wealth process development until retirement: $\{c_t \equiv 0 : t < T_R\}$. Therefore, the dynamic budget constraint (3.3.2) for $t < T_R$ becomes

$$\mathrm{d}W_t = \left(r_t + \boldsymbol{\omega}_t' \boldsymbol{\Sigma} \boldsymbol{\lambda}\right) W_t \,\mathrm{d}t + W_t \boldsymbol{\omega}_t' \boldsymbol{\Sigma} \,\mathrm{d}\boldsymbol{Z}_t. \tag{3.5.1}$$

However, note that the actual consumption in the accumulation phase is not equal to zero. The martingale method relies on the replacement of the dynamic budget constraint by an equivalent static budget constraint, which can be interpreted as only maximizing over consumption strategies that are budget-feasible: the current market value of the consumption strategy must be less than or equal to the available wealth. Therefore, the domain of consumption strategies is restricted to only admissible strategies by implementing the static budget constraint and the maximization problem reduces to only the consumption decision variable. We elaborate more on the martingale method in a terminal wealth problem setting in Chapter 5. The optimal static maximization problem can now be written as:

$$\max_{\{c_s:\, T_R \leq s \leq T_D\}} \mathbb{E}_0 \left[\int_{T_R}^{T_D} e^{-\delta s} \frac{c_s^{1-\gamma}}{1-\gamma} \, \mathrm{d}s \right] \quad \text{subject to} \tag{3.5.2}$$

$$V_0 \equiv \mathbb{E}_0 \left[\int_{T_R}^{T_D} M_s c_s \, \mathrm{d}s \right] \leq W_0, \tag{3.5.3}$$

where

$$W_0 = \mathbb{E}_0 \left[\int_0^{T_D} M_s K(s) \, \mathrm{d}s \right] = \int_0^{T_R} P_0^s p_s Y_s \, \mathrm{d}s. \tag{3.5.4}$$

The value of W_0 is stated as described in Section 3.3. In line with the previous sections, p_t denotes the fixed premium percentage, Y_t is the labor income rate and M_t represents the pricing kernel at time t, as stated in (3.2.8). In equation (3.5.3), V_0 stands for the value of the liabilities at time 0. The parameter δ represents the subjective rate of time preference of consumption, which is an addition to the model posed by Brennan and Xia (2002). Next, we focus on solving this stated maximization problem.

3.5.2 Optimal consumption decision

The Lagrangian function corresponding to the above maximization problem can be given as:

$$\mathcal{L}(c_s, \varphi) = \mathbb{E}_0 \left[\int_{T_R}^{T_D} e^{-\delta s} \frac{c_s^{1-\gamma}}{1-\gamma} \, \mathrm{d}s \right] - \varphi \left(\mathbb{E}_0 \left[\int_{T_R}^{T_D} M_s c_s \, \mathrm{d}s \right] - W_0 \right), \tag{3.5.5}$$

where φ represents the Langrange multiplier. Taking the first-order derivative of (3.5.5) with respect to c_t for each state of the economy, the first-order derivative with respect to φ, equating those to zero and solving for c_t, eventually yields the following optimal consumption strategy over the entire lifetime:

$$c_s^* = \begin{cases} (1-p_s) Y_s & \text{for} \quad 0 < s \leq T_R \\ e^{-\delta s/\gamma} M_s^{-1/\gamma} \left(\int_{T_R}^{T_D} e^{-\delta s/\gamma} \Theta\left(0, s, 1 - 1/\gamma\right) \mathrm{d}s \right)^{-1} W_0 & \text{for } T_R < s \leq T_D \end{cases}, \tag{3.5.6}$$

where

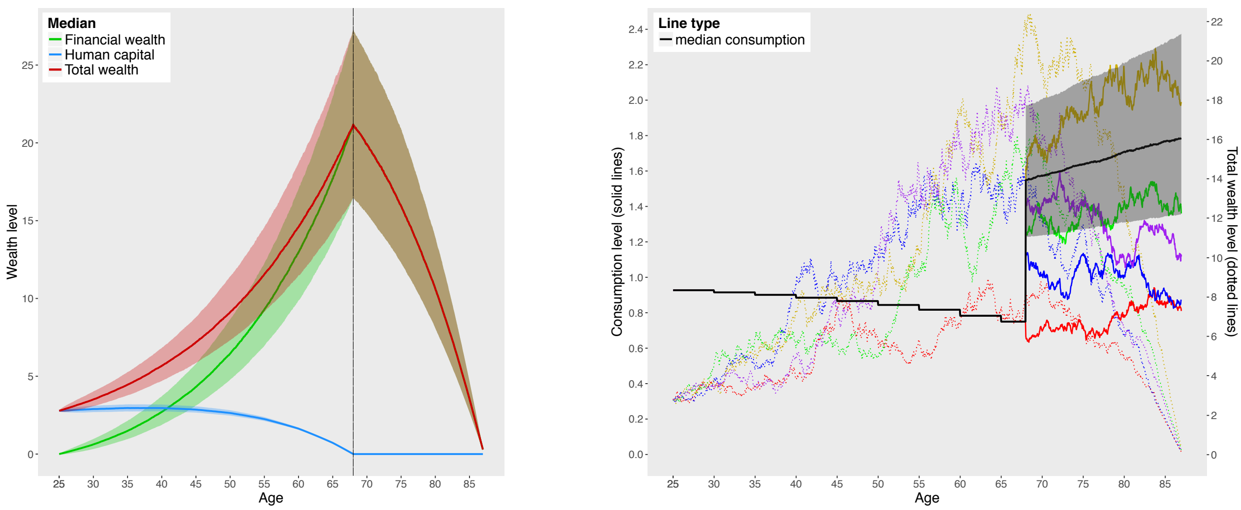
$$\Theta(t, s, x) \equiv \mathbb{E}_t \left[\left(\frac{M_s}{M_t} \right)^x \right], \qquad t < s < T_D, \tag{3.5.7}$$

for which an explicit expression, dependent on r_t, is derived in Appendix G. The full derivation of (3.5.6) can be found in Appendix J. Optimal consumption with regard to the accumulation phase, $\{c_s^* : 0 < s < T_R\}$, is exogenous and equal to the part of labor income that is not utilized for paying fixed mandatory pension premiums. The optimal stochastic consumption path within retirement follows from the actual derivation provided in this section.

From the above equation (3.5.6), we can observe that a potential stochastic labor income component in human capital instead of risk-free labor income, does not affect the endogenous optimal consumption path within retirement. Only the consumption level itself would be affected through the component W_0, in which the expectation of Y_t is present. However, The exogenous consumption in the accumulation phase, is affected through fluctuations in Y_t. Also, adding components to human capital would only affect the consumption level through W_0.

Now that the optimal consumption strategy is known, we are able to simulate the optimal wealth processes and consumption decisions under the chosen benchmark parameter set, for which the medians are displayed in Figure 3.1. One can observe in Figure 3.1a that the optimal financial and total wealth path are most volatility around the retirement date, whereas the interquartile range for the human capital is smallest overall, since all the variation originates from the interest rate. Total wealth is increasing up until retirement, since no endogenous consumption takes place before retirement. Regarding the optimal consumption decision (3.5.6), we can observe in Figure 3.1b that the median replacement rate is quite high, which is partly a result of the chosen financial market benchmark parameters. The median consumption within retirement is increasing, due to the fact that the denominator of (3.5.6), composed from M_t and the integral over the function Θ, is decreasing over time. Also, a significant variation is observed within retirement, whereas the stepwise decreasing consumption in the accumulation phase is deterministic and a result of the exogenous maximum contribution rates. In general, we can deduce from Figure 3.1b that a favorable development of the total wealth account because of a positive economic scenario, results in a higher retirement consumption possibility and vice versa.

Figure 3.1: Optimal median wealth and consumption processes



(a) Wealth development (b) Consumption strategy

Medians are based on $n = 10,000$ scenario paths. The shaded areas in both figures represent the interquartile range of the simulated scenarios corresponding to the median with the same color. In figure (a), the vertical line represents the retirement age. In figure (b), besides the median consumption (in black), 5 random scenarios for the consumption strategy are displayed in color (solid), accompanied by the total wealth development (dotted) in the same 5 scenarios. In figure (b), read consumption on the left y-axis and total wealth on the right y-axis.

3.5.3 Investment strategy in terms of total wealth

Now that the optimal consumption strategy is identified, the focus is moved towards the question of how to finance this optimal consumption strategy. For that purpose, the budget condition is added to the described model. This budget condition states that at all times, the total wealth account of the individual must match the market value of the individual's pension liabilities. Recall that the static budget condition holding at equality, represents this restriction at time $t = 0$. Therefore, the budget condition can be expressed as:

$$W_t = V_t, \quad 0 < t < T_D. \tag{3.5.8}$$

Note that this budget condition also must hold in optimality. Therefore, in order to identify this investment strategy that enables us to consume in an optimal way, we are interested in the market value of optimal total liabilities over the lifetime. This can be expressed as:

$$\begin{aligned} V_t^* &= \mathbb{E}_t\left[\int_{\max\{t,T_R\}}^{T_D} c_s^* \frac{M_s}{M_t}\,\mathrm{d}s\right] \\ &= M_t^{-\frac{1}{\gamma}} \varphi^{-\frac{1}{\gamma}} \int_{\max\{t,T_R\}}^{T_D} e^{-\delta s/\gamma}\Theta(t,s,1-1/\gamma)\,\mathrm{d}s. \end{aligned} \tag{3.5.9}$$

where the second equality follows from substituting the optimal consumption strategy (3.5.6) and the definition of Θ. Now we define $V_t^* = f(t, r_t, M_t)$ and applying a second-order Taylor series expansion (in a sense a multivariate version of Itô's lemma), results in the following dynamics for the market value of optimal liabilities (see Appendix K for the derivation):

$$\frac{\mathrm{d}V_t^*}{V_t^*} = \bar{\mu}_t^{V^*}(t, r_t, M_t)\,\mathrm{d}t + \frac{\lambda_S}{\gamma}\,\mathrm{d}Z_t^S + \left(\frac{\lambda_r}{\gamma} + \frac{1-\gamma}{\gamma}\sigma_r D_t^{V^*}\right)\mathrm{d}Z_t^r, \tag{3.5.10}$$

with

$$D_t^{V^*} \equiv \frac{\int_{\max\{t,T_R\}}^{T_D} e^{-\delta s/\gamma} D(s-t)\Theta(t,s,1-1/\gamma)\,\mathrm{d}s}{\int_{\max\{t,T_R\}}^{T_D} e^{-\delta s/\gamma}\Theta(t,s,1-1/\gamma)\,\mathrm{d}s}. \tag{3.5.11}$$

$D_t^{V^*}$ can be seen as the duration of remaining lifetime consumption, since it is a duration weighted average of a quantity which represents the depletion speed of wealth. The time-varying drift term $\bar{\mu}_t^{V^*}(t, r_t, M_t)$ is specified in Appendix K.

Now that the dynamics of the optimal pension liabilities is known, we can determine the investment strategy regarding total wealth that finances these liabilities, through the budget equation. The dynamics of total wealth, also known as the dynamic budget constraint, can be expressed as follows for the specified model:

$$\mathrm{d}W_t = \begin{cases} (r_t + \boldsymbol{\omega}_t'\boldsymbol{\Sigma\lambda})\,W_t\,\mathrm{d}t + W_t\boldsymbol{\omega}_t'\boldsymbol{\Sigma}\,\mathrm{d}\boldsymbol{Z}_t & \text{for} \quad 0 < t \le T_R \\ (r_t + \boldsymbol{\omega}_t'\boldsymbol{\Sigma\lambda})\,W_t\,\mathrm{d}t + W_t\boldsymbol{\omega}_t'\boldsymbol{\Sigma}\,\mathrm{d}\boldsymbol{Z}_t - c_t\,\mathrm{d}t & \text{for } T_R < t < T_D \end{cases}, \tag{3.5.12}$$

where $\boldsymbol{\Sigma}$ and $\boldsymbol{\omega}_t$ are defined in (3.3.2). Choosing the portfolio fractions of total wealth in such a way that wealth replicates the market value of optimal liabilites, yields the optimal wealth path W_t^*. In other words, a perfect replicating portfolio is created by allocating total wealth to the available assets in such a way that the diffusion parts of the dynamics of the total optimal liabilities $\mathrm{d}V_t^*$ (3.5.10) match the diffusion parts of the dynamics of the total wealth portfolio $\mathrm{d}W_t$ (3.5.12). This results in the following optimal total wealth allocations (see Appendix K for the derivation):

$$\boldsymbol{\omega}_t^* = \frac{1}{\gamma}\begin{pmatrix} \frac{\lambda_S}{\sigma_S} \\ -\frac{\lambda_r}{\sigma_r D(h)} \end{pmatrix} + \left(1 - \frac{1}{\gamma}\right)\begin{pmatrix} 0 \\ \frac{D_t^{V^*}}{D(h)} \end{pmatrix}, \tag{3.5.13}$$

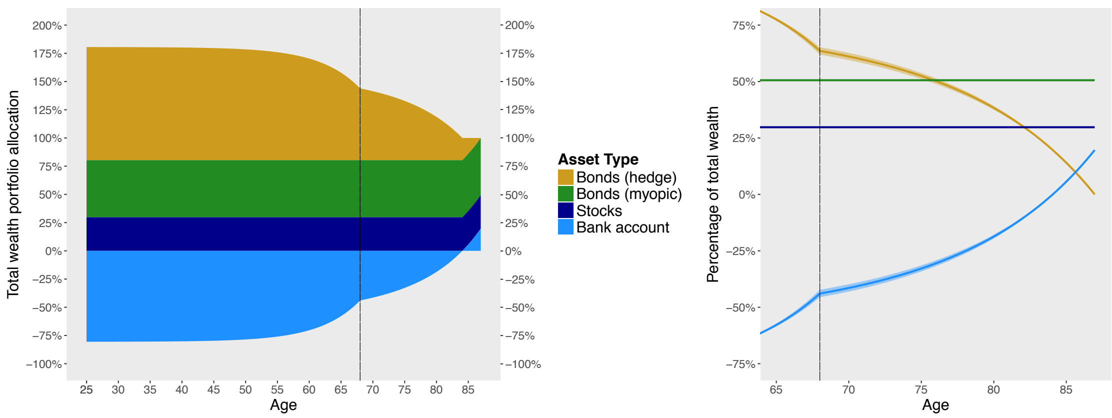
where $D_t^{V^*}$ is defined in (3.5.11). Recall that the optimal fraction in the risk-free bank account is residually determined by $\omega_t^{B*} = 1 - \boldsymbol{\omega}_t^{*\top}\boldsymbol{\iota}$. We can see from (3.5.13) that $\boldsymbol{\omega}_t^*$ is a convex combination of two components, of which the weights are determined by the degree of risk aversion of the individual. The first component is the mean-variance tangency portfolio, as originally formulated by Markowitz (1952). The second component is a portfolio which minimizes the risk resulting from changes in the term structure of interest rates. Therefore, the first component can also be described as a speculative term which is held by the investor with an instantaneous investment horizon (myopic investor). Then, the second term is the hedging term which is held by an infinitely risk averse investor. We can see that this myopic investor can be defined as the log-utility investor, ($\gamma = 1$), since in this case the hedge component of the portfolio vanishes. For the investor with no risk tolerance ($\gamma \to \infty$), the speculative part of the portfolio is nullified and the duration of the bond portfolio is matched to the duration of the remaining lifetime consumption (liabilities) $D_t^{V^*}$ to immunize future consumption for interest rate risk, as was also described by Sørensen (1999), only in a final wealth setup. The individual therefore trades off the desire to pick up risk premia and providing an intertemporal hedge via the degree of risk aversion γ. The median optimal investment strategy is displayed for the benchmark parameter set in Figure 3.2.

Regarding the optimal portfolio fractions on an asset level rather than a portfolio level, the following can be observed from (3.5.13) . The allocation of wealth in stocks is purely speculative and constant of time. However, the allocation in nominal bonds is serving both a speculative goal as well as a intertemporal hedge goal with respect to interest rate risk exposure that is present in the value of the liabilities. The allocations of total wealth that serve a speculative goal are time-independent, whereas the allocation for hedge purposes decreases as the investment horizon shortens. The latter can be attributed to a decreasing duration of lifetime consumption as one gets closer to the deterministic date of death T_D.

It can also be seen from (3.5.13) and Figure 3.2 that the investment strategy which finances the optimal consumption strategy, is stochastic. This stochasticity appears in the risk hedge portfolio where $D_t^{V^*}$, which is affected by the state variable r_t, plays a role. Therefore, the state of the economy regarding the state variable r_t partly drives the hedge demand of the individual. Consequently, a general deterministic, horizon-dependent investment strategy that does not take into account the economic state would underperform compared to this optimal strategy. From Figure 3.2b, one can observe that the variation in the bond hedge portfolio is reasonably small, since the only source of uncertainty with respect to this portfolio component, is the instantaneous interest rate. In the period where a short position in the risk-free asset is taken to partly finance this bond hedge component, the variation also affects the risk-free asset holding. The bond hedge portfolio part is observed to be decreasing over time, since $D_t^{V^*}$ is a decreasing function which equals zero at T_D as there is no future more consumption to (partly) hedge at this point.

Lastly, note that the maximization problem exhibits a structural break resulting from the fixed exogenous consumption and saving in the accumulation phase, and free choice of consumption and savings in the retirement phase. Within retirement, the derived optimal asset allocations of total wealth (3.5.13) do coincide with the investment strategy reported by Brennan and Xia (2002). However, with respect to the accumulation phase, the theoretically optimal portfolio compositions do differ due to this structural break, which explains the kink observed in the median portfolio composition in 3.2a. Brennan and Xia (2002) also do not account for any form of human capital in their model, which is discussed in the next section.

Figure 3.2: Optimal median investment strategy total wealth



(a) Portfolio composition (b) Asset fractions

Medians are based on $n = 10,000$ scenario paths. The vertical line in both figures represents the retirement age. In figure (b), the shaded areas represent the 2.5%-97.5% percentile range of the simulated scenarios corresponding to the median with the same color.

3.5.4 Investment strategy in terms of financial wealth

Next, we consider the fact that total wealth is composed of financial wealth and human capital and the resulting implications for the investment strategy of financial capital. Firstly, we need to obtain the dynamics of human capital, which are derived in Appendix L and can be stated as follows:

$$\mathrm{d}H_t = -K(t)\,\mathrm{d}t + \left(r_t - \sigma_r D_t^H \lambda_r\right) H_t\,\mathrm{d}t - \sigma_r D_t^H H_t\,\mathrm{d}Z_t^r, \quad \text{with} \tag{3.5.14}$$

$$D_t^H \equiv \frac{\int_0^{T_D - t} D(h) P_t^h K(t+h)\,\mathrm{d}h}{\int_0^{T_D - t} P_t^h K(t+h)\,\mathrm{d}h}, \tag{3.5.15}$$

where D_t^H can be interpreted as the interest rate sensitivity or 'duration' of the remaining human capital at time t. Because the human capital was assumed to be bond-like, the same structure can be observed in (3.5.14) as in the dynamics of the rolling bond (3.2.14). The main difference however is that human capital depletes over time, which is represented by the component $-K(t)\,\mathrm{d}t$. The dynamics of financial wealth can be determined in the same fashion as the dynamics of total wealth, only instead of depletion by consumption, financial wealth grows over time with the fixed premium payments. We define the dynamics of financial wealth similar to the financial wealth specification used by Cairns et al. (2006, p. 847) amongst others. Now, using the assumption that total wealth consists of financial wealth and human wealth, Itô's lemma and the concept of replication, the fractions of financial wealth $\boldsymbol{\theta}_t^{*\top} = (\theta_t^{S*}, \theta_t^{P*})$ that finance optimal consumption, can be expressed as (see Appendix M for the full derivation):

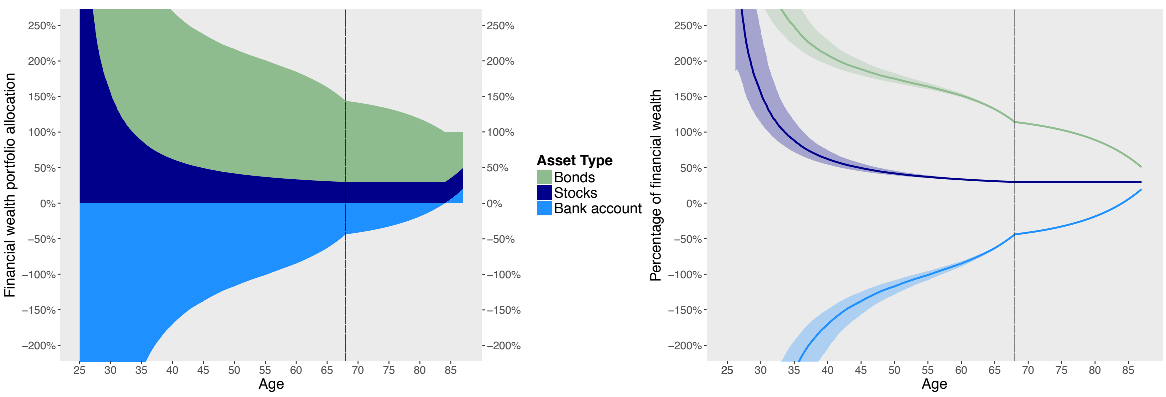
$$\boldsymbol{\theta}_t^* = \left(1 + \frac{H_t}{F_t}\right)\boldsymbol{\omega}_t^* - \frac{H_t}{F_t}\begin{pmatrix} 0 \\ \frac{D_t^H}{D(h)} \end{pmatrix}, \tag{3.5.16}$$

where $\boldsymbol{\omega}_t^*$ is defined in (3.5.13) and D_t^H in (3.5.15). From Expression (3.5.16), we can observe that the investment strategy for actual financial capital is affected by one's market value of human capital over the lifetime. The specification of human capital and the assumption of the components it consists of, is therefore significantly influencing the investment strategy of financial wealth. The median optimal investment strategy corresponding to 3.5.16 is displayed for the benchmark parameter set in Figure 3.3

First of all, it can be observed that in case human capital is depleted, the investment strategies for human capital and financial wealth coincide. Regarding stock exposure, the allocation of financial wealth towards equity is significantly increased when there exists a relatively large human capital-to-financial wealth ratio. Regarding the allocation in bonds, two effects play a role: the increasing effect of a large human wealth-to-financial wealth ratio on the total wealth bond allocation is moderated by the second term in (3.5.16). This can be explained by the fact that human capital is also subject to interest rate risk, just like bonds. We cannot make a intuitive distinction between a hedge portfolio and a speculative portfolio anymore.

With respect to the variability in the investment strategy, we can observe from Figure 3.3b that compared to the total wealth asset allocations, the variability is significantly larger due to the fact that the human capital versus financial wealth ratio depends on both the economic scenario and past returns. This variability is especially large in the first few years of the working life, since only a small amount of financial wealth is present versus a large amount of human capital. Similar to the median portfolio composition of total wealth, a short position is taken in the risk-free asset for almost the entire lifetime.

Figure 3.3: Optimal median investment strategy financial wealth



(a) Portfolio compostion (b) Asset fractions

Medians are based on $n = 10,000$ scenario paths. The vertical line in both figures represents the retirement age. In figure (b), the shaded areas represent the interquartile range of the simulated scenarios corresponding to the median with the same color.

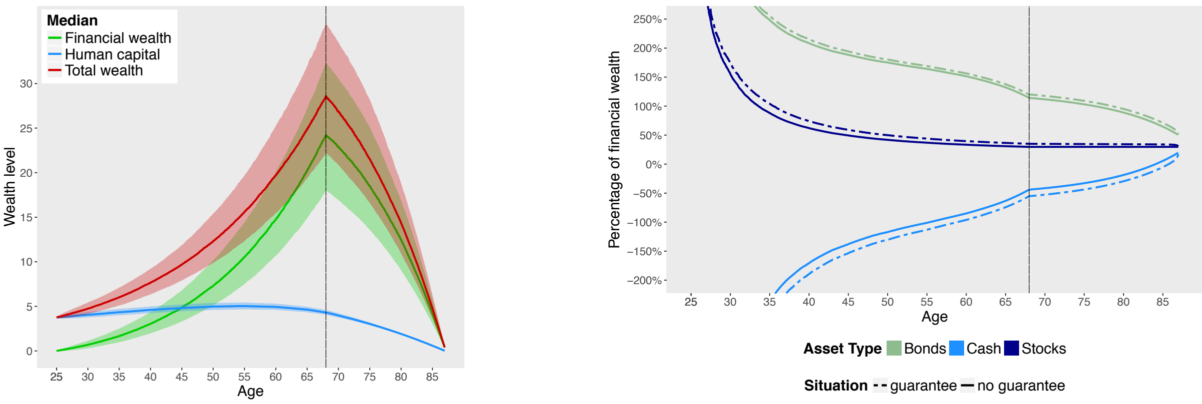
3.5.5 Taking into account guaranteed state pension

Human capital is only composed of labor income in this chapter up until now. Suppose the individual receives a risk-free state pension within retirement, such as the AOW in the Netherlands. The inclusion of such a guarantee would naturally impact the optimal consumption and investment allocation strategy. If this guarantee is assumed to be risk-free, similar to the labor income, it can be included as a component of the individual's human capital, as it essentially represents a stream of guaranteed payments within retirement up until time of death. The model framework is set up in such a way that we only have to adjust the definition of $K(u)$, stated in (3.3.5). Suppose the guarantee equals 30% of the last earned labor income before retirement, then the model modification would be represented by:

$$K(u) = \begin{cases} p_u Y_u & \text{for } u \in [0, T_R) \\ 0.3 Y_{T_R} & \text{for } u \in [T_R, T_D] \end{cases}. \tag{3.5.17}$$

As was discussed in Section 3.5.2, a change in the human capital definition would only manifest itself through the initial wealth level W_0, which results in the same optimal consumption path expressed as a fraction of the total wealth process. Therefore, the optimal investment strategy regarding total wealth remains the same with or without an inclusion of a guarantee. However, the optimal investment strategy regarding actual financial wealth does change. The effect of including the defined guarantee within retirement on the median wealth processes and the median financial wealth asset fractions, is displayed in Figure 3.4.

Figure 3.4: Effect of a guarantee on the median processes



(a) Wealth development (b) Asset fractions financial wealth

Medians are based on $n = 10,000$ scenario paths. The vertical line in both figures represents the retirement age. In figure (a), the shaded areas represent the interquartile range of the simulated scenarios corresponding to the median with the same color. In figure (b), the fractions of the available asset types with respect to the financial wealth portfolio are compared for a setting with and without a guarantee. The situation with guarantee corresponds to Figure 3.3b.

In Figure 3.4a, it can be observed that human capital is indeed not depleted at time of retirement, which results in a non-overlapping financial and total wealth within retirement. Analyzing Figure 3.4a, we see an increase in the equity holding over the life cycle, resulting from a larger market value of human capital. A guarantee within retirement therefore enables the individual to take a relatively more risky overall position. One might suspect the bond holding to drop over the entire life cycle, since the individual now possesses an extra bond-like asset. But due to the earlier discussed twofold effect of human capital on the bond position, the net bond position slightly increases under these benchmark parameters.

The impact of the guarantee on the financial wealth position is clearly present. Therefore, when composing a framework to determine the best investment strategy for a top-up plan, it is of importance to realize that components such as the underlying occupational pension scheme and the state pension actually affect the desired investment strategy.

Chapter 4

Annuity purchase at retirement: a tractable specification

In this chapter, the lifetime asset allocation problem is discussed in a setting where the individual is obliged to convert to a life annuity pension at the retirement date. Whereas the consumption strategy in Chapter 3 is a decision variable only within retirement, the consumption strategy in this setting is determined exogenously altogether. Therefore, we switch from an intertemporal consumption problem to a terminal wealth problem that solely embodies the accumulation phase. These two types of approaches are essentially equivalent, meaning that the optimal intertemporal consumption strategy could be derived from optimal terminal wealth. However, since the set of decision variables is reduced to solely a portfolio allocation strategy, a terminal wealth setting is naturally intuitive.

As was previously described in Section 2.2, the feature of a top-up plan that entails buying into the underlying occupational pension scheme at the retirement date, can be interpreted as purchasing a single life annuity. In general, a constant life annuity can be described as a financial contract that provides the purchaser with a constant periodic payment for his remaining lifetime in exchange for an initial lump sum. The longevity risk of the individual is thereby transferred to the annuity provider, which in most cases is an insurer or a pension fund. For the annuity purchaser, the annuity price, also called the annuity factor, is affected by the term structure of interest rates: a fall in bond yields generally increases the annuity price and vice versa. Interest rate risk is therefore an important factor in the annuitization risk and can be appropriately hedged by purchasing long-term bonds. From the perspective of the annuity provider, an overall longevity improvement increases the annuity costs.

An annuity purchase can have different welfare effects for different types of people. For a risk loving individual, a life annuity purchase at retirement is generally utility decreasing, since potential higher returns on risky assets within retirement cannot be achieved. On the contrary, a very risk averse person would prefer a life annuity, since it provides a benefit guarantee. Individuals can also exhibit different views on how to handle the implicit risks of fully annuitizing their wealth at the retirement date, which is expressed in their different desired investment strategies in the accumulation phase. As was discussed in Section 2.1, Cairns et al. (2006) develop a framework in which a long-horizon investor hedges a minimum guarantee subject to interest rate risk resulting from a single life annuity purchase at retirement. In this chapter, we will discuss this model framework, tailored to our financial market and human capital, in more detail and describe how this ultimately leads to a theoretically tractable expression for the annuity factor. This expression is then applied in Chapter 5.

In Section 4.1, the model specification and potential investment strategy for a single annuity purchase is discussed. Then, in Section 4.2, we address a simplification for the rather complex life annuity factor. Lastly, in Section 4.3, the simplification results and its implications are discussed.

4.1 Model specification

In this section, the developed model by Cairns et al. (2006) is reviewed within the context of our financial market setting as described in Section 3.2. Note that this model framework is originally set up for a general one-factor short rate process. First, we state the general underlying assumptions that we take into account in their model specification. Then, the terminal utility of the model is discussed, which is followed by the derivation method of the optimal investment strategy. Within this derivation, one could identify a potential tractable expression for the annuity factor.

4.1.1 Assumptions

Let the available financial market be as described in Section 3.2 and the individual's life cycle characteristics and preferences be in accordance with Section 3.1. However, compared to the assumptions underlying Chapter 3, a revision is necessary with respect to the labor income component of human capital. In the framework of Cairns et al. (2006), this labor income is assumed to be cash-like, meaning that the growth rate of the salary payments comprises the instantaneous interest rate amongst others. Consequently, the labor income becomes stochastic. Nonetheless, it is still assumed to exhibit no exposure to the financial market risk factors and therefore can be interpreted as locally risk-free. Additional to the labor income revision, the individual is now obliged to convert the accumulated pension wealth at time T_R, of which he can determine the asset allocation strategy for the period $[0, T_R]$. This results in a constant consumption rate within retirement, determined by the annuity purchase at time T_R. Therefore, it is sufficient to consider the lifetime portfolio choice problem solely in the accumulation phase, including the conversion time T_R.

4.1.2 Terminal utility maximization

As mentioned in Section 2.1, the model of Cairns et al. (2006) incorporates a form of habit formation; The individual is assumed to maximize the expected utility of the pension right obtained at retirement, evaluated in the terminal salary. The terminal salary acts as measure for the purchased life annuity pension. This pension-to-salary ratio can be interpreted as a 'distorted' replacement rate. Consequently, the individual's terminal utility can be stated as:

$$
\begin{aligned}
&u\left(\frac{W_{T_R}}{\bar{a}_{T_R}}/Y_{T_R}\right) = u\left(\frac{Q_{T_R}}{\bar{a}_{T_R}}\right), \quad \text{with} \\
&Q_t \equiv \frac{W_t}{Y_t},
\end{aligned}
\tag{4.1.1}
$$

where u is the power utility function as stated in (3.1.1). W_t represents the total wealth, which evolves in accordance with (3.3.1), in which $\{c_t \equiv 0 : 0 < t < T_D\}$. As preceding, the labor income is represented by Y_t and $\bar{a}_t$ is the annuity factor at time t . Note that in expression

(4.1.1), the denominator should be equal to $(1-p_{T_R})Y_{T_R}$ in order to speak of the maximization of a 'clear' replacement rate, in which the actual consumption before and after retirement are compared. However, as Cairns et al. (2006) point out, the factor $(1-p_{T_R})$ does not affect the optimization problem that follows in any way, because the premium contribution rate p_t is deterministic and exogenously determined. Therefore, $(1-p_{T_R})$ results in a constant multiplier for utility at terminal time T_R in case of power utility. Consequently, this factor can be neglected in (4.1.1) and one can still interpret the results as if it where to represent a pure replacement rate in terminal utility. In accordance with the assumptions described in Section 4.1.1, the dynamics of labor income can now be expressed as:

$$\frac{\mathrm{d}Y_t}{Y_t} = (r_t + \mu_y(t))\,\mathrm{d}t, \quad 0 < t < T_R, \tag{4.1.2}$$

where $\mu_y(t)$ is the deterministic growth rate of labor income over the career path and r_t is the instantaneous interest rate. As previously mentioned, the symbol $\bar{a}_t$ in (4.1.1) represents the annuity factor, or the market-consistent annuity rate at time t. This quantity equals the actuarial present value of a life annuity that pays continuously a constant level of one unit per year, for a generic retirement date t. Therefore, the annuity factor can be stated as:

$$\bar{a}_t = \mathbb{E}_t\left[\int_0^\infty {}_h\pi_x \frac{M_{t+h}}{M_t}\,\mathrm{d}h\right] = \int_0^\infty {}_h\pi_x P_t^h\,\mathrm{d}h, \tag{4.1.3}$$

where P_t^h again is the nominal zero-coupon bond price for maturity h at time t and ${}_h\pi_x$ is the h-year survival probability of an individual aged x. One can observe that $\bar{a}_t$ is a function of the instantaneous short rate r_t. Nonetheless, the relationship between the short rate and the annuity factor is rather complex.

4.1.3 Optimal investment strategy

For determining the optimal investment strategy, Cairns et al. (2006) apply the previously mentioned dynamic programming method for stochastic control problems. This method was also originally used by Merton (1969) and relies on the Hamilton-Jacobi-Bellman (HJB) equation and Bellman's principle of optimality (Pham, 2009).

Firstly, we need to identify the state variables and their dynamics. In our specified setting, there exist two state variables, Q_t and r_t. The dynamics of r_t is known and specified in (3.2.1). Applying the multivariate Itô's lemma derived in Appendix K, we can obtain the dynamics of Q_t, in accordance with Cairns et al. (2006). In matrix notation, the dynamics of the state variables can be expressed as:

$$\begin{aligned}\begin{pmatrix}\mathrm{d}Q_t\\ \mathrm{d}r_t\end{pmatrix} &= \begin{pmatrix}Q_t\left(-\mu_y(t)+\boldsymbol{\omega}_t'\boldsymbol{\Sigma\lambda}\right)\\ \kappa(\bar{r}-r_t)\end{pmatrix}\mathrm{d}t + \begin{pmatrix}Q_t\omega_t^S\sigma_S & -Q_t\omega_t^P\sigma_r D(h)\\ 0 & \sigma_r\end{pmatrix}\mathrm{d}\boldsymbol{Z}_t\\ &\equiv \boldsymbol{\mu}(q_t, r_t, \boldsymbol{\omega}_t)\,\mathrm{d}t + \boldsymbol{\sigma}(q_t,\boldsymbol{\omega}_t)\,\mathrm{d}\boldsymbol{Z}_t.\end{aligned} \tag{4.1.4}$$

The expected terminal utility is a function of these two state variables and the optimal indirect utility function, also often called the value function, for a set of admissible strategies $\mathcal{W}$, can be expressed as:

$$v(t,q,r) = \sup_{\boldsymbol{\omega}\in\mathcal{W}}\left\{\mathbb{E}_t\left[u(Q_{T_R}^{(\boldsymbol{\omega})}, r_{T_R})|Q_t = q, r_t = r\right]\right\},\ (t,q,r)\in[0,T_R)\times\mathbb{R}_+\times\mathbb{R}. \tag{4.1.5}$$

Terminal utility, as specified in (4.1.1), is expressed as function of the state variables, and $Q^{(\boldsymbol{\omega})}_{T_R}$ denotes the end of the path of state variable Q given a certain implemented investment strategy $\boldsymbol{\omega} = \{\boldsymbol{\omega}_t : t \in [0, T_R]\}$. From the state variable dynamics in (4.1.4), we can deduce the HJB equation corresponding to this optimal control problem (Pham, 2009):

$$\begin{cases} \frac{\partial v}{\partial t} + \sup_{\boldsymbol{\omega} \in \mathcal{W}} \{\mathcal{L}^{\boldsymbol{\omega}} v\} & = 0, \qquad (t, q, r) \in [0, T_R) \times \mathbb{R}_+ \times \mathbb{R} \\ v(T_R, q, r) & = u(q, r), \ \ (q, r) \in \mathbb{R}_+ \times \mathbb{R} \end{cases}, \tag{4.1.6}$$

where $\mathcal{L}^{\boldsymbol{\omega}}$ represents the second-order partial differential operator to the controlled state variable process (4.1.4), which can be expressed as:

$$\begin{aligned} \mathcal{L}^{\boldsymbol{\omega}} &= \sum_i \boldsymbol{\mu}_i(\boldsymbol{x}, \boldsymbol{\omega}) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j} \left(\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\omega}) \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\omega})'\right)_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}, \quad \boldsymbol{x} = (q, \ r)', \ i, j = \{1, 2\} \\ &= \boldsymbol{\mu}(q, r, \boldsymbol{\omega}) D_{q,r} + \frac{1}{2} \mathrm{tr}\left(\boldsymbol{\sigma}(q, \boldsymbol{\omega}) \, \boldsymbol{\sigma}(q, \boldsymbol{\omega})' \, D^2_{q,r}\right), \end{aligned} \tag{4.1.7}$$

with

$$D_{q,r} \equiv \left(\frac{\partial}{\partial q}, \ \frac{\partial}{\partial r}\right)', \quad D^2_{q,r} \equiv \begin{pmatrix} \frac{\partial^2}{\partial q^2} & \frac{\partial^2}{\partial q \partial r} \\ \frac{\partial^2}{\partial r \partial q} & \frac{\partial^2}{\partial r^2} \end{pmatrix}. \tag{4.1.8}$$

Taking the first-order derivative of $\mathcal{L}^{\boldsymbol{\omega}} v$ with respect to $\boldsymbol{\omega}$, equating to zero and solving for $\boldsymbol{\omega}$ yields the following optimal portfolio allocation strategy, dependent on the value function:

$$\boldsymbol{\omega}^*(t, q, r | v) = \boldsymbol{\Sigma}'^{-1} \left(-\boldsymbol{\lambda} \frac{v_q}{q v_{qq}} - \sigma_r \boldsymbol{e}_2 \frac{v_{qr}}{q v_{qq}}\right), \tag{4.1.9}$$

where $\boldsymbol{e}_i$ denotes the i-th unit vector and v_q represents the partial derivative of the value function with respect to q, and so on.

In order to find an expression for the optimal asset allocation strategy, we need to find an expression for $v(t, q, r)$. The value function $v(t, q, r)$ develops according to the partial differential equation (PDE) that results from substituting (4.1.9) in the HJB equation (4.1.6), which satisfies the boundary condition $u(q, r)$. As both Munk and Sørensen (2010) and Cairns et al. (2006) point out, in the case of time-additive power utility, it is well-known that the indirect utility function (4.1.5) is of the form

$$v(t, q, r) = \frac{q^{1-\gamma}}{1-\gamma} f(t, r)^{\gamma} = u(q) f(t, r)^{\gamma}, \tag{4.1.10}$$

where the general function $f(t, r)$ depends on both investment horizon and the state variable r_t. Using this functional form and the PDE of $v(t, q, r)$, Cairns et al. (2006) show that the function $f(t, r_t)$ satisfies a certain parabolic PDE. This parabolic PDE uniquely identifies a probability measure $\widehat{\mathbb{P}}$ under which the time t conditional expectation of the boundary condition $f(T_R, r) = \left((1-\gamma) u(q, r) q^{\gamma-1}\right)^{-\gamma}$ forms an expression for $f(t, r_t)$, according to the Feynman-Kac theorem (Etheridge, 2002, p. 170). Note that this conditional expectation expression for $f(t, r_t)$ does not necessarily have to result in an explicit expression.

Since the state variable Q_t is independent of r_t, the functional form of the annuity directly determines the expression for $f(t, r_t)$. This annuity expression and the potential simplification is discussed in the next section.

4.2 Simplification of the annuity price

The annuity factor $\bar{a}_t$ is previously expressed in (4.1.3). From this representation, we can observe that the annuity factor is essentially a zero-coupon bond portfolio composed of infinitely many bonds with different maturities.

In Section 3.2, it was mentioned that for a Vasicek one-factor model, all bonds with different maturities are perfectly correlated regarding an interest rate shock. Therefore, only one particular zero-coupon bond is needed to replicate any of the other bonds in this financial market. For example, let there be two different rolling bonds with different constant maturities of respectively h years and $\tilde{h}$ years. Then, one can easily derive that the dynamics of the rolling bond with maturity $\tilde{h}$ can be expressed as a convex combination of the locally risk-free asset dynamics and the bond dynamics with maturity h (based on Boulier, Huang, and Taillard (2001)):

$$\frac{\mathrm{d}P_t^{\tilde{h}}}{P_t^{\tilde{h}}} = \left(1 - \frac{D(\tilde{h})}{D(h)}\right)\frac{\mathrm{d}B_t}{B_t} + \frac{D(\tilde{h})}{D(h)}\frac{\mathrm{d}P_t^h}{P_t^h}. \tag{4.2.1}$$

Therefore, the price of a bond with maturity $\tilde{h}$ can be replicated by a cash position and a position in the bond with maturity h.

With this fact in mind, the particular bond portfolio that represents the annuity factor, could be approximated by a position in one particular rolling bond with constant maturity denoted by $\hat{h}$. This would result in an expression of the following form, base on (3.2.11):

$$\begin{aligned} &\bar{a}_t \approx e^{\xi - \psi r_t}, \quad \text{with} \\ &\psi \equiv D(\hat{h}) \quad \text{and} \quad \xi \equiv \log x - a(\hat{h}), \end{aligned} \tag{4.2.2}$$

where x denotes the position in the particular bond and the functions a and D as stated in respectively (3.2.12) and (3.2.13). The expression (4.2.2) is of exponential form because of the affine term structure that is implied by the Vasicek model.

Absent the above motivation, this form of annuity approximation is also explored by Cairns et al. (2006), which yields the following expression of the boundary condition in (4.1.6):

$$v(T_R, q, r) = u(q, r) = \frac{1}{1-\gamma}\left(\frac{q}{\bar{a}_{T_R}}\right)^{1-\gamma} = u(q)e^{(\gamma-1)(\xi-\psi r)}, \tag{4.2.3}$$

According to Equation (4.1.10), the expression above uniquely defines the boundary condition for the function f as

$$f(T_R, r) = e^{\frac{\gamma-1}{\gamma}(\xi - \psi r)}. \tag{4.2.4}$$

Because the dynamics of the short rate is governed by an Ornstein-Uhlenbeck process, the short rate r_t is normally distributed. The function $f(T_R, r)$ is therefore log-normally distributed, for which the explicit conditional expectation under measure $\widehat{\mathbb{P}}$ is well-known. This yields an explicit expression for the value function $v(t, q, r)$ in (4.1.10) and therefore an explicit expression for $\boldsymbol{\omega}^*(t, q, r|v)$ in (4.1.9). Also Li and Zhang (2018) apply a variant of this annuity factor approximation.

Concluding, this type of annuity factor simplification yields a closed-form investment strategy in the discussed model framework with a Vasicek one-factor model. Next, we discuss the actual estimation of the approximated annuity factor and its accuracy.

4.2.1 Benchmark parameter values

For the estimation of the simplified annuity expression, we use the same benchmark parameter value set as described in Section 3.4 as a basis. The only revision is with regard to the deterministic time of death T_D. We now assume that the individual survives until the retirement date T_R, after which he has a certain probability of dying. At the age of 120, the probability of dying is equal to one ($T_D^u = 95$). Since only the accumulation phase $t \in [0, T_R]$ is of interest, the incorporation of mortality risk after the retirement date does not influence the framework in any other way than through the annuity factor. The mortality rates for the Dutch population are assumed to be deterministic and are acquired from the Koninklijk Actuarieel Genootschap (2016). We apply the mortality rates for a Dutch male, taking into account macro longevity risk.

With regard to the net pension top-up plan, the participant pool exhibits certain mortality characteristics that differ from a representative sample of the Dutch population. APG therefore estimated correction factors for the previously mentioned mortality rates that fit the net pension population (APG, 2018). These correction factors are also incorporated in the annuity price[1].

Note that the labor income process (4.1.2) differs from the constant labor income in Chapter 3. Therefore, we still need to establish benchmark parameter values for this process. We assume a benchmark growth rate that equals $\{\mu_y(t) \equiv 0 : 0 < t < T_R\}$, with $Y_0 = 1$. Therefore, instead of deterministically equal to one, the labor income of initially one now varies with the short rate.

4.2.2 Estimation procedure

In order to obtain benchmark parameters values for ξ and ψ, we use Monte Carlo simulation in order to estimate a linear regression model. To acquire a sense of the accuracy of the estimation, a bootstrap method is applied with regard to the linear model parameter estimates and the fit. The applied method for process simulation and numerical approximation is described in Section 3.4. The estimation procedure is as follows.

1) Simulate J scenario paths of r_t and the corresponding pricing kernel realizations M_t for $t \in [0, T_R]$. Fortnightly time steps ($\Delta t = \frac{1}{24}$) are used so that the approximation for $\bar{a}_t$ in step 2) is sufficiently accurate.

2) Calculate for each scenario j the corresponding numerical approximation of the annuity factor at time t, $\bar{a}_{t,j} = \int_0^{T_D^u - T_R} {}_h\pi_{25+T_R} P_{t,j}^h \, \mathrm{d}h$,

3) Estimate for each scenario j the following linear regression model that corresponds to (4.2.2):

[1]These correction factors are available upon request.

$$\log \bar{a}_{t,j} = \xi_j - \psi_j r_{t,j} + \varepsilon_{t,j},$$

with $\varepsilon_{t,j}$ a white noise process.

4) Obtain the benchmark parameter value estimates as the average of the regression estimates over the J scenarios:

$$\bar{\hat{\xi}} = \frac{1}{J}\sum_{j=0}^{J}\hat{\xi}_j, \quad \bar{\hat{\psi}} = \frac{1}{J}\sum_{j=0}^{J}\hat{\psi}_j.$$

This procedure can be interpreted as a J non-overlapping blocks bootstrap, where the time series in each block starts in an equilibrium situation, because $r_{0,j} = \bar{r}$ for all j. The estimation results for $J = 10,000$ replications is displayed in Table 4.1.

Table 4.1: Bootstrap estimation results

Parameter	Mean	Std. Dev.
ξ	2.768	0.001
	(94,742.766)	(41,251.592)
ψ	4.107	0.036
	(4,108.115)	(881.117)
R^2	0.9999	0.0000
SE of $\hat{\varepsilon}$	0.0007	0.0003

Based on $J = 10,000$ replications. Column 'Mean' displays the average over the J replications and column 'St. Dev.' displays the sample standard deviation, which is essentially the bootstrap estimate of the standard error of the statistic. Value between brackets is the t-statistic for the corresponding parameter estimate above.

These estimates imply that the annuity factor acts like a position in a zero-coupon bond with maturity $\hat{h} \approx 6.69$ years of $x \approx 17.51$ units. In the next section, this estimation procedure output is further discussed.

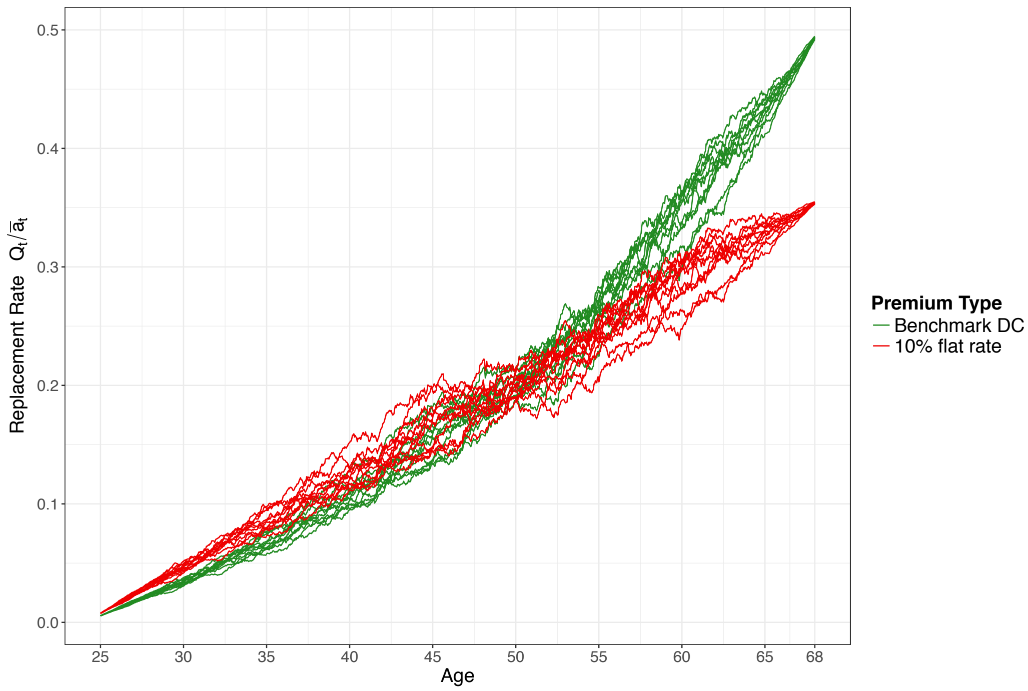
4.3 Discussion

Firstly, the model fit and parameter accuracy of the simplified annuity factor estimated in Section 4.2, is discussed. From Table 4.1, one can observe that the parameter estimates vary only little over the scenarios, which suggests reliable parameter estimates. Regarding the simplification itself, it can be seen that the log linear model almost perfectly captures the actual annuity factor, since the R^2 is almost one in all scenarios and the standard error of the residuals is also almost zero. The parameter estimates are in general highly significant. We also varied the parameters of the short rate dynamics to test whether this estimation procedure is generally robust. The parameter estimates remain accurate and highly significant in case of a shift in the relevant underlying benchmark parameters. Since this tractable annuity simplification performs very well, it is used in further model development in the next chapter.

Although it is not particularly addressed by Cairns et al. (2006), the inclusion of the instantaneous short rate in the growth rate of labor income results in the independence of the state variables Q_t and r_t. This is a mathematically desirable feature. Another consequence is that labor income is to be considered locally risk-free, which results in a mathematically desirable expression for the human capital value. However, this cash-like specification of human capital is rejected by empirical evidence in later research on the labor income specification by e.g. Munk and Sørensen (2010). Therefore, we choose to remain with the original specification of nominal bond-like human capital regarding the model development in the next chapter.

However, the particular model choice of Cairns et al. (2006) does provide an intuitive insight in the investment strategy for the accumulation phase. Using the benchmark parameter values and estimation output discussed in Section 4.2, the optimal investment strategy for a highly risk averse individual is implemented and the resulting scenarios for the replacement rate are displayed in Figure 4.1 (green). Additionally, the optimal replacement rate development is also displayed for a different premium contribution rate: a 10% flat rate (red). We can observe that in both cases, the individual wants to completely eliminate risk regarding the annuity purchase and the resulting replacement rate, therefore targeting a specific replacement rate at $T_R = 43$. This leads to the scenario paths of the replacement rate converging to a predetermined, targeted replacement rate. However, over the accumulation phase a significant variation is observed due to the dynamic investment strategy reacting to the economic state (previous returns on financial wealth) and the inflow of premiums in order to obtain the desired replacement rate.

Figure 4.1: Development of the replacement rate for a highly risk averse individual, $\gamma = 1000$



In this figure, 10 scenarios of the replacement rate process $\frac{Q_t}{\bar{a}_t}$ are displayed for both the benchmark DC premium contribution rate and a 10% flat rate, based on the same financial market shocks. The applied parametrization is in accordance with Section 4.2.1.

The incorporation of the simplified annuity factor does indeed reflect the desired result for a highly risk averse individual: complete replication of the annuity purchase evaluated in the labor income.

Chapter 5

Investment strategy for top-up plans

As discussed in Chapter 2, the annuitization in the net pension top-up plan is subject to a particular funding ratio surcharge. For this top-up plan, only taking into account interest rate risk regarding the life annuity purchase results in a partial identification of the desired investment strategy for the accumulation phase. In this chapter, we discuss the lifetime asset allocation problem where upon retirement the pension wealth is converted into a life annuity, subject to a surcharge that equals the actual funding ratio of the underlying occupational pension scheme at retirement.

Similar to Chapter 4, the lifetime asset allocation problem is reduced to the accumulation phase and treated as a terminal wealth problem. The martingale method is applied to derive a closed-form solution of the investment strategy that yields the highest expected utility of pension wealth. Regarding the investment strategy of financial wealth in the top-up plan, the asset allocation problem is further reduced to the horizon of the top-up plan.

Firstly, the general assumptions are discussed and a model specification of the funding ratio is determined. Secondly, the optimal investment strategy of total wealth corresponding to the stated terminal wealth problem is derived. Thirdly, the human capital specification and the implied investment strategy of financial wealth in the net pension top-up plan is established. Appropriate sensitive analyses for the investment strategies with respect to the benchmark parameter values are performed to create insight in the robustness and dependencies. Lastly, the derived model and benchmark results are discussed.

5.1 General assumptions

The financial market as described in Section 3.2, is adopted in this chapter. The individual's preferences and life-cycle characteristics are assumed to meet the criteria as stated in Section 3.1, subject to three revisions. The first general revision states mandatory conversion of the accumulated financial wealth at time T_R into a life annuity pension, similar to Chapter 4. In this chapter, the annuity factor is additionally subject to a surcharge that equals the actual funding ratio at T_R. This mandatory conversion means it is sufficient to consider the lifetime portfolio choice problem solely in the accumulation phase, including the conversion time T_R. The second revision is with respect to the mortality assumptions of the individual: survival is guaranteed up until the retirement date T_R, after which there exist a certain probability of dying. These positive mortality probabilities that appear outside of the model timeframe, only affect the asset allocation problem through the market value of the annuity factor. Let this annuity factor be represented by the explored simplification in Chapter 4, stated in Equation (4.2.2).

The third revision is with respect to the components of human capital. Besides labor income, other additional components implicated by a top-up pension plan are also present in the human capital definition. Human capital is still considered to be bond-like, with possibly exposure to the different existing risk factors. These additional components in the human capital definition are further discussed in Section 5.5.

5.2 Modeling the funding ratio

In order to represent the funding ratio surcharge on the annuity purchase, we need to appropriately capture the movements of the funding ratio. Preferably, a direct relationship between the funding ratio movement and the constant long-term investment strategy of the occupational pension fund in question needs to be established. The following approach is taken with regard to modeling the funding ratio dynamics.

The funding ratio can in general be simplified to the fraction of assets (A_t) and liabilities (L_t) of the pension fund, which change over time because of amongst others investment returns, received premiums, benefit payments and a changing population. The value of the assets of the pension fund at time t can be represented by a certain portfolio consisting of the available assets in the financial market, similar to the individual wealth at time t discussed in Chapter 3. We choose to model the liabilities of the pension fund as a rolling zero-coupon bond with a maturity chosen equal to the average duration of the pension fund liabilities in question. This results in the following representation of the funding ratio G for $0 < t \leq T_R$:

$$G_t = \frac{A_t}{L_t} = \frac{\hat{\boldsymbol{\eta}}_t' \boldsymbol{V}_t}{P_t^l}, \quad \text{with} \quad \hat{\boldsymbol{\eta}}_t = \begin{pmatrix} \hat{\eta}_t^S \\ \hat{\eta}_t^P \\ \hat{\eta}_t^B \end{pmatrix}, \quad \boldsymbol{V}_t = \begin{pmatrix} S_t \\ P_t^h \\ B_t \end{pmatrix}, \tag{5.2.1}$$

where $\hat{\boldsymbol{\eta}}_t$ denote the number of the different assets held by the pension fund at time t. The time to maturity of the liabilities, denoted by l, is assumed constant. This results in a duration of the liabilities equal to $D(l)$. The underlying assumption is that the outflow of benefits equals the inflow of the pension premiums in the fund for all $t \in [0, T_R]$. We also take the asset mix of the pension fund as constant over time, as it represents a long-term rigid view on the financial market. Then, the funding ratio dynamics can be expressed as:

$$\begin{aligned} \frac{\mathrm{d}G_t}{G_t} &= \mu_g \,\mathrm{d}t + \boldsymbol{\sigma}_g' \,\mathrm{d}\boldsymbol{Z}_t, \quad \text{with} \qquad (5.2.2) \\ \mu_g &\equiv \hat{\omega}^S \lambda_S \sigma_S - \hat{\omega}^P \left(\lambda_r \sigma_r D(h) + \sigma_r^2 D(h) D(l)\right) + \lambda_r \sigma_r D(l) + \sigma_r^2 D^2(l), \\ \boldsymbol{\sigma}_g &= \begin{pmatrix} \sigma_g^S \\ \sigma_g^r \end{pmatrix} \equiv \begin{pmatrix} \hat{\omega}^S \sigma_S \\ \sigma_r D(l) - \hat{\omega}^P \sigma_r D(h) \end{pmatrix}, \end{aligned}$$

where $\hat{\omega}^S$ ($\hat{\omega}^P$) denotes the constant fraction of the pension fund assets invested in the stock price index (nominal zero-coupon bond). We refer to Appendix N for the derivation of (5.2.2). Note that the drift and volatility of the funding ratio dynamics are time- and state-invariant as a consequence of this particular specification for the funding ratio.

5.3 Terminal wealth problem

The individual has the objective of maximizing the expected utility of terminal wealth at the retirement date. This terminal wealth equals the purchased life annuity including the funding ratio surcharge. Since the martingale method is applied again to solve this maximization problem, the general derivation will be similar to Chapter 3. For the simple Merton model, Grebenchtchikova, Molenaar, Schotman, and Werker (2017) also discuss a general terminal wealth problem, which is taken as a starting point for the problem statement and treatment.

In case terminal wealth solely equals the accumulated financial wealth in the individual's pension account, the static maximization problem can be stated as:

$$\max_{W_{T_R}} \mathbb{E}_0 \left[\frac{W_{T_R}^{1-\gamma}}{1-\gamma} \right] \quad \text{subject to} \tag{5.3.1}$$

$$W_0 = \mathbb{E}_0 \left[W_{T_R} M_{T_R} \right]. \tag{5.3.2}$$

For a given initial wealth W_0 and level of risk aversion γ, the individual first determines his optimal final wealth $W_{T_R}^*$. Secondly, given the optimal final wealth, the optimal investment strategy that realizes this optimal final wealth is determined. Note that the implicit numeraire of this economy is the money market account, as follows from e.g. Appendix E, in which the origin of the pricing kernel specification M_t is discussed.

However, terminal wealth equals the purchased life annuity at retirement in our setting. This is accounted for by changing the numeraire of the economy to the product of the annuity factor and the funding ratio. A change of numeraire is a quite common technique in derivative pricing, e.g. a similar annuity factor is the numeraire of choice for the pricing of swaptions. This change of numeraire is applied in the life-cycle setting to acquire the appropriate terminal wealth problem that represents the lifetime asset allocation problem of a net pension top-up plan participant. We define terminal wealth under the new numeraire as

$$\widetilde{W}_{T_R} = \frac{W_{T_R}}{\bar{a}_{T_R} G_{T_R}}, \tag{5.3.3}$$

where total wealth W_t is represented by (3.3.1). The annuity factor $\bar{a}_t$ and the funding ratio G_t are stated in respectively (4.2.2) and (5.2.1). From the static budget constraint (5.3.2) or the asset pricing equation, we can uniquely identify the pricing kernel under the new numeraire as

$$\widetilde{M}_t = M_t \bar{a}_t G_t, \tag{5.3.4}$$

where the pricing kernel under the money market account numeraire is stated in (3.2.8). Under the pricing kernel specification (5.3.4), the fundamental asset pricing equation,

$$\widetilde{W}_t = \frac{1}{\widetilde{M}_t} \mathbb{E}_t[\widetilde{W}_{T_R} \widetilde{M}_{T_R}], \tag{5.3.5}$$

and therefore the static budget constraint, holds. The change in numeraire results in the following terminal wealth problem specification:

$$\max_{\widetilde{W}_{T_R}} \mathbb{E}_0 \left[\frac{\widetilde{W}_{T_R}^{1-\gamma}}{1-\gamma} \right] \quad \text{subject to} \tag{5.3.6}$$

$$\widetilde{W}_0 \widetilde{M}_0 = \mathbb{E}_0 \left[\widetilde{W}_{T_R} \widetilde{M}_{T_R} \right]. \tag{5.3.7}$$

The Lagrangian function corresponding to this maximization problem can be expressed as

$$\mathcal{L}(\widetilde{W}_{T_R}, \varphi) = \mathbb{E}_0 \left[\frac{\widetilde{W}_{T_R}^{1-\gamma}}{1-\gamma} \right] + \varphi \left(\widetilde{W}_0 \widetilde{M}_0 - \mathbb{E}_0 \left[\widetilde{W}_{T_R} \widetilde{M}_{T_R} \right] \right), \tag{5.3.8}$$

where φ denotes the Lagrange multiplier. Taking similar steps as in Appendix J regarding the intertemporal consumption problem in Chapter 3, the optimal final wealth can be expressed as

$$\widetilde{W}_{T_R}^{*} = \frac{\widetilde{W}_0 \widetilde{M}_0}{\mathbb{E}_0 \left[\widetilde{M}_{T_R}^{1-\frac{1}{\gamma}} \right]} \widetilde{M}_{T_R}^{\frac{-1}{\gamma}}. \tag{5.3.9}$$

Now that we obtained the optimal terminal wealth, the optimal wealth path for $t \in [0, T_R]$ can be determined by substituting $\widetilde{W}_{T_R}^{*}$ into the asset pricing equation (5.3.5). This results in the following general expression for the optimal wealth path:

$$\widetilde{W}_t^{*} = \frac{\widetilde{W}_0 \widetilde{M}_0}{\widetilde{M}_t} \frac{\mathbb{E}_t \left[\widetilde{M}_{T_R}^{1-\frac{1}{\gamma}} \right]}{\mathbb{E}_0 \left[\widetilde{M}_{T_R}^{1-\frac{1}{\gamma}} \right]}. \tag{5.3.10}$$

Next, we first briefly discuss the benchmark parameter choices regarding this model setup. Then, the optimal investment strategy that finances the above optimal final wealth path is determined.

5.3.1 Benchmark parameter values

The same benchmark parameter values as described in Section 3.4 are used with regard to the financial market, the individual's preferences and the model timeline. With respect to the mortality rates for the individual, the benchmark data is taken as stated in Section 4.2.1. The corresponding benchmark paramater values for the annuity factor price resulting from the estimation procedure in Section 4.2, can be found in Table 4.1.

The introduced funding ratio process also contains parameters for the asset allocation of the pension fund and duration of the pension liabilities. Benchmark values of these parameters are chosen based on the characteristics of the largest pension fund for which APG manages the net pension plan, which is ABP. For the year 2017, the asset mix of ABP consisted of roughly 60% equity and 40% fixed income securities (ABP, 2018b). The average duration of the liabilities was approximately 18 years. For the start of the funding ratio process, we assume an equilibrium situation in which the assets exactly equal the liabilities of the pension fund. The resulting benchmark values are reported in Table 5.1.

parameter	symbol	value
Asset allocation pension fund		
Equity exposure	$\hat{\omega}^S$	0.6
Fixed income securities exposure	$\hat{\omega}^P$	0.4
Initial funding ratio	G_0	1.0
Maturity of pension liabilities (years)	l	18.0

Table 5.1: Benchmark parameter values funding ratio

5.4 Optimal investment strategy total wealth

To determine the investment strategy that yields the highest expected utility of pension wealth, we first need to obtain an explicit expression for the optimal wealth path (5.3.10). This is realized in the following manner. First of all, applying Itô's lemma to the annuity factor $\bar{a}_t$ results in the following geometric Brownian motion dynamics with a time-independent diffusion coefficient:

$$\frac{\mathrm{d}\bar{a}_t}{\bar{a}_t} = \left(-\psi\kappa(\bar{r} - r_t) + \frac{1}{2}\psi^2\sigma_r^2\right)\mathrm{d}t - \psi\sigma_r\,\mathrm{d}Z_t^r. \tag{5.4.1}$$

Then, using the annuity factor dynamics, the dynamics of the pricing kernel under the money market account numeraire (3.2.9) and the dynamics of the funding ratio (5.2.2), the closed-form expression of the pricing kernel under the new numeraire becomes

$$\begin{aligned} \frac{\widetilde{M}_{t+s}}{\widetilde{M}_t} &= \exp\left\{-(1-\psi\kappa)\int_t^{t+s} r_u\,\mathrm{d}u - \left(\frac{1}{2}\|\tilde{\boldsymbol{\nu}}\|^2 + c_1\right)s - \int_t^{t+s}\tilde{\boldsymbol{\nu}}'\,\mathrm{d}\boldsymbol{Z}_u\right\}, \\ \widetilde{M}_0 &= \bar{a}_0 G_0, \end{aligned} \tag{5.4.2}$$

where

$$\tilde{\boldsymbol{\nu}} \equiv \boldsymbol{\lambda} - \boldsymbol{\sigma}_g + \psi\sigma_r\boldsymbol{e}_2, \tag{5.4.3}$$

$$c_1 \equiv \psi\kappa\bar{r} - \sigma_r^2 D(l)D(h) - \sigma_r^2 D^2(l) - \frac{1}{2}\psi^2\sigma_r^2 - \lambda_r\psi\sigma_r + \sigma_g^r\psi\sigma_r. \tag{5.4.4}$$

The vector $\boldsymbol{e}_i$ denotes the i-th unit vector. For $\widetilde{M}_t$, a similar functional form as the pricing kernel under the money market account numeraire is chosen. We refer to Appendix O for the derivation of (5.4.2). Then, we are able to find the following explicit expression for the conditional expectation of the pricing kernel (5.4.2) to a general power:

$$\begin{aligned} \widetilde{\Theta}(t,s,x;r_t) &\equiv \mathbb{E}_t\left[\left(\frac{\widetilde{M}_s}{\widetilde{M}_t}\right)^x\right], \quad t < s < T_R \\ &= \exp\left\{-x\left(\tilde{a}(s-t) + D(s-t)(1-\kappa\psi)r_t\right)\right\}, \end{aligned} \tag{5.4.5}$$

where

$$\tilde{a}(u) \equiv -\frac{1}{2}(x-1)\left\|\boldsymbol{\lambda}\right\|^2 u + \left(\bar{r} - \frac{x\sigma_r\lambda_r}{\kappa}\right)(1-\kappa\psi)(u-D(u)) - \frac{x\sigma_r^2(1-\kappa\psi)^2}{\kappa^2}\left(\frac{1}{2}u - D(u) + \frac{1}{4}D(2u)\right) + c_1 u. \tag{5.4.6}$$

We refer to Appendix P for the derivation of the function $\widetilde{\Theta}(t,s,x;r_t)$. Applying this function yields the following explicit expression for the optimal terminal wealth path (5.3.10):

$$\begin{aligned}\widetilde{W}_t^* &= \widetilde{W}_0\left(\frac{\widetilde{M}_t}{\widetilde{M}_0}\right)^{-\frac{1}{\gamma}}\frac{\widetilde{\Theta}(t,T_R,1-\frac{1}{\gamma};r_t)}{\widetilde{\Theta}(0,T_R,1-\frac{1}{\gamma};r_0)}\\ &= \widetilde{W}_0\exp\left\{\frac{1}{\gamma}\left((1-\psi\kappa)\int_0^t r_u\,\mathrm{d}u + \left(\frac{1}{2}\left\|\tilde{\boldsymbol{\nu}}\right\|^2 + c_1\right)t + \tilde{\boldsymbol{\nu}}'\boldsymbol{Z}_t\right)\right\}\times\\ &\quad \exp\left\{\frac{1-\gamma}{\gamma}\Big(\tilde{a}(T_R-t) - \tilde{a}(T_R) + (1-\psi\kappa)(D(T_R-t)r_t - D(T_R)r_0)\Big)\right\}.\end{aligned} \tag{5.4.7}$$

Note that the diffusion terms of this optimal terminal wealth process originate from the standard two-dimensional Brownian motion in the pricing kernel dynamics and the corresponding Brownian motion that governs the short rate process r_t. Applying a multivariate version of Itô's formula, derived in Appendix K, results in the following dynamics for the optimal terminal wealth process (See Appendix Q for a formal derivation):

$$\mathrm{d}\log\widetilde{W}_t^* = \mu^{\log\tilde{W}_t^*}(t,r_t)\,\mathrm{d}t + \frac{1}{\gamma}\tilde{\boldsymbol{\nu}}'\,\mathrm{d}\boldsymbol{Z}_t - \left(1-\frac{1}{\gamma}\right)(1-\psi\kappa)D(T_R-t)\sigma_r\,\mathrm{d}Z_t^r, \tag{5.4.8}$$

where $\mu^{\log\tilde{W}_t^*}(t,r_t)$ is the drift of the process expressed as a general function of the investment horizon and the short rate. Note that the log dynamics stated above has a level-independent drift and volatility. This optimal wealth path development leads to the desired optimal terminal wealth $\widetilde{W}_{T_R}^*$ over time.

Now that the development of the optimal terminal wealth path is known, the question remains on how to allocate the individual's assets over time to finance these desired pension liabilities. We can identify the optimal investment strategy following steps similar to Chapter 3. The total wealth of the individual under the new numeraire (5.3.3) contains the decision variable $\boldsymbol{\omega}_t$, which stems from the total wealth expression under the money market account numeraire. The development of this process under the new numeraire is proven to satisfy the following SDE (See Appendix Q for the proof):

$$\mathrm{d}\log\widetilde{W}_t = \mu^{\log\tilde{W}_t}(t,r_t)\,\mathrm{d}t + \left(\boldsymbol{\Sigma}'\boldsymbol{\omega}_t - \boldsymbol{\sigma}_g + \psi\sigma_r\boldsymbol{e}_2\right)'\mathrm{d}\boldsymbol{Z}_t, \tag{5.4.9}$$

where $\mu^{\log\tilde{W}_t}(t,r_t)$ is the drift of the process expressed as a general function of the investment horizon and the short rate. Note that once again these log dynamics exhibit a level-independent drift and volatility.

Applying the concept of replication or liability hedging, we are able to derive the optimal investment strategy for the terminal wealth problem (5.3.6, 5.3.7). The optimal portfolio fractions $\boldsymbol{\omega}_t^*$ are chosen such that the individual's pension wealth process $\widetilde{W}_t$, for which the log dynamics dependent on the decision variable are stated in (5.4.9), replicates the optimal terminal wealth path $\widetilde{W}_t^*$. Therefore, the diffusion terms of the dynamics (5.4.9) and (5.4.8) must be identical. This optimality condition results in the following optimal investment strategy for $t \in [0, T_R]$:

$$\omega_t^{S*} = \frac{1}{\gamma}\frac{\lambda_S}{\sigma_S} + \left(1 - \frac{1}{\gamma}\right)\hat{\omega}^S, \tag{5.4.10}$$

$$\omega_t^{P*} = -\frac{1}{\gamma}\frac{\lambda_r}{\sigma_r D(h)} + \left(1 - \frac{1}{\gamma}\right)\left(\frac{D(T_R - t)}{D(h)} + \frac{\psi(1 - \kappa D(T_R - t))}{D(h)} - \frac{D(l)}{D(h)} + \hat{\omega}^P\right), \tag{5.4.11}$$

where the the optimal asset allocation in the locally risk-free asset is residually determined as

$$\omega_t^{B*} = 1 - \omega_t^{S*} - \omega_t^{P*}. \tag{5.4.12}$$

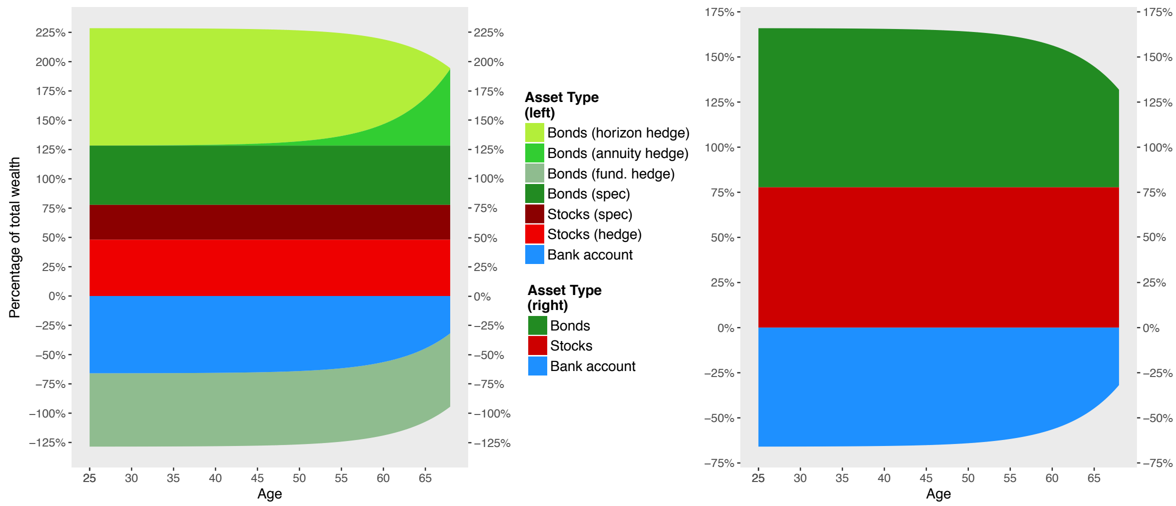
This investment strategy is displayed in Figure 5.1 for the benchmark parameters values discussed in Section 5.3.1. Next, we discuss the components that constitute this optimal investment strategy of total wealth.

On a portfolio level, we observe that the optimal investment strategy $\boldsymbol{\omega}_t^*$ consists of a convex combination of two components, of which the weights are determined by the degree of risk aversion. The first component is the mean-variance tangency portfolio. The second component represents a hedging portfolio which minimizes the individual's risk with respect to changes in the investment opportunity set and the mandatory annuity purchase. These identified components are similar to the composition of the optimal investment strategy of the investment-linked drawdown account, stated in Section 3.5.3. An infinitely risk averse investor would therefore target a particular pension entitlement with certainty, ignoring potential risk premia benefits, as discussed in Section 4.3. Note that in contrast with the stochastic investment strategy in Section 3.5.3, this optimal investment strategy is deterministic and therefore independent of the level of the state variables r_t and $\widetilde{W}_t$. This is partly a result of the choice of the annuity simplification proposed in Chapter 4.

Regarding the optimal portfolio fractions on an asset level, we can first of all observe from (5.4.10) that the allocation in stocks is once again constant over both time and the economic state space. Note that besides the speculative demand, the stock allocation now also consists of a hedging component that originates from the funding ratio surcharge on the annuity purchase at retirement. This hedging component is equal to the stock exposure in the asset mix of the underlying pension fund. The optimal allocation in bonds stated in (5.4.11) also serves both a speculative and an intertemporal hedge purpose. In contrast to the speculative demand, the hedge demand of the bond is time-dependent and generally decreases as the investment horizon shortens, as can be observed in Figure 5.1. The hedge demand of the bond can intuitively be split up into three components: the first term and second term coincide with the first and second component respectively, and the last two terms form the third component.

The first component represents a hedging term for possible changes in the term structure of interest rates and therefore equals a zero-coupon bond with a maturity of the remaining investment horizon. However, since this zero-coupon bond is not available in the market (unless $h = T_R - t$ at a particular time), the desired duration is obtained by taking a corresponding position in the available zero-coupon bond. This component eventually reduces to zero at time of retirement, as can be observed in Figure 5.1.

Figure 5.1: Optimal investment strategy total wealth



The left figure displays the optimal investment strategy split up into the identified components that serve different purposes. The right figure is the corresponding net allocation in the available asset menu. In both figures, the color green corresponds with the zero-coupon bond, red with the equity index and blue with the risk-free asset.

The second component denotes a hedge demand with respect to the mandatory annuity purchase at retirement. We can see that the desired hedge position is a function of the annuity factor parameter ψ, the speed of mean reversion κ and once again the Vasicek duration of the remaining investment horizon. The term $(1 - \kappa D(T_R - t))$ should always lie between zero and one under a correct parameterization, therefore always resulting in a non-negative annuity factor hedge demand. This hedge demand is generally increasing as one approaches retirement, which can be observed in Figure 5.1. Furthermore, since the Vasicek duration is also a function of the speed of mean reversion, we have the following limiting behavior of this term:

$$\lim_{\kappa \to 0} (1 - \kappa D(T_R - t)) = 1, \quad \lim_{\kappa \to \infty} (1 - \kappa D(T_R - t)) = 0, \quad \lim_{T_R \to \infty} (1 - \kappa D(T_R - t)) = 0.$$

Therefore, in case of very low mean reversion of the short rate, the individual prefers to start hedging the annuity risk despite of a large remaining investment horizon and vice versa. In case the remaining investment horizon is (unreasonably) large, the mean reversion becomes insignificant and the individual prefers to disregard the annuity risk when composing the contemporary hedge portfolio. The annuity risk in general is hedged by taking the appropriate position in the available zero-coupon bond that matches the implied duration exposure of the annuity purchase, which is represented by the estimated coefficient ψ.

The third component once again denotes a horizon-independent hedging demand that originates from the funding ratio surcharge, similar to the stock hedging demand. We can observe two opposing effects. Firstly, the term $D(l)/D(h)$ reduces the bond hedge demand. This can be explained by the fact that the dynamics of the funding ratio is partially determined by the development of the value of the pension liabilities. Since we chose to model these pension liabilities as a zero-coupon bond with a particular duration, these liabilities act as an implicit hedging position for the individual. For example, a situation occurs where the value of the pension fund portfolio increases as result of a rise in bond prices. Then, the pension liabilities develop parallel to the bond portfolio resulting from a perfect correlation of bonds with different maturities in the Vasicek model. This development of the value of liabilities tempers the overall funding ratio increase and therefore the surcharge on the annuity purchase. Secondly, the bond exposure in the asset mix of the underlying pension fund increases the hedging demand with respect to the funding ratio surcharge. Hence, the two described effects exhibit opposing directions with regard to the desired bond demand in the hedge portfolio.

Next, we perform a sensitivity analysis on this optimal investment strategy with respect to a selection of assumed benchmark parameter values.

5.4.1 Sensitivity analysis

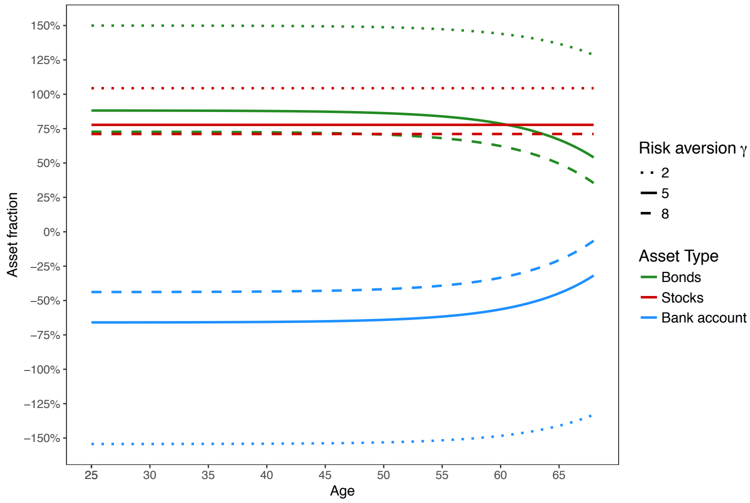
In order to test the robustness of the optimal investment strategy displayed in Figure 5.1 with respect to the assumed benchmark parameter value set, a brief sensitivity analysis is performed with respect to a number of relevant parameters.

First of all, we take into consideration the only behavioral parameter present in the investment strategy, namely the degree of risk aversion (γ). In practice, it has been proven difficult to accurately estimate the degree of risk aversion for a certain individual or group of individuals. The benchmark degree of risk aversion is also solely based on convention in comparable works on lifetime asset allocation. Therefore, a significant uncertainty regarding this parameter value for the net pension top-up plan participants is present. One could obtain a more precise benchmark value by estimating the degree of risk aversion for this group by an appropriate survey for example. A mismatch between the assumed and the actual degree of risk aversion can indeed result in significant welfare losses, as reported by .e.g. Bovenberg et al. (2007). Therefore, the effect of a possible deviation of the benchmark degree of risk aversion on the optimal investment strategy is interesting to analyze.

In Figure 5.2 asset fractions over time for the three different asset types are displayed for three different degrees of risk aversion, ceteris paribus the remaining parameter value set discussed in Section 5.3.1. Note that the benchmark value of the degree of risk aversion is included. Regarding the equity allocation, we observe only a slight decrease for a significantly higher degree of risk aversion. For the same relative deviation resulting in a lower degree of risk aversion, the asset increase is significantly higher. The same effect can be detected for the bond allocation, only manifested even stronger. The shifts for the bond allocation are not entirely parallel, since the degree of risk aversion affects the allocation in the time-dependent hedging portfolio. For $\gamma = 2$, the curvature of the glide path near the retirement date appears to be less present than for $\gamma = 5$.

Next we consider the financial market benchmark parameter values. Regarding the risk premia and the equity volatility, the effects on the optimal investment strategy are straightforward since their presence is restricted to the mean-variance tangency portfolio. However, the effect of parameter value deviations with respect to the short rate process is quite complex. This complexity is partly caused by the fact that the estimated coefficient of the annuity factor simplification ψ is also affected by parameter value changes in the short rate process. Therefore, we focus on the effects of the assumed benchmark values for speed of mean reversion (κ), long-run equilibrium short rate ($\bar{r}$) and the short rate volatility (σ_r).

Figure 5.2: Optimal investment strategy total wealth for different γ



In this figure, the line color corresponds with the asset type and the line type indicates the degree of risk aversion of the glide path.

In order to analyze the effect of an inaccurate parameter value for $\{\kappa, \bar{r}, \sigma_r\}$ on the investment strategy, the following steps are taken. First of all, the effect of the three parameter values on the estimated coefficient for ψ needs to be mapped. Therefore, the procedure stated in Section 4.2.2 is repeated varying one particular parameter value ceteris paribus the remaining benchmark parameter value set. Secondly, the effect on the investment strategy of a unilateral deviation of either $\kappa, \bar{r}$ or σ_r can be determined. Since the bond allocation exhibits horizon-dependence, the effect needs to be determined for the participant's entire working life. The resulting bond fractions for different parameter values over the entire horizon are displayed in Appendix R.

Regarding the speed of mean reversion, the following can be deduced from Figure R.1a. The chosen benchmark value appears to result in a relatively low glide path. A true speed of mean reversion that is higher than the benchmark value, would result in a bond allocation deficiency over the entire horizon. This can be partly explained by the fact that the implicit hedge component of the pension fund liabilities decreases in this situation. On the other hand, a low speed of mean reversion tends to increase the desired bond fraction. This increase in particular applies to the young, partly in order to appropriately hedge the annuity risk. A low speed of mean reversion implies low predictability regarding the interest rate and leads to more uncertainty of the governing interest rate at a particular future retirement date. Therefore, a decreasing speed of mean reversion implies an increasing estimated coefficient of ψ.

From Figure R.1b, one can conclude that a deviation from the benchmark value for the long-run equilibrium short rate only slightly impacts the desired bond fraction path. This parameter does not appear in the optimal investment strategy (5.4.11) and only affects the investment strategy through the estimated coefficient of ψ. In general, an increasing long-run equilibrium short rate exerts a decreasing effect on the annuity factor. This leads to a smaller estimated coefficient ψ, since a bond with a lower duration would suffice to accurately represent the annuity factor. In figure R.1b, this effect is observed by a lower bond fraction near retirement for lower values of $\bar{r}$.

Considering the value for the volatility of the short rate, in Figure R.1c it can be observed that the shape of the bond fraction path over the investment horizon remains quite identical for different values of σ_r. However, a true parameter value which is slightly higher than the benchmark value seems to imply a significant lower bond fraction over the entire horizon. This effect is mainly attributed to the fact that σ_r also affects the desired mean-variance tangency portfolio. In general, an increase in σ_r would also result in an increase of the estimated coefficient ψ, since this shift implies more uncertainty regarding the annuity factor. However, this effect is rather marginal compared to the impact of σ_r on the speculative bond demand.

All in all, taking into account the magnitude of parameter deviations needed to acquire substantial deviations in the bond fraction allocation path, we can conclude that the investment strategy with regard to the short rate process benchmark values is rather robust. General uncertainty with respect to the degree of risk aversion would possibly exert a larger impact on the investment strategy.

5.5 Human capital specification

The specification of the existing human capital eventually affects the optimal financial wealth allocation. Therefore, a realistically defined human capital process is a requirement for an adequate investment strategy. In Chapter 3, we considered human capital to be bond-like as well as constant over time and deterministic. With only a labor income component, this resulted in a definition of human capital (3.3.4) with corresponding dynamics (3.5.14). As discussed in Chapter 4, we retain the bond-like human capital specification, backed by empirical evidence. Therefore, the short rate is not incorporated in the growth rate of the labor income process. In case the human capital definition remains as stated above, we can then easily derive that under the new numeraire the financial wealth allocation is obtained by the same adjustments to the total wealth allocation as under the cash numeraire, given in (3.5.16).

However, employees generally do not start out in a top-up plan immediately at the beginning of their career. Therefore, in order to distinguish between periods of one's career in which he is eligible or not eligible for participation in a top-up plan, the labor income needs to exhibit growth. Consequently, we redefine the labor income component subject to a growth rate and possible exposures to the risk factors present in the stated economy. In accordance with e.g. Munk and Sørensen (2010), the dynamics of labor income are defined as

$$\frac{\mathrm{d}Y_t}{Y_t} = \mu_y(t)\,\mathrm{d}t + \boldsymbol{\sigma}_y'\,\mathrm{d}\boldsymbol{Z}_t, \quad 0 < t < T_R, \tag{5.5.1}$$

where $\boldsymbol{\sigma}_y^\top = \left(\rho_{ys},\ \sqrt{1-\rho_{ys}^2}\right)$ and the drift term is equal to the growth rate of the labor income $\mu_y(t)$, which is a deterministic function of time. The term ρ_{ys} denotes the correlation between the stock and the labor income. Note that the labor income process is fully spanned and therefore the assumption of a complete financial market remains valid. In case this redefined labor income is the sole component of human capital, following Equation (3.3.4), human capital can be expressed for $t \in [0, T_R]$ as:

$$H_t = Y_t \int_t^{T_R} \exp\{\hat{a}(s-t) - D(s-t)r_t\}, \quad \text{with} \tag{5.5.2}$$

$$\hat{a}(s-t) = -a(s-t) - \boldsymbol{\lambda}'\boldsymbol{\sigma}_y(s-t) - \frac{\sigma_g^r \sigma_r}{\kappa}\Big((s-t) - D(s-t)\Big) + \log p_s + \int_t^s \mu_y(u)\,\mathrm{d}u,$$

where p_t denotes the premium contribution rate as percentage of the labor income at time t. Applying the multivariate version of Itô's lemma given in Appendix K on human capital defined as a function of the short rate and the labor income process, results in the following human capital dynamics:

$$\frac{\mathrm{d}H_t}{H_t} = \mu^{H_t}(t, r_t, Y_t)\,\mathrm{d}t + \left(\sqrt{1-\rho_{ys}^2} - D_t^H \sigma_r\right)\mathrm{d}Z_t^r + \rho_{ys}\,\mathrm{d}Z_t^S, \quad \text{with} \tag{5.5.3}$$

$$D_t^H \equiv \frac{\int_0^{T_R-t} D(h)\exp\{\hat{a}(h) - D(h)r_t\}\,\mathrm{d}h}{\int_0^{T_R-t} \exp\{\hat{a}(h) - D(h)r_t\}\,\mathrm{d}h}. \tag{5.5.4}$$

Then, according to the composition of total wealth (3.3.3), the optimal asset allocation for financial wealth can easily be derived to be

$$\boldsymbol{\theta}_t^* = \left(1 + \frac{H_t}{F_t}\right)\boldsymbol{\omega}_t^* - \frac{H_t}{F_t}\begin{pmatrix} \frac{\rho_{ys}}{\sigma_S} \\ \frac{D_t^H}{D(h)} - \frac{\sqrt{1-\rho_{ys}^2}}{\sigma_r D(h)} \end{pmatrix}, \tag{5.5.5}$$

following a similar derivation as in Appendix M. The quantity D_t^H is given in (5.5.4).

Nonetheless, since we are interested in the investment strategy for the top-up plan, stochasticity in the labor income process brings forth complications with regard to the identification of the time point of entry in the top-up plan (T_E). A deterministic point of entry keeps the timelines tractable and in order to accomplish this, we assume $\boldsymbol{\sigma}_y \equiv \mathbf{0}$. Under this assumption, (5.5.5) almost reduces to (3.5.16), between which the deterministic growth rate in $\hat{a}(s-t)$ is the sole distinction. Now that the point of entry is deterministic and therefore known ex ante, guaranteed pension benefits with regard to the second pillar pension are also tractable in this model setup. As discussed in Section 2.2.1, a top-up pension plan implies different forms of guaranteed pension benefits. First of all, the first pillar pension (AOW) is a state guaranteed pension benefit. Secondly, the underlying occupational pension plan that the top-up plan participant buys into at the retirement date, also forms a pension benefit basis besides the accumulated pension right in the top-up plan. The effect of such guarantees on the financial wealth investment strategy is also discussed in Section 3.5.5, only for a pension that consists of an investment-linked drawdown account.

In this model setup, we treat both the first and second pillar pension as guaranteed pension benefits. The second pillar yearly pension benefit can then be expressed as

$$H^{(2)} = \zeta\left(\int_0^{T_E} [Y_u - \underline{\Upsilon}]_+\,\mathrm{d}u + (\overline{\Upsilon} - \underline{\Upsilon})(T_R - T_E)\right), \tag{5.5.6}$$

where ζ denotes the pension right accrual rate in the occupational pension scheme over the pensionable salary. The cut-off point of the pensionable salary is represented by $\overline{\Upsilon}$, whereas the

AOW offset is denoted by $\underline{\Upsilon}$. This leads to the following specification of the guaranteed yearly pension benefit amount:

$$H^{(1,2)} = AOW + H^{(2)}. \tag{5.5.7}$$

The actual component of human capital that consists of labor income for $t \in [0, T_R]$ with respect to the net pension top-up plan, is therefore equal to

$$H_t^{np} = \mathbb{E}_t \left[\int_0^{T_R - t} K(t+h) \frac{M_{t+h}}{M_t} \, \mathrm{d}h \right], \quad \text{with} \tag{5.5.8}$$

$$K(u) = \begin{cases} 0 & \text{for } u \in [0, T_E) \\ p_u(Y_u - \overline{\Upsilon}) & \text{for } u \in [T_E, T_R) \end{cases}. \tag{5.5.9}$$

In the next section, we discuss the benchmark parameter values with respect to these components of human capital.

5.5.1 Benchmark parameter values

First of all, the benchmark parameter values with respect to the human capital process are an addition to the parameter values discussed in Section 5.3.1. The chosen values are displayed in Table 5.2.

The AOW amount is taken with respect to a single individual in the year 2018[1] and the pensionable salary cut-off is coherent with the net pension discussed in Section 2.2.1. Both the second pillar pension accrual rate and the AOW offset are taken as most recently reported by ABP (2018b), the largest pension fund to facilitate a net pension plan. Regarding the parameters of the income process, the following simplistic calibration is performed. Based on the 2018 income data of all participants of ABP, the average age of the individuals who would be eligible for the net pension for the first time, is 52 years, which implies $T_E = 27$. The initial labor income at age 25 and the deterministic growth rate are chosen in such a way, that the individual is eligible to participate in the net pension for a corresponding 16 years (since $T_R = 43$). Note that the deterministic time of entry T_E is therefore implied by the choice of $\mu_y(t)$ and Y_0. One can observe that for the benchmark values in Table 5.2, the corresponding T_E is indeed equal to 27. Regarding the fiscal maximum pension contribution rates (p_t) for the net pension, the benchmark values are provided in the left column of Table B.1 in Appendix B.

By opting a constant $\mu_y(t)$, we choose for a linear growth pattern of the labor income. A linear income growth pattern is generally inaccurate to represents one's career path. However, in this setup, it is merely a benchmark choice that determines the weight assigned to the guaranteed pension and therefore the pension wealth resulting from the net pension top-up plan. Further calibration of the labor income career pattern of the population of potential top earners could be performed in order to accurately represent the labor income process. For now, this is outside of the scope of this thesis.

In the next section, we consider the implications of the identified human capital components on the optimal investment strategy in case of the newly introduced numeraire.

[1] `https://www.svb.nl/int/nl/aow/hoogte_aow/bedragen/`

parameter	symbol	value
First and second pillar pension		
State pension (yearly, €)	AOW	14,176
Pensionable salary cut-off (€)	$\overline{\Upsilon}$	105,075
State pension offset (€)	$\underline{\Upsilon}$	13,350
Accrual rate second pillar pension	ζ	0.01875
Income process		
Income growth rate (continuously)	$\mu_y(t)$	log(1.03)
Initial income (yearly, €)	Y_0	47,000

Table 5.2: Benchmark parameter values human capital

5.5.2 Optimal investment strategy financial wealth in a top-up plan

Both the specified guarantees and the part of the labor income that is eligible for the net pension, affect the optimal investment strategy for financial wealth in the top-up plan. As a result of the changed numeraire of the economy, taking into account guarantees is not straightforward. Recall that for the terminal wealth problem in this chapter, the utility of the pension right at the retirement date is maximized. The pension benefit guarantee therefore acts as a constant yearly amount under the new numeraire, to be added to the converted pension wealth resulting from the top-up plan at T_R. Contrary to the financial wealth and human capital eligible for the net pension top-up plan, this guaranteed amount is not subject to the mandatory conversion regime implied by the numeraire. Consequently, total wealth under the new numeraire cannot easily be split up into the different identified components. Evaluating the composition of total wealth under the cash numeraire appears to be more suitable for the situation at hand. Therefore, total wealth under the cash numeraire, taking into account the conversion regime and the guarantees, can be expressed as follows:

$$\begin{aligned} W_t^{(\boldsymbol{\omega}_t^*)} &= F_t^{(\boldsymbol{\theta}_t^*)} + H_t^{np} + H^{(1,2)}(\bar{a}_t G_t) \\ &\equiv F_t^{(\boldsymbol{\theta}_t^*)} + H_t^{np} + \bar{A}_t, \qquad\qquad T_E \leq t \leq T_R, \end{aligned} \tag{5.5.10}$$

where $W_t^{(\boldsymbol{\omega}_t^*)}$ represents total wealth subject to the optimal investment strategy specified in (5.4.10) and (5.4.11). $F_t^{(\boldsymbol{\theta}_t^*)}$ represents the financial wealth process subject to an optimal investment strategy. Note that the defined quantity $\bar{A}_t$ reduces to the guaranteed yearly pension entitlement $H^{(1,2)}$ after conversion. Regarding financial and net pension human wealth, the following boundary conditions apply:

$$\begin{cases} F_{T_E}^{(\boldsymbol{\theta}_t^*)} = 0 \\ H_{T_R}^{np} = 0 \end{cases}, \tag{5.5.11}$$

which implies that total wealth at time T_R solely consists of the ex ante known pension guarantee and the accrued financial wealth in the top-up plan.

Determining the dynamics of (5.5.10) and applying the concept of replication, yields the following optimal asset allocation strategy regarding financial wealth in the top-up plan for $t \in [T_E, T_R)$

(see Appendix S for the full derivation):

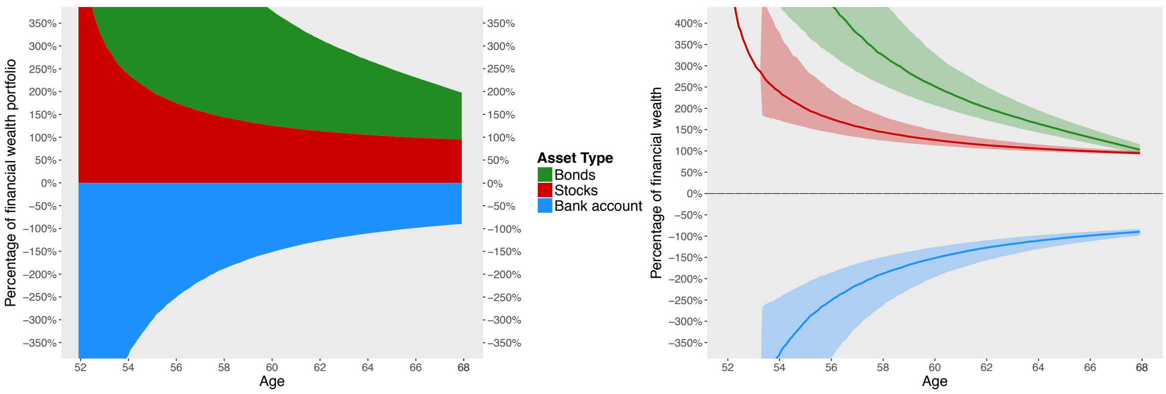
$$\boldsymbol{\theta}_t^* = \left(1 + \frac{H_t^{np}}{F_t} + \frac{\bar{A}_t}{F_t}\right)\boldsymbol{\omega}_t^* - \frac{H_t^{np}}{F_t}\begin{pmatrix} 0 \\ \frac{D_t^H}{D(h)} \end{pmatrix} - \frac{\bar{A}_t}{F_t}\begin{pmatrix} \hat{\omega}^S \\ \frac{\psi\sigma_r - \sigma_g^r}{\sigma_r D(h)} \end{pmatrix}, \tag{5.5.12}$$

where

$$D_t^H \equiv \frac{\int_0^{T_R - t} D(h) P_t^h K(t+h)\,\mathrm{d}h}{\int_0^{T_R - t} P_t^h K(t+h)\,\mathrm{d}h}, \tag{5.5.13}$$

with $K(u)$ as defined in (5.5.9). The median of this optimal investment strategy resulting from a simulation study with the discussed benchmark parameter values, is displayed in Figure 5.3. From (5.5.12), we can observe that the strategy for the top-up plan's financial wealth is affected by the market value of the labor income eligible for the top-up plan, the actual accrued financial wealth in the top-up plan and the market value of the guarantee under the cash numeraire. Contrary to the optimal financial strategy of financial wealth for the IDC scheme specified in Chapter 3, the guarantee plays a direct role in the asset allocation instead of manifesting through the remaining duration of human capital D_t^H.

Figure 5.3: Optimal median investment strategy financial wealth



(a) Portfolio composition (b) Asset fractions

Medians are based on $n = 10,000$ scenario paths. In figure (b), the shaded areas represent the interquartile range of the simulated scenarios corresponding to the median with the same color. The interquartile range for the first one and a half year in the top-up plan is not displayed because of heavy fluctuations of this stochastic investment strategy.

In Figure 5.3, the classical life-cycle pattern of the asset holdings can be observed, similar to the life-cycle pattern for the IDC scheme in Figure 3.3. However, one particular difference occurs. At time of retirement, the median optimal portfolio composition for financial wealth does not coincide with the median optimal portfolio composition for total wealth in Figure 5.1. This is caused by taking into account the guarantee $H^{(1,2)}$, which implies the total human capital is not depleted at retirement. When comparing the total and financial wealth strategies in Figures 5.1 and 5.3 at T_R, one can conclude that taking into account the guarantee would lead to approximately 20% additional equity holdings and 40% additional bond holdings. Hence, the guarantee actually implies an increase in the desired bond fraction instead of acting as a substitute. This effect, only smaller, is also found and addressed in Section 3.5.5.

With respect to the desired equity allocation of financial wealth, a general decreasing pattern is observed for the median portfolio composition. The equity allocation at the time of entry in the top-up plan starts out quite high and seems to converge over time judging from the volatility pattern. This effect can be explained by the fact that both the ratios $\bar{A}_t/F_t$ and H_t^{np}/F_t are large at the start of the top-up plan due to a small amount of accrued pension wealth in this plan. A slower decrease over the lifetime is observed for the desired bond fraction, which is caused by the interplay between the dynamics of the guarantee and the remaining duration of human capital eligible for the top-up plan.

Note that this investment strategy for financial wealth is a stochastic investment strategy. In Figure 5.3b, it becomes clear that the variability for the first few years after time of entry is significantly large. Then, the variability reduces over age to zero at T_R. Compared to Figure 3.3b, the variability is larger for all ages, which can be attributed to the fact that the pension guarantee under the cash numeraire exhibits significant uncertainty. Once again, a significant short position is taken for the entire duration of the top-up plan in order to finance the desired exposure to equity and the available bond.

5.5.3 Sensitivity with respect to the entry time

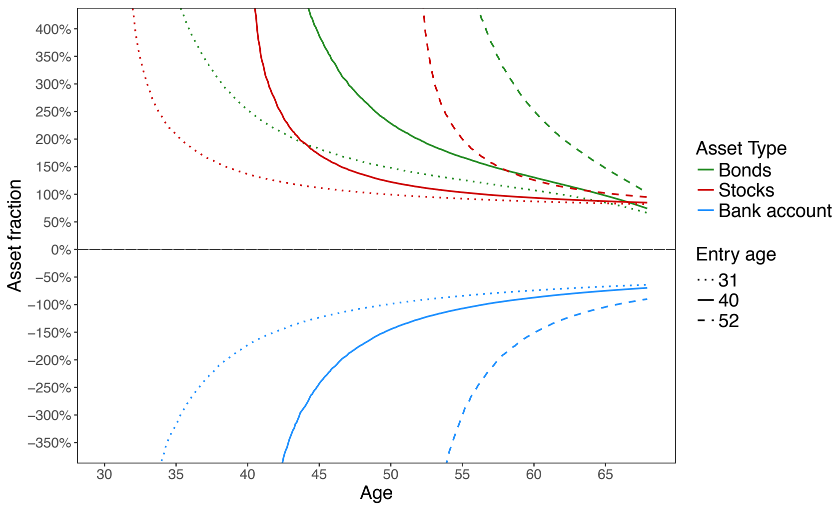
Generally, participants in the net pension top-up plan do not all enter the plan at exactly the same age. A substantial spread is observed in the data for ABP. However, APG operates a uniform life-cycle investment strategy with regard to these plans for the foreseeable future. Therefore, the impact of the entry age on the desired financial wealth asset allocation in the top-up plan is an interesting factor to analyze.

In order to establish an earlier entry time of the individual in the top-up plan, we choose to retain the same labor income growth rate and increase the initial income Y_0. In other words, an upward parallel shift of the labor income process is created, resulting in an extended window of eligibility for the top-up plan and a larger second pillar pension guarantee. In Figure 5.4, the median optimal asset fractions of financial wealth are displayed for $T_E \in \{6, 15, 27\}$, corresponding with initial incomes of respectively $Y_0 \in \{47,000;\ 67,000;\ 87,000\}$. The ages of the participants at the three different T_E will respectively be 31, 40 and 52, for which the latter corresponds with the benchmark scenario. We can observe that the optimal asset allocation substantially differs for the three different times of entry.

One can observe from Figure 5.4 that the desired asset fractions differ substantially for a participant who just entered the top-up plan compared to a participant that has been in the plan for more than five years. This is a consequence of the fact that the participant who just entered the plan, has little financial wealth corresponding to this top-up plan. Therefore, this participant strives for a relatively higher exposure to equity. Regarding the fraction of equity, to a certain extent a converge over time is displayed for all three entry ages. The equity fraction for entry age 31 acts as a lower bound over time for the equity fractions corresponding to the higher entry ages. For the entry ages 31 and 40, only minor rebalancing of the equity fraction is needed from age 50 until T_R. This converging pattern over time is weaker for the bond fraction. Therefore, the equity fractions for the three different entry ages exhibit a smaller spread at T_R compared to the bond fractions.

Note that the portfolio compositions for the three entry times are not identical copies of each other at different horizons. Different implicit positions in the pension guarantee and different present values of labor income eligible for the top-up plan cause heterogeneity in the shape of glide paths for the three entry times. As the time of entry increases, the bond and equity fractions undergo a relatively quicker descent over the remaining horizon. The bond fractions for entry ages 40 and 51 also exhibit a significant drop near the retirement date, which results in a lower fraction in bonds compared to equity. This pattern seems absent for the high entry age 52. One explanation could be that for lower entry ages, the opportunity occurs to spread the hedging of interest rate risk with respect to annuity purchase at T_R over a relatively larger time window. Therefore, the participant can afford a lower exposure to bonds near retirement. Next, we direct our attention towards the resulting replacement rates for the different horizons in the top-up plan.

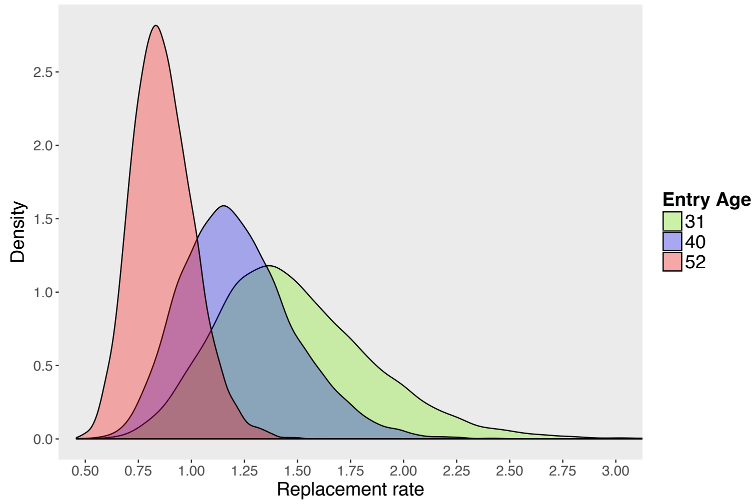
Figure 5.4: Optimal median investment strategy financial wealth for different T_E



Medians are based on $n = 10,000$ *scenario paths. The line color corresponds with the asset type and the line type indicates the entry age in the top-up plan. The benchmark entry age is 52.*

In Figure 5.5, simulated densities of the replacement rate for the three different entry times are displayed. The replacement rate is the ratio of the acquired pension benefit at time of retirement versus the last earned labor income. These densities correspond to the application of the optimal strategy of financial wealth (5.5.12) on the same scenario set on which the medians in Figure 5.4 are based. Because of the pension benefit guarantee, the replacement rate never falls below a threshold of approximately 50% for all three times of entry in the top-up plan. All three the densities are right-skewed, for which the skewness is negatively correlated with the entry age. The volatility of the replacement rate is the lowest for entry age 52. This can be explained by the fact that over a shorter horizon in the top-up plan, the portfolio returns cannot significantly diverge. One can observe that a for an entry age of 52 the mean replacement rate is approximately 80%. This is the lowest of the three entry ages. The mean replacement rate for entry age 31 in the top-up plan is significantly higher, with approximately 135%. Of course, the early entry individual also acquires a higher second pillar pension guarantee in absolute terms. But, on the other hand, this individual is also subject to possible negative portfolio returns in the top-up plan. Apparently, being able to access a DC top-up plan at a young age, in which significant risk is taken the first few years, yields a higher relative pension wealth in this setup.

Figure 5.5: Simulated density of the replacement rate for different T_E



The simulated densities for $T_E \in \{6, 15, 27\}$ *are based on the endpoints* (T_R) *of* $n = 10,000$ *scenario paths.*

Recall that the labor income specification is deterministic in the results above. Therefore, the treated maximization problem implicitly represents a problem in which habit formation is present, similar to the study of Cairns et al. (2006) discussed in Chapter 4. So, the optimal total and financial wealth strategies for a goal function in which the expectation of the replacement rate is maximized at T_R, are equal to the strategies discussed above.

5.6 Discussion and concluding remarks

In this section, we discuss the drawbacks and complications of the developed life-cycle model setup and the resulting benchmark investment strategy regarding the top-up plan. These limitations often imply opportunities for further research. First of all, assuming log normality for the stock price index (the Black-Scholes assumption) seems a misspecification based on empirical evidence of e.g. volatility clustering. Therefore, in reality the optimal investment strategies might be slightly different. However, one could argue that over long investment horizons, the Black-Scholes assumption is sufficiently accurate to capture the desired exposures to the type of asset that is equity. Secondly, it is well-known that a one-factor interest rate model cannot accurately capture the entire yield curve. This simplification could also possibly have an impact on the optimal investment strategy.

In the life-cycle model, the funding ratio specification is chosen in such a way, that the returns of the process exhibit independent, identically distributed log normal returns. In practice, a large set of regulatory restrictions and policies keeps the funding ratio from falling below a certain threshold and a particular solvency goal is targeted. This behavior might have an impact on the desired funding ratio related components in the hedge portfolio. Nonetheless, one could argue that the impact of naively modeling the funding ratio is only small, as extremely well and extremely poor scenario developments of the funding ratio balance each other out. On a related note, concerning the human capital specification, the second pillar occupational pension is specified as a guarantee and therefore risk-free in our model. However, in case the pension fund's portfolio has poor performance and the funding ratio falls, cuts and withholding indexation of the second pillar pension benefits might occur in practice. Therefore, the desired add-on of risk as a result of the pension guarantees, might be less in reality due to an increase in uncertainty.

For the benchmark optimal portfolio composition of financial wealth in the top-up plan, one should realize that this life-cycle strategy carries substantial dependence on the chosen benchmark parameter values. In order to capture this fact and create insight in the effect of certain benchmark choices, we performed a number of sensitivity analyses with respect to these parameter values. However, the goal of this thesis is to create a life-cycle model in which clear economic intuition is embedded and dependencies can be studied, rather than to create an instantly applicable life-cycle strategy. Although the benchmark parameter values are chosen with care and substantial reasoning, further calibration of certain quantities might be necessary, for example with respect to the level of risk aversion.

Taking into consideration the benchmark optimal investment strategies of both financial and total wealth, we observe significant short positions over the entire horizon. As it is common practice for pension funds not to work with leveraged portfolios when it comes to DC asset allocation, a mapping of the desired exposures into a 'feasible' life-cycle strategy needs to take place. For example, the short position in cash and long position in the available bond can be replicated by a portfolio of swaps. Moreover, in our financial market, we only have access to one zero-coupon bond with a constant maturity. In practice, there exists of course a wide variety of bonds with different maturities and risk characteristics that can be used to obtain the desired optimal duration and risk exposure resulting from the life-cycle model.

As was mentioned, APG operates a uniform life-cycle investment strategy with regard to top-up plans for the foreseeable future. In this fact, the pension fund manager does not stand alone, as there barely exist options for tailored (and therefore stochastic) life-cycle funds in the Netherlands. However, we observed for instance a large difference in desired life cycles for individuals that enter the top-up plan at different ages. Therefore, a uniform (benchmark case) life-cycle strategy is far from optimal. A possible better proposition could be a weighted average of optimal glide paths, where the weights are based on the ages of the participant pool of the top-up plan. This choice of uniform versus tailored life-cycle pensions also appears when the median optimal life-cycle is implemented. The optimal investment strategy of financial wealth in this chapter is a stochastic investment strategy. In case the median of these simulated investment strategies is implemented, one could end up in a scenario where the median optimal life-cycle performs very poorly. In case it is possible to conduct a stochastic life-cycle strategy based on a limited amount of information, this could possibly lead to significant welfare gains, as shown by e.g. Cairns et al. (2006).

Chapter 6

Investment strategy for top-up plans: a numerical approach

In this chapter, the fact that the funding ratio surcharge at retirement is subject to a minimum value, is taken into account. This last main characteristic of the net pension top-up plan is abstracted away from in Chapter 5. As discussed in Chapter 2, the funding ratio surcharge consists of the larger of the fund's actual funding ratio at the time of retirement, and the fund's minimum required solvency (MVEV). This results in a call-option-type expression in the numeraire of the economy, for which the dynamics are not defined. Also, no suitable theoretical moment conditions for future returns on the call option can be derived. Therefore, we have to resort to numerical methods to determine a preferred investment strategy for pension wealth in the top-up plan in case the minimum surcharge is taken into account.

This guaranteed minimum funding ratio surcharge can influence the optimal investment strategy in the following way. For example, the economic scenario path has caused the funding ratio level to drop far below the minimum surcharge with only a small remaining investment horizon. Then, it is very likely that the individual has to face the minimum surcharge at retirement, which could result in a decline of the preferred hedge demand with respect to the funding ratio level.

This chapter can be viewed as an extension of Chapter 5. Therefore, the general setup and assumptions of Chapter 5 are adopted along with the discussed benchmark parameters. Firstly, the methodology regarding the numerical method and the intuition it rests on, is discussed. Secondly, the results of the optimization program are discussed, including the sensitivity of the results

6.1 Methodology

The acquired pension wealth at retirement taking into account the minimum funding ratio surcharge, can be expressed as:

$$\widetilde{\widetilde{W}}_{T_R} = \frac{W_{T_R}}{\bar{a}_{T_R} \max\{\Psi, G_{T_R}\}}, \tag{6.1.1}$$

where the minimum required solvency is denoted as Ψ. In case $G_t > \Psi$ for all t, the funding ratio surcharge involves the actual funding ratio, hence represents the situation discussed in Chapter 5. In case $G_t < \Psi$ for all t, the funding ratio surcharge consists of the constant term Ψ, which does not affect the optimization problem of terminal wealth that follows as Ψ is exogenously determined. It would result in a constant multiplier in case of power utility, which can be neglected. Therefore, an investment strategy that yields the highest expected utility of terminal wealth $W_{T_R}/\bar{a}_{T_R}$, represented by $\{\breve{\boldsymbol{\omega}}_t^* : t \in [0, T_R]\}$, is optimal in this case. This investment strategy can easily be derived by equating the funding ratio volatility to zero and following the theory presented in Chapter 5. The resulting optimal asset allocation for total wealth can then be expressed as:

$$\breve{\boldsymbol{\omega}}_t^* = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda} + \left(1 - \frac{1}{\gamma}\right) D(h)^{-1}\Big(D(T_R - t)(1 - \psi\kappa) + \psi\Big)\boldsymbol{e}_2, \tag{6.1.2}$$

where $\boldsymbol{e}_i$ denotes the i-th unit vector.

In general, the switching regime between the two described situations ($G_t > \Psi$ for all t and $G_t < \Psi$ for all t) determines the optimal investment strategy of total wealth W_t. This optimal strategy will therefore be a combination of the optimal investment strategies corresponding to the two separate situations. Since in both situations the optimal investment strategy is only horizon-dependent and G_t exhibits a level-independent expected growth rate, the economic state will not affect the preferred investment strategy. We express the investment strategy corresponding to the maximization of the expected utility of $\widetilde{\widetilde{W}}_{T_R}$ as a linear combination of the optimal investment strategies in the two possible situations:

$$\tilde{\boldsymbol{\omega}}_t(\alpha) \equiv \alpha\boldsymbol{\omega}_t^* + (1 - \alpha)\breve{\boldsymbol{\omega}}_t^*, \tag{6.1.3}$$

where $\breve{\boldsymbol{\omega}}_t^*$ is stated in (6.1.2) and $\boldsymbol{\omega}_t^*$ consists of the elements stated in (5.4.10) and (5.4.11). Note that the investment strategy is a function of the weight α. Intuitively, the weight α that yields the optimal investment strategy $\tilde{\boldsymbol{\omega}}_t^*$ should be between zero and one; The individual would never opt to hedge more than 100% of the funding ratio surcharge risk, even if it were to concern a highly risk averse individual and vice versa.

Next, the method for determining the optimal value of weight α is described. This method partly consists of a Monte Carlo simulation of the goal function, dependent on α. The applied method for process simulation and numerical approximation is described in Section 3.4. The optimization routine consists of the following steps:

1) Simulate $n = 10{,}000$ scenario paths of the short rate r_t for $t \in [0, T_R]$.

2) Determine the corresponding scenario paths for the funding ratio G_t and the annuity factor $\bar{a}_t$. Also determine the total wealth scenario paths subject to the investment strategy (6.1.3), which for scenario $i = 1, \ldots, n$ can be expressed as $W_{t,i}^{(\tilde{\boldsymbol{\omega}}_t(\alpha))}$.

3) Based on the n scenarios, calculate the estimate of $\mathbb{E}\left[u\left(\widetilde{\widetilde{W}}_{T_R}^{(\tilde{\boldsymbol{\omega}}_t(\alpha))}\right)\right]$, which is stated as follows:

$$\bar{u}^{(\alpha)} = \frac{1}{n}\sum_{i=1}^{n} u\left(\frac{W_{T_R,i}^{(\tilde{\boldsymbol{\omega}}_t(\alpha))}}{\bar{a}_{T_R,i}\max\{\Psi, G_{T_R,i}\}}\right).$$

4) Determine the certainty equivalent return (*cer*) based on the certainty equivalent (*ce*) corresponding to the mean of the utility of terminal wealth calculated in step 3). This

certainty equivalent return can be expressed as a function of α:

$$cer(\alpha) = \frac{ce(\alpha)}{\widetilde{\widetilde{W}}_0} - 1 = \frac{u^{-1}\left(\bar{u}^{(\alpha)}\right)}{\widetilde{\widetilde{W}}_0} - 1.$$

5) Determine the weight which maximizes the certainty equivalent return on pension wealth:

$$\alpha^* = \underset{\alpha \in \mathbb{R}}{\operatorname{argmax}}\, cer(\alpha).$$

6) Repeat steps 1) - 5) $m = 250$ times.

7) Repeat steps 1) - 6) for each $\Psi \in \{0.1j : j = 0, 1, \ldots, 20\}$.

8) Repeat steps 1) - 7) for each $\gamma \in \{2, 5, 8, 10\}$.

In general, the number of scenario paths n is chosen to be large such that the accuracy of the estimate regarding the first moment of the utility of pension wealth is sufficient. The optimization procedure is replicated m times in order to obtain insight in the robustness of the optimization results and therefore its reliability. As goal function, the certainty equivalent return rather than the certainty equivalent itself is chosen. This is because a return is more evident to interpret and therefore also gives meaning to a specified tolerance level in the optimization function.

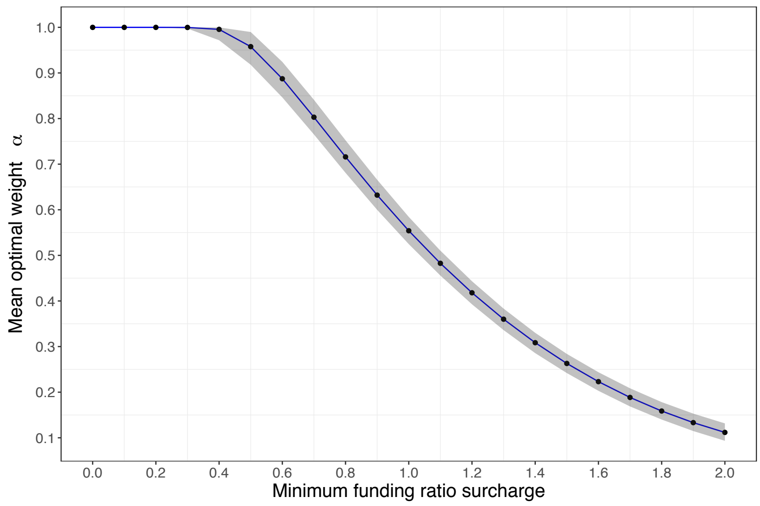
The optimal weight is analyzed along the dimensions of two parameters, namely the minimum required solvency and the level of risk aversion, which can be observed in steps 7) and 8). Naturally, the level of the minimum funding ratio surcharge affects the optimal investment strategy. For example, consider a relatively high minimum required solvency compared to the current funding ratio value. Then, it becomes more likely the individual will end up in the scenario where this minimum acts as the funding ratio surcharge at retirement. Therefore, the optimal weight α^* should be relatively low. The functional form of the relationship of Ψ and α^* is of interest because of this direct impact. Secondly, as concluded in Section 5.4.1, the optimal investment strategy of total wealth in Chapter 5 exhibits relatively high sensitivity with respect to the individual's risk aversion. The sensitivity of the optimal investment strategy corresponding to the maximization problem in this chapter, could consequently also be significant with respect to the level of risk aversion. With this in mind, the optimization routine also incorporates variability over the risk aversion in order to analyze the possible spread in the results. In the next section, the results of the optimization routine are discussed.

6.2 Results

For the benchmark risk aversion $\gamma = 5$, the relationship between the mean of optimal weight α^* and the minimum funding ratio surcharge Ψ is depicted in Figure 6.1. First of all, we can observe that the previously stated intuition regarding the results seems to hold. For relatively low values of Ψ, smaller than 0.4, it is optimal to fully invest in the strategy where the funding ratio surcharge risk is accounted for. This result is intuitive, as almost surely the minimum required solvency is exceeded by the funding ratio level at time of retirement. However, for relatively high values of Ψ, it is not optimal to fully commit to the investment strategy where the surcharge risk is completely ignored. This can be attributed to the fact that the funding ratio starts out at value one and exhibits a positive drift, which yields apparently a substantial probability of the funding ratio exceeding these high values of Ψ at retirement.

Secondly, we can observe from Figure 6.1 that the decline of α^* as Ψ increases, is generally steeper for $0.5 < \Psi < 1$ than for $\Psi > 1$, where G_0 is equal to one. This phenomenon could be explained by the concave shape of the power utility function and the distribution of the funding ratio's probability mass at retirement. Also, the volatility of the obtained optimal weights is highest for values of Ψ near 0.5 and exhibits a slightly decreasing pattern in Ψ for values larger than 0.5. The following explanation can be provided for this phenomenon; For Ψ near 0.5, the effect of a single scenario path that results in the minimum funding ratio surcharge at retirement, has a significant impact on the optimal investment strategy weight through $\bar{u}^{(\alpha)}$. Therefore, a larger spread of α^* resulting from the optimization routine can be observed in this area.

Figure 6.1: Optimal weight α^* for different minimum funding ratio surcharges Ψ



The mean of optimal weight α^* based on m replications, is depicted for different values of Ψ. The shaded area denotes the 2.5 - 97.5 percentile range of the m replications.

Next, we take into consideration the sensitivity of the optimal weight for different levels of risk aversion. From Table 6.1, we can detect a negative relationship between the level of risk aversion and the optimal weight for relatively low values of Ψ. However, for relatively high values of Ψ, the correlation between the level of risk aversion and the optimal weight is positive. Note that the optimal weight spread between $\gamma = 10$ and $\gamma = 2$ is significantly larger for $\Psi = 0.6$ than for $\Psi = 1.4$.

For a highly risk averse individual, the observed decline of α^* in Ψ for relatively low values of Ψ is steeper compared to a risk loving individual. On the other hand, for relatively high values of Ψ, the optimal weight level for a highly risk averse individual is relatively higher compared to a risk loving individual. Regarding the volatility of the optimized weights, for a risk loving individual there seems to be a generally wider spread of α^*. This can be explained by the fact that the mean-variance tangency portfolio is substantially larger for this individual compared to a more risk averse individual. This yields a more volatile portfolio performance and consequently more volatile optimal weights. In spite of that, the reported standard deviations are sufficiently small, thus the results of the optimization routine can be viewed as reliable. All in all, the optimal weight α^* is rather robust with respect to the level of risk aversion.

Table 6.1: Optimal weight α^* for different levels of γ and Ψ

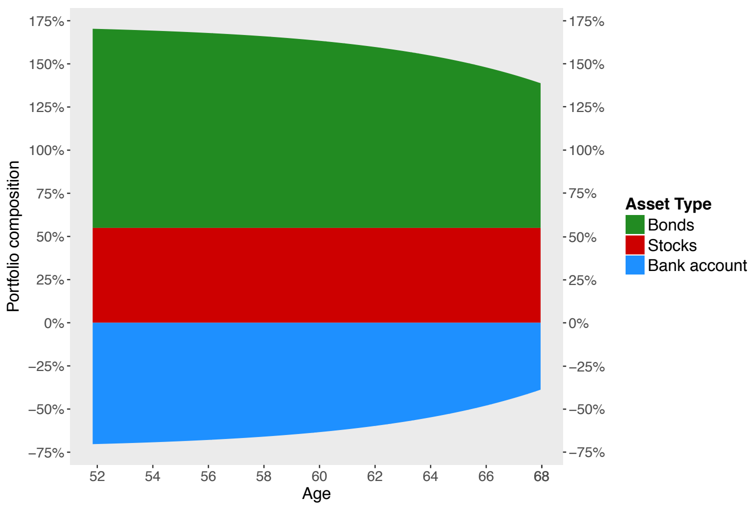
	level of risk aversion (γ)							
	2		5		8		10	
Ψ	Mean	St. Dev.	Mean	St. Dev.	Mean	St. Dev	Mean	St. Dev.
0.2	0.9965	0.0085	0.9999	0.0001	0.9999	0.0000	0.9999	0.0000
0.6	0.9433	0.0263	0.8873	0.0209	0.8321	0.0318	0.8019	0.0361
1.0	0.6181	0.0291	0.5540	0.0153	0.5386	0.0169	0.5352	0.0189
1.4	0.3180	0.0287	0.3085	0.0117	0.3413	0.0113	0.3627	0.0134
1.8	0.1509	0.0281	0.1587	0.0106	0.2060	0.0117	0.2395	0.0153

Values for the optimal weight α^* corresponding to investment strategy (6.1.3) are displayed in this table for different levels of risk aversion (γ) and the minimum funding ratio surcharge (Ψ). The mean and standard deviation correspond the the m replications in the optimization procedure. Note that for $\gamma = 5$, the mean values correspond to Figure 6.1

What is the actual resulting composition of the optimal investment portfolio of total wealth subject to the benchmark parameter values set? In order to answer this question, we must first determine the benchmark value for Ψ. As discussed in Section 2.2, the minimum required solvency for pension fund ABP in 2017 was equal to approximately 104%. We therefore choose the benchmark value $\Psi = 1.04$. The corresponding mean optimal weight α^* is approximately 0.525. For these values, the optimal asset allocation for total wealth is displayed in Figure 6.2 with regard to the top-up plan horizon.

Comparing this portfolio composition to the optimal portfolio when abstracting away from the minimum funding ratio surcharge, displayed in Figure 5.1, one can observe the following. The adjustment in the stock allocation is trivial, since only approximately half of the stock hedge demand is retained in Figure 6.2. However, the desired bond demand has increased over the entire horizon by including the minimum surcharge. This can be partially explained by the fact that the implicit hedge position that is represented by the liabilities of the pension fund, is simultaneously cut.

Figure 6.2: Optimal asset allocation total wealth for the top-up plan horizon



The portfolio composition of total wealth is displayed for $t \in [T_E, T_R]$ based on $\alpha^* \approx 0.525$, corresponding with the benchmark parameter values, amongst which $\Psi = 1.04$ and $\gamma = 5$.

Chapter 7

Conclusion

This chapter concludes the research on optimal life-cycle investment design for Dutch top-up plans. Firstly, we provide a summary of the performed life-cycle model development and discuss the main findings. Secondly, the resulting recommendations with regard to the revision of the life-cycle strategy for top-up plans is discussed. Lastly, relevant suggestions are provided for the extension of this model in future research.

7.1 Summary

In the Netherlands, a gradual shift to a more individualized pension system can be observed, in which the individual faces complex investment decisions and bears the consequences of these decisions. An approach from individual perspective when considering strategic asset allocation over the lifetime, is therefore all the more relevant. Top-up plans such as the net pension represent a class of pension schemes where the general defined benefit and defined contribution approaches to pensions collide. Adapted versions of such plans can potentially become a tool to facilitate more freedom of choice regarding pension accrual in the future.

In this thesis, we developed a model framework in which the optimal life-cycle investment strategy for the net pension top-up plan and similar defined contribution schemes is analyzed. This framework assumes an expected utility maximizing individual regarding pension wealth, operating in a complete financial market with two sources of risk: equity risk and interest rate risk. The explored utility function is the isoelastic utility function. This model captures the main features of the net pension top-up plan. Pension wealth in the top-up plan is build up through fiscally maximized premiums. Upon retirement, the accrued wealth in the top-up plan is converted into a fixed life annuity subject to a funding ratio surcharge. This funding ratio surcharge exhibits a certain minimum level. For each of these features, we explicitly identified the desired components of both a speculative and a hedge portfolio within the proposed framework. Sensitivity analyses have been performed with respect to a sensibly chosen set of benchmark parameter values in order examen the robustness of the the resulting investment strategies. Also, the top-up plan's implicit characteristic of a relatively large build-up of pension rights in the underlying occupational DB pension scheme, is taken into account. This implies a holistic perspective on retirement benefits and savings across the three pension pillars.

We have built up this framework in the following way. Firstly, we discussed a general individual defined contribution scheme which embodies an investment linked-drawdown over the entire lifetime. The financial market, general assumptions about the individual and wealth processes concerning the pension scheme were discussed. Since pension premiums were fixed during the accumulation phase, the consumption was also exogenously determined in this period. We determined the optimal consumption and asset allocation decision subject to this structural break and discussed the impact of bond-like human capital and a pension guarantee on the life-cycle asset allocation. We identified a hedge portfolio for the accumulation phase that consists of a bond position with a duration equal to the weighted average duration of the remaining consumption in the retirement phase. Secondly, we explored a tractable specification for the annuity factor in order to suitably model an annuity purchase at retirement. Thereafter, we specified a terminal wealth problem for the accumulation phase that incorporated a life annuity purchase at retirement subject to a funding ratio surcharge, where we abstracted away from a certain guaranteed minimum surcharge. A change of numeraire technique was applied for this specification and the optimal investment strategy of total wealth was determined and analyzed. Besides the well-known speculative (myopic) portfolio, we identified a minimum risk portfolio that is composed of components in both equity and the zero-coupon bond. These components have been elaborately discussed. Again, the total portfolio is a convex combination of these two portfolios where the weights are dependent on the level of risk aversion. By including a bond-like human capital specification, the asset allocation for the financial wealth in the top-up plan was then determined and analyzed. Lastly, a numerical approach was taken to create insight in the effect of a minimum funding ratio surcharge on the desired life-cycle glide path. This minimum had a clear downward effect on the equity hedge demand and an upward effect on the bond hedge demand.

The developed life-cycle model enabled us to create insight in the interdependencies between different quantities. We showed that the asset mix of the underlying occupational pension fund is directly linked to the hedge demand of the individual. Moreover, the liabilities of the pension fund also represent an implicit hedge position of the individual, which has a decreasing effect on the bond hedge demand. We also showed that the bond hedge demand with respect to the annuity factor at retirement is affected by the speed of mean reversion of the interest rate. Regarding financial wealth in the top-up plan, the assumption on the type of human capital significantly impacts the desired asset allocation. Also the assumed labor income growth has a significant impact on the human capital eligible for the top-up plan, as it implicitly determines the expected time of entry into the plan. Besides that, the level of mean reversion of the individual generally impacts the optimal investment strategy significantly.

7.2 Recommendations

This thesis provides a theoretical background on the life-cycle principle applied to the individual defined contribution scheme that is the net pension top-up plan. Asset managers can use the provided model framework as a handle for decision making and take the theoretical results and analyses into consideration when composing a life-cycle glide path for similar top-up plans. APG has prompted a revision of the existing policies of top-up plans, including the net pension and other similarly defined schemes. This revision is based on recent changes in legislation, which replaced a fixed surcharge on the annuity rate with a variable funding ratio surcharge as described in this thesis. Based on the results and analyses in this thesis, the following recommendations can be made regarding the revised composition of a uniform and deterministic life-cycle investment strategy for these plans.

Firstly, a general increase in equity holdings over the lifetime seems desirable, as the asset mix of the underlying occupation pension fund directly creates a previously absent equity demand in the hedge portfolio for an individual in the top-up plan. The minimum funding ratio surcharge may prevent this equity demand to be as substantial as the equity position of the underlying pension fund, but it still remains a significant proportion. In case we observe the top-up plan not as a stand-alone product, the implicit guarantees of the top-up plan also exhibit a general increasing effect on the demand for (risky) equity. Secondly, the interest rate sensitivity with respect to the annuity purchase at retirement should be hedged appropriately over the entire life cycle. Because of a constant uncertainty about the governing interest rate at retirement, it is quite common to hold a bond component for this purpose in the hedge portfolio even if the investment horizon is still large. In the current strategy, the build-up of the duration hedge regarding the annuity purchase is tilted too heavily towards the end of the life-cycle. Thirdly, an accurate calibration of the level of risk aversion concerning the participant pool is of the utmost importance to acquire the desired life-cycle strategy. Even more so when a one-size-fits-all strategy is exercised. This risk aversion level affects all the previously described components of the entire asset allocation in a significant manner, as came forth from our sensitivity analyses.

APG currently operates a uniform life-cycle investment strategy with regard to top-up plans. However, we observed that a uniform life-cycle strategy can substantially differ from the optimal strategy. In our framework, we provided a stochastic life-cycle investment strategy that is individually tailored. This investment strategy for top-up plans is independent of economic state variables and can be implemented with a limited amount of basic information. This basic information consists of the individual's accrued financial wealth in both the top-up plan and the occupational pension plan, age and time of entry in the top-up plan. Additionally, a simple prognosis of future labor income and premium contributions completes the information needed in order to offer a stochastic life-cycle regime. This information provision and assumptions are generally of low cost and the resulting individual strategies can easily be incorporated in the monthly rebalancing of the life-cycle assets that already takes place.

7.3 Further research

The research on optimal life-cycle investment design performed in this thesis, can be extended along many different paths. The most relevant extensions of our framework are as follows. First of all, a specification could be explored where the funding ratio is (asymmetrically) mean reverting to better represent the legislation regarding the solvency of pension funds in the Netherlands. Secondly, inflation could be incorporated as a source of uncertainty in the financial market. This would facilitate indexation modelling of second pillar pension benefits, which we considered to be a nominal guarantee in our framework. Thirdly, incorporating a stochastic form of human capital might lead to a slightly different asset allocation. In this case, one needs to deal appropriately with the challenge of a stochastic entry time in the top-up plan. Fourthly, analyzing the robustness of the investment strategies with respect to different specifications of the short rate process, could provide valuable insight in the systemic risk of model misspecification. With regard to the straightforward numerical approach in Chapter 6, one could investigate the possibility of a more thorough numerical backward induction method and its effect on the resulting median asset allocation. Lastly, we could explore different methods to map the determined stochastic asset allocation strategy in a uniform, one-size-fits all strategy and its corresponding welfare losses.

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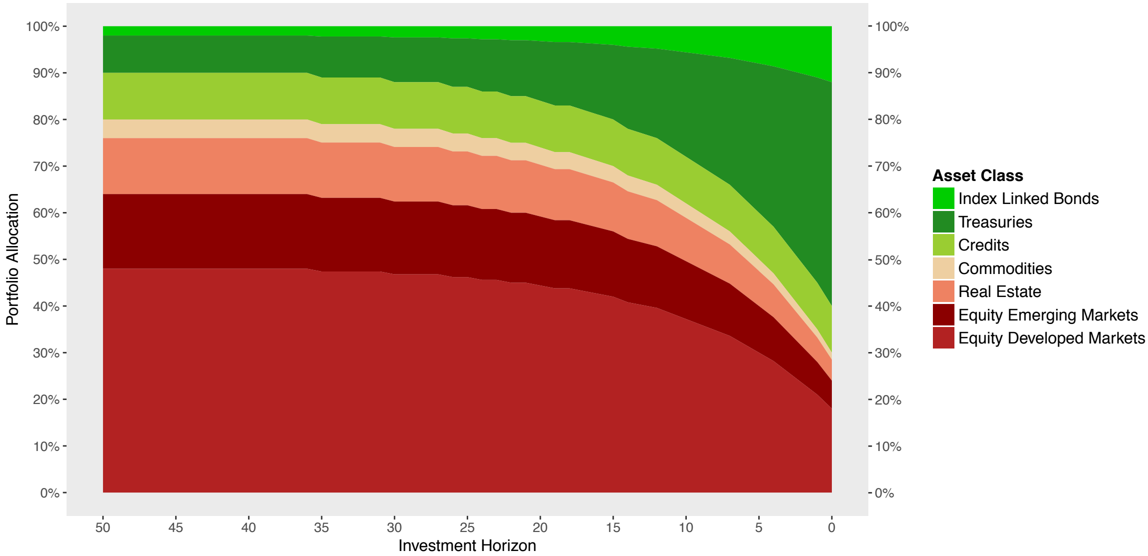
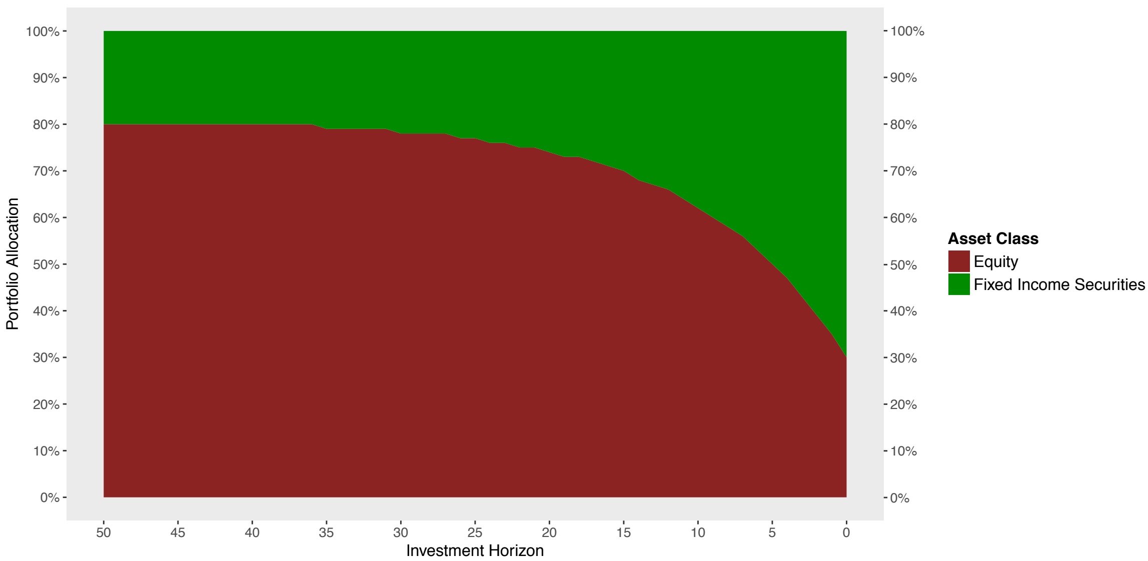
Appendix

A Current glide path composition of the net pension



(a) All asset classes



(b) Reduced asset classes

Figure A.1: Current glide path net pension. Obtained from *Pensioenreglement 2018* (p. 127-130), by ABP (2018b).

B Dutch fiscal rules regarding DC schemes

	percentage of pension basis	
age category	Net Pension	regular DC scheme
15-19	3.8	5.7
20-24	4.4	6.3
25-29	5.0	7.3
30-34	5.8	8.5
35-39	6.6	9.9
40-44	7.6	11.5
45-49	8.8	13.4
50-54	10.2	15.6
55-59	11.8	18.3
60-64	13.5	21.7
65-67	15.0	25.0

Table B.1: Fiscal maximum premium contribution of pension basis, obtained from *Loonheffingen, inkomstenbelasting. Pensioenen; beschikbare-premieregelingen en premie en kapitaalovereenkomsten en nettopensioenregelingen* (2017, November 23).

C Derivation of cond. expectation and variance of r_t

The well-known transformation $f(t, r_t) = e^{\kappa t}(r_t - \bar{r})$ is used to find the closed-form solution of the Ornstein-Uhlenbeck process. Applying Itô's lemma results in the following dynamics of $f(t, r_t)$:

$$\begin{aligned} \mathrm{d}f(t, r_t) &= \frac{\partial f(t, r_t)}{\partial t}\,\mathrm{d}t + \frac{\partial f(t, r_t)}{\partial r_t}\,\mathrm{d}r_t + \frac{1}{2}\frac{\partial^2 f(t, r_t)}{\partial r_t^2}\sigma_r^2\,\mathrm{d}t \\ &= \kappa e^{\kappa t}(r_t - \bar{r})\,\mathrm{d}t + e^{\kappa t}\,\mathrm{d}r_t + \frac{1}{2}\sigma_r^2\,\mathrm{d}t \cdot 0 \\ &= \kappa e^{\kappa t}(r_t - \bar{r})\,\mathrm{d}t + \kappa e^{\kappa t}(\bar{r} - r_t)\,\mathrm{d}t + e^{\kappa t}\sigma_r\,\mathrm{d}Z_t^r \\ &= e^{\kappa t}\sigma_r\,\mathrm{d}Z_t^r, \end{aligned} \tag{C.1}$$

where the third equality follows from substituting the dynamics of r_t described in (3.2.1). Using $\int_0^t \mathrm{d}f(t, r_t) \equiv f(t, r_t) - f(0, r_0)$, the integral form of the stochastic differential equation above becomes:

$$f(t, r_t) = f(0, r_0) + \sigma_r \int_0^t e^{\kappa u}\,\mathrm{d}Z_u^r.$$

Substituting $f(t, r_t) = e^{\kappa t}(r_t - \bar{r})$ and $f(0, r_0) = r_0 - \bar{r}$, yields the following expression for r_t:

$$\begin{aligned} r_t &= e^{-\kappa t}r_0 + \bar{r}(1 - e^{-\kappa t}) + \sigma_r \int_0^t e^{-\kappa(t-u)}\,\mathrm{d}Z_u^r \\ &= r_0 + (\bar{r} - r_0)(1 - e^{-\kappa t}) + \sigma_r \int_0^t e^{-\kappa(t-u)}\,\mathrm{d}Z_u^r. \end{aligned} \tag{C.2}$$

The integral $\int_0^t e^{-\kappa(t-u)}\,\mathrm{d}Z_u^r$ is an Itô integral and therefore a continuous martingale with respect to the natural filtration of the brownian motion $\{Z_u^r\}_{0 \leq u \leq T_D}$.

This implies $\mathbb{E}\left[\int_0^t e^{-\kappa(t-u)}\,\mathrm{d}Z_u^r\right] = 0$, which leads to the following conditional expectation of r_t:

$$\begin{aligned}\mathbb{E}_0\left[r_t\right] &= r_0 + (\bar{r} - r_0)(1 - e^{-\kappa t}) + 0 \\ &= e^{-\kappa t} r_0 + (1 - e^{-\kappa t})\bar{r},\end{aligned} \tag{C.3}$$

which is dependent on the known value of r_0.

The conditional variance of r_t can be described as follows:

$$\begin{aligned}\mathbb{V}_0\left[r_t\right] &= \sigma_r^2\ \mathbb{V}_0\left[\int_0^t e^{-\kappa(t-u)}\,\mathrm{d}Z_u^r\right] = \sigma_r^2\ \mathbb{E}_0\left[\left(\int_0^t e^{-\kappa(t-u)}\,\mathrm{d}Z_u^r\right)^2\right] \\ &= \sigma_r^2\ \mathbb{E}_0\left[\int_0^t e^{-2\kappa(t-u)}\,\mathrm{d}u\right] = \sigma_r^2 \int_0^t e^{-2\kappa(t-u)}\,\mathrm{d}u \\ &= \frac{1}{2\kappa}\sigma_r^2 e^{-2\kappa t}\left(e^{2\kappa t} - 1\right),\end{aligned} \tag{C.4}$$

where the third equality follows from the Itô isometry property and the fourth equality follows from Fubini's theorem and the fact that $e^{-2\kappa(t-u)}$ is non-stochastic. In general, if an Itô integral $J(f)_t \equiv \int_0^t f(s)\,\mathrm{d}Z_s$ contains a non-stochastic function $f(s)$, then $J(f)_t \sim \mathcal{N}\left(0,\ \int_0^t f(s)^2\,\mathrm{d}s\right)$. The expressions (C.3) and (C.4) can be generalized with time t instead of time 0 and a horizon $s \geq 0$, leading to the expressions of $\mathbb{E}_t[r_{t+s}]$ and $\mathbb{V}_t[r_{t+s}]$ in (3.2.2) and (3.2.3).

D Derivation of stock price index S_t

Define $f(t, S_t) = \log S_t$. Applying Itô's lemma to $f(t, S_t)$ results in the following dynamics:

$$\begin{aligned}\mathrm{d}f(t, S_t) &= \frac{\partial f(t, S_t)}{\partial t}\,\mathrm{d}t + \frac{\partial f(t, S_t)}{\partial S_t}\,\mathrm{d}S_t + \frac{1}{2}\frac{\partial^2 f(t, S_t)}{\partial S_t^2}(\mathrm{d}S_t)^2 \\ &= \frac{1}{S_t}\,\mathrm{d}S_t - \frac{1}{2S_t^2}\sigma_S^2 S_t^2\,\mathrm{d}t \\ &= \left(\mu_t - \frac{1}{2}\sigma_S^2\right) dt + \sigma_S dZ_t^S \\ &= \nu_t\,\mathrm{d}t + \sigma_S\,\mathrm{d}Z_t^S,\end{aligned} \tag{D.1}$$

where $(\mathrm{d}S_t)^2$ denotes the quadratic variation of the process S_t. Note that μ_t represents the expected relative rate of return of the stock price index at time t and ν_t represents the expected log rate of return. Using $\int_t^{t+s}\mathrm{d}\log S_u \equiv \log S_{t+s} - \log S_t$, we write the above differential equation in the following integral form for $s \geq 0$:

$$\log S_{t+s} = \log S_t + \int_t^{t+s} \nu_u\,\mathrm{d}u + \sigma_S \int_t^{t+s} \mathrm{d}Z_u^S. \tag{D.2}$$

Rewriting this expression results in

$$\begin{aligned}S_{t+s} &= S_t \exp\left\{\int_t^{t+s} \nu_u\,\mathrm{d}u + \sigma_S(Z_{t+s}^S - Z_t^S)\right\}, \\ &\text{with known}\quad \{S_k\}_{0\leq k\leq t} \quad\text{and}\quad S_0 > 0.\end{aligned} \tag{D.3}$$

Because the Brownian motion increment $(Z^S_{t+s} - Z^S_t) \sim \mathcal{N}(0, s)$ and r_{t+u} is normally distributed, the log return of stock price index is normally distributed and the stock price return itself is therefore log-normally distributed. The following expressions for the conditional expectation of the stock price returns can be derived.

$$\begin{aligned}
\mathbb{E}_t\left[\frac{S_{t+s}}{S_t}\right] &= \mathbb{E}_t\left[\exp\left\{\int_t^{t+s} \nu_u \,\mathrm{d}u + \sigma_S\sqrt{s}X\right\}\right], \quad \text{where } X \sim \mathcal{N}(0,1) \\
&= \mathbb{E}_t\left[\exp\left\{\int_t^{t+s} \nu_u \,\mathrm{d}u\right\}\right] \mathbb{E}_t\left[e^{\sigma_S\sqrt{s}X}\right] \\
&= \mathbb{E}_t\left[\exp\left\{\int_t^{t+s} r_u + \lambda_S\sigma_S - \frac{1}{2}\sigma_S^2 \,\mathrm{d}u\right\}\right] e^{s\frac{1}{2}\sigma_S^2} \\
&= e^{s\lambda_S\sigma_S}\mathbb{E}_t\left[\exp\left\{\int_t^{t+s} r_u \,\mathrm{d}u\right\}\right].
\end{aligned} \tag{D.4}$$

For the second equality, we use the fact that r_{t+u} and Z^S_{t+u} are independent. Both the expectation of a log-normal random variable and the definition of ν_t (expressed in (D.1)) are used for the third equality. From (D.4) we see that that the instantaneous interest rate plays a key role in the conditional expectation of the stock price index process for horizon s.

E Derivation of pricing kernel M_t

The functional form of the pricing kernel or stochastic discount factor can be derived using Girsanov's Theorem and the resulting risk-neutral valuation measure. Since we have multiple assets in our financial market, we focus on a multifactor generalization of this theorem (Etheridge, 2002, p. 166).

Theorem E.1. ***Multifactor Girsanov Theorem***
Let $\{Z^i_t\}_{t\geq 0}, i = 1, \ldots, n$, be an n-independent $\mathbb{P}$-Brownian motion ,with natural filtration $\{\mathcal{F}\}_{t\geq 0}$ and let $\{\lambda^i_t\}_{t\geq 0}, i = 1, \ldots, n$, be n processes adapted to $\{\mathcal{F}\}_{t\geq 0}$, satisfying

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left\{\frac{1}{2}\int_0^T \sum_{i=1}^n (\lambda^i_s)^2 \,\mathrm{d}s\right\}\right] < \infty.$$

Then, the processes $\{X^i_t\}_{0\geq t\geq T}, i = 1, \ldots, n$ defined by

$$X^i_t = \int_0^t \lambda^i_s \,\mathrm{d}s + Z^i_t, \quad \textit{or equivalently} \quad \mathrm{d}X^i_t = \lambda^i_t \,\mathrm{d}t + \mathrm{d}Z^i_t,$$

are all martingales under probability measure $\mathbb{P}^L$, defined for all events $\boldsymbol{A} \in \mathcal{F}_t$ by

$$\mathbb{P}^{(L)}[\boldsymbol{A}] = \mathbb{E}^{\mathbb{P}}\left[L_t \mathbb{1}_{\boldsymbol{A}}\right], \quad L_t = \exp\left\{-\sum_{i=1}^n \left(\frac{1}{2}\int_0^t (\lambda^i_s)^2 \,\mathrm{d}s + \int_0^t \lambda^i_s \,\mathrm{d}Z^i_s\right)\right\},$$

where

$$L_t = \left.\frac{\mathrm{d}\mathbb{P}^{(L)}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t}$$

is the Radon-Nikodym derivative of $\mathbb{P}^{(L)}$ with respect to the equivalent probability measure $\mathbb{P}$.

Since the financial market is complete, there exists one unique equivalent risk-neutral measure, denoted $\mathbb{Q}$, under which the discounted prices are all martingales. Taking $\lambda_t^1 \equiv \lambda_S$ and $\lambda_t^2 \equiv \lambda_r$ in Theorem E.1, defines the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ in our financial market:

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left\{-\int_0^t \frac{1}{2}\|\boldsymbol{\lambda}\|^2\,\mathrm{d}u - \int_0^t \boldsymbol{\lambda}'\,\mathrm{d}\mathbf{Z}_u\right\}, \tag{E.1}$$

where $\|\cdot\|$ represents the Euclidean vector norm and λ_S (λ_r) denotes the market price of equity (interest rate) risk. This can be demonstrated for e.g. the stock price process (3.2.4):

$$\begin{aligned}\frac{\mathrm{d}S_t}{S_t} &= (r_t + \lambda_S\sigma_S)\,\mathrm{d}t + \sigma_S\,\mathrm{d}Z_t^S \\ &= (r_t + \lambda_S\sigma_S)\,\mathrm{d}t + \sigma_S(\mathrm{d}X_t^S - \lambda_S\,\mathrm{d}t) \\ &= r_t\,\mathrm{d}t + \sigma_S\,\mathrm{d}X_t^S,\end{aligned}$$

where X_t^S is a standard Brownian motion under measure $\mathbb{Q}$ and the substitution of Z_t^S is based on Girsanov's theorem. Then, the $\mathbb{Q}$-expectation of the discounted stock price yields:

$$\mathbb{E}_0^{\mathbb{Q}}\left[\exp\left\{-\int_0^t r_u\,\mathrm{d}u\right\}S_t\right] = S_0 \quad \left(= \mathbb{E}_0^{\mathbb{P}}\left[\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t}\exp\left\{-\int_0^t r_u\,\mathrm{d}u\right\}S_t\right]\right). \tag{E.2}$$

The financial deflator or stochastic discount factor at time t for a cashflow to be paid at time $s \geq t$ therefore equals the product of the Radon-Nikodym derivative (E.1) and the discount factor:

$$\begin{aligned}\frac{M_s}{M_t} &= \frac{\exp\left\{-\int_0^s r_u\,\mathrm{d}u\right\}\cdot\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_s}}{\exp\left\{-\int_0^t r_u\,\mathrm{d}u\right\}\cdot\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t}} \\ &= \exp\left\{-\int_t^s r_u\,\mathrm{d}u - \frac{1}{2}\int_t^s\|\boldsymbol{\lambda}\|^2\,\mathrm{d}u - \int_t^s\boldsymbol{\lambda}'\,\mathrm{d}\mathbf{Z}_u\right\},\end{aligned} \tag{E.3}$$

which is equivalent to equation (3.2.8). In case $\mathbf{V}_t^\top = (S_t, P_t^h, B_t)$, discounting with M_t and taking the expectation under measure $\mathbb{P}$ results in

$$\mathbb{E}_0^{\mathbb{P}}\left[M_t\mathbf{V}_t\right] = \mathbf{V}_0, \tag{E.4}$$

where $M_t\boldsymbol{V_t}$ is a three-dimensional martingale under measure $\mathbb{P}$.

F Derivation of pricing kernel dynamics

Taking the logarithm of (E.3) results in

$$\int_t^s \mathrm{d}\log M_u = -\int_t^s r_u\,\mathrm{d}u - \frac{1}{2}\int_t^s\|\boldsymbol{\lambda}\|^2\,\mathrm{d}u - \int_t^s\boldsymbol{\lambda}'\,\mathrm{d}\mathbf{Z}_u, \qquad s \geq t. \tag{F.1}$$

Therefore, the dynamics of $\log M_t$ can be described as

$$\mathrm{d}\log M_t = r_t\,\mathrm{d}t - \frac{1}{2}\|\boldsymbol{\lambda}\|^2\,\mathrm{d}t - \boldsymbol{\lambda}'\,\mathrm{d}\mathbf{Z}_t. \tag{F.2}$$

Define $f(t, \log M_t) = e^{\log M_t}$. Applying Itô's lemma to $f(t, \log M_t)$ results in the following dynamics:

$$\begin{aligned} \mathrm{d}f(t, \log M_t) &= \frac{\partial f(t, \log M_t)}{\partial t}\,\mathrm{d}t + \frac{\partial f(t, \log M_t)}{\partial \log M_t}\,\mathrm{d}\log M_t + \frac{1}{2}\frac{\partial^2 f(t, \log M_t)}{\partial \log^2 M_t}(\mathrm{d}\log M_t)^2 \\ &= M_t\,\mathrm{d}\log M_t + \frac{1}{2}M_t(\mathrm{d}\log M_t)^2 \\ &= M_t(-r_t\,\mathrm{d}t - \frac{1}{2}\left\|\boldsymbol{\lambda}\right\|^2\mathrm{d}t - \boldsymbol{\lambda}'\,\mathrm{d}\mathbf{Z}_t) + \frac{1}{2}M_t\left\|\boldsymbol{\lambda}\right\|^2\mathrm{d}t \\ &= M_t(-r_t\,\mathrm{d}t - \boldsymbol{\lambda}'\,\mathrm{d}\mathbf{Z}_t), \end{aligned} \tag{F.3}$$

where the third equality follows from the quadratic variation of $\log M_t$:

$$\begin{aligned} (\mathrm{d}\log M_t)^2 &= (\boldsymbol{\lambda}'\,\mathrm{d}\mathbf{Z}_t)(\boldsymbol{\lambda}'\,\mathrm{d}\mathbf{Z}_t)' \\ &= \boldsymbol{\lambda}'\boldsymbol{\lambda}\,\mathrm{d}t \\ &= \left\|\boldsymbol{\lambda}\right\|^2\mathrm{d}t. \end{aligned} \tag{F.4}$$

Therefore,

$$\frac{\mathrm{d}M_t}{M_t} = -r_t\,\mathrm{d}t - \boldsymbol{\lambda}'\,\mathrm{d}\mathbf{Z}_t, \tag{F.5}$$

which is equal to Equation (3.2.9).

G Expression for adapted cond. expectation pricing kernel

We derive an expression for the following conditional expectation of a power function of the pricing kernel:

$$\Theta(t,s,x) \equiv \mathbb{E}_t\left[\left(\frac{M_s}{M_t}\right)^x\right], \qquad t < s < T_D.$$

Below the following derivation, brief commentary is provided on the steps taken in the derivation.

$$\begin{aligned}
\mathbb{E}_t\left[\left(\frac{M_s}{M_t}\right)^x\right] &= \mathbb{E}_t\left[\exp\left\{-x\int_0^{s-t}\left(r_{t+u}+\frac{1}{2}\left\|\boldsymbol{\lambda}\right\|^2\right)\mathrm{d}u - x\int_0^{s-t}\boldsymbol{\lambda}'\,\mathrm{d}\mathbf{Z}_{t+u}\right\}\right]\\
&= \mathbb{E}_t\left[\exp\left\{-x\int_0^{s-t}\mathbb{E}_t\left[r_{t+u}\right]+\frac{1}{2}\left\|\boldsymbol{\lambda}\right\|^2\mathrm{d}u\right\}\times\right.\\
&\qquad\left.\exp\left\{-\sigma_r x\int_0^{s-t}D(s-t-u)\,\mathrm{d}Z^r_{t+u} - x\int_0^{s-t}\boldsymbol{\lambda}'\,\mathrm{d}\mathbf{Z}_{t+u}\right\}\right]\\
&= \exp\left\{x\left(-r_tD(s-t)-\bar{r}(s-t-D(s-t))-\frac{1}{2}\left\|\boldsymbol{\lambda}\right\|^2(s-t)\right)\right\}\times\\
&\qquad \mathbb{E}_t\left[\exp\left\{-x\int_0^{s-t}\left(\boldsymbol{\lambda}+\sigma_rD(s-t-u)\mathbf{e}_2\right)'\mathrm{d}\mathbf{Z}_{t+u}\right\}\right]\\
&= \exp\left\{x\left(-r_tD(s-t)-\bar{r}(s-t-D(s-t))-\frac{1}{2}\left\|\boldsymbol{\lambda}\right\|^2(s-t)\right)\right\}\times\\
&\qquad \exp\left\{\frac{1}{2}x^2\int_0^{s-t}\left\|\boldsymbol{\lambda}+\sigma_rD(u)\mathbf{e}_2\right\|^2\mathrm{d}u\right\}\\
&= \exp\left\{-xr_tD(s-t)-x\bar{r}(s-t-D(s-t))+\frac{1}{2}x(x-1)\left\|\boldsymbol{\lambda}\right\|^2(s-t)\right\}\times\\
&\qquad \exp\left\{\frac{1}{2}x^2\int_0^{s-t}2\sigma_r\lambda_rD(u)+\sigma_r^2D(u)^2\,\mathrm{d}u\right\}\\
&= \exp\left\{-xr_tD(s-t)-x\bar{r}(s-t-D(s-t))+\frac{1}{2}x(x-1)\left\|\boldsymbol{\lambda}\right\|^2(s-t)\right\}\times\\
&\qquad \exp\left\{\frac{x^2\sigma_r\lambda_r}{\kappa}(s-t-D(s-t))\right\}\times\exp\left\{\frac{x^2\sigma_r^2}{2}\int_0^{s-t}D(u)^2\,\mathrm{d}u\right\}\\
&= \exp\left\{-xr_tD(s-t)-x\bar{r}(s-t-D(s-t))+\frac{1}{2}x(x-1)\left\|\boldsymbol{\lambda}\right\|^2(s-t)\right\}\times\\
&\qquad \exp\left\{\frac{x^2\sigma_r\lambda_r}{\kappa}(s-t-D(s-t))+\frac{x^2\sigma_r^2}{\kappa^2}\left(\frac{1}{2}(s-t)-D(s-t)+\frac{1}{4}D(2s-2t)\right)\right\}\\
&= \exp\left\{-x\left(a(s-t)+D(s-t)r_t\right)\right\},
\end{aligned}\tag{G.1}$$

where

$$a(u) \equiv -\frac{1}{2}(x-1)\left\|\boldsymbol{\lambda}\right\|^2u+\left(\bar{r}-\frac{x\sigma_r\lambda_r}{\kappa}\right)(u-D(u))-\frac{x\sigma_r^2}{\kappa^2}\left(\frac{1}{2}u-D(u)+\frac{1}{4}D(2u)\right), \tag{G.2}$$

$$D(u) \equiv \frac{1}{\kappa}\left(1-e^{-\kappa u}\right). \tag{G.3}$$

The vector $\boldsymbol{e}_i$ used above, denotes the i-th unit vector: a zero-valued vector except for entry i, which is equal to one. In the second equality, the expression for r_t derived in appendix C (C.2,C.3) is used:

$$\begin{aligned} r_{t+s} &= r_t + (\bar{r} - r_t)(1 - e^{-\kappa s}) + \sigma_r \int_t^{t+s} e^{-\kappa(t+s-u)}\, \mathrm{d}Z_u^r \\ &= \mathbb{E}_t[r_{t+s}] + \sigma_r \int_t^{t+s} e^{-\kappa(t+s-u)}\, \mathrm{d}Z_u^r. \end{aligned}$$

For the fourth equality, the Itô isometry property and the expectation of the lognormal distribution is used, as well as the following function property of D:

$$\int_0^n D(\kappa, n-u)\, \mathrm{d}u = \int_0^n D(\kappa, u)\, \mathrm{d}u.$$

For the seventh equality, the integral of $D(u)^2$ is solved in the following way:

$$\begin{aligned} D(u)^2 &= \frac{1}{\kappa^2}\left(1 - 2e^{-\kappa u} + e^{-2\kappa u}\right) \\ &= \frac{1}{\kappa^2}\left(1 - e^{-\kappa u} + 1 - e^{-\kappa u} - \left(1 - e^{-2\kappa u}\right)\right) \\ &= \frac{1}{\kappa}\left(2D(u) - D(2u)\right), \end{aligned}$$

so that the integral can be written as

$$\begin{aligned} \frac{1}{2}\int_0^h D(u)^2 du &= \frac{1}{2\kappa}\int_0^h 2D(u)\, \mathrm{d}u - \frac{1}{2\kappa} D(2u)\, \mathrm{d}u \\ &= \frac{1}{\kappa}\left(\frac{1}{\kappa}(h - D(h)\right) - \frac{1}{2\kappa}\left(\frac{1}{\kappa}\left(h - \frac{1}{2}D(2h)\right)\right) \\ &= \frac{1}{\kappa^2}\left(\frac{1}{2}h - D(h) + \frac{1}{4}D(2h)\right). \end{aligned}$$

Note that for $x = 1$, we obtain $\Theta(t, s, 1) = P_t^{(s-t)}$ the price of a nominal zero-coupon bond with maturity $(s - t)$. In case $x = \frac{\gamma - 1}{\gamma}$, the expression of $\int_t^{T_D} \Theta\left(t, s, \frac{\gamma-1}{\gamma}\right) \mathrm{d}s$ is equivalent to the quantity $Q(t, T_D)$ in the paper of Brennan and Xia (2002, p. 1213, eq. 33) after some rewriting, when inflation is absent.

H Derivation of bond price dynamics

Note that in Appendix G the price of a zero-coupon bond at time t, maturing at time $t + h$, is expressed as

$$\begin{aligned} P_t^h &= \Theta(t, t+h, 1) \\ &= e^{-a(t+h-t) - D(t+h-t) r_t} \end{aligned} \tag{H.1}$$

To derive the dynamics of a rolling zero-coupon bond with constant maturity h, we consider the maturity date $t + h$ to be fixed, whereas time t is treated as a variable in (H.1). Therefore, we consider $P_t^h = f(t, r_t)$ and applying Itô's lemma results in:

$$\begin{aligned}
\frac{\partial f(t,r_t)}{\partial t} &= P_t^h \left(\frac{\partial a(s-t)}{\partial t}\Big|_{s=t+h} - r_t \frac{\partial D(s-t)}{\partial t}\Big|_{s=t+h} \right) \\
&= P_t^h \left(r_t + \kappa(\bar{r} - r_t)D(h) - \lambda_r \sigma_r D(h) - \frac{1}{2}\sigma_r^2 D^2(h) \right), \\
\frac{\partial f(t,r_t)}{\partial r_t} &= -D(h)P_t^h, \\
\frac{\partial^2 f(t,r_t)}{\partial r_t^2} &= D^2(h)P_t^h,
\end{aligned} \tag{H.2}$$

$$\begin{aligned}
\mathrm{d}f(t,r_t) &= \frac{\partial f(t,r_t)}{\partial t}\,\mathrm{d}t + \frac{\partial f(t,r_t)}{\partial r_t}\,\mathrm{d}r_t + \frac{1}{2}\frac{\partial^2 f(t,r_t)}{\partial r_t^2}\sigma_r^2(\mathrm{d}Z_t^r)^2 \\
&= \left(r_t + \kappa(\bar{r} - r_t)D(h) - \lambda_r \sigma_r D(h) - \frac{1}{2}\sigma_r^2 D^2(h) \right) P_t^h\,\mathrm{d}t - \\
&\quad D(h)P_t^h\left(\kappa(\bar{r} - r_t)\,\mathrm{d}t + \sigma_r\,\mathrm{d}Z_t^r\right) + \frac{1}{2}\sigma_r^2 D^2(h)P_t^h\,\mathrm{d}t \\
&= \left(r_t - \lambda_r\sigma_r D(h)\right)P_t^h\,\mathrm{d}t - \sigma_r D(h)P_t^h\,\mathrm{d}Z_t^r.
\end{aligned} \tag{H.3}$$

Therefore, we obtain, as stated in (3.2.14),

$$\frac{\mathrm{d}P_t^h}{P_t^h} = \left(r_t - \lambda_r\sigma_r D(h)\right)\mathrm{d}t - \sigma_r D(h)\,\mathrm{d}Z_t^r. \tag{H.4}$$

The expression for the partial derivative with respect to t given in (H.2), is obtained by straightforward but tedious calculus. Note that

$$\frac{1}{P_t^h}\frac{\partial f(t,r_t)}{\partial t} = \frac{\mathrm{d}\log P_t^h}{\mathrm{d}t},$$

can be interpreted as the time t instantaneous forward interest rate with respect to horizon h (r_t^h). Then, we can also express the bond price as

$$P_t^h = \exp\left\{-\int_0^h r_t^u\,\mathrm{d}u\right\}, \tag{H.5}$$

based on the fundamental relationship of continuous time interest rates (Hull, 2012, p. 683). The instantaneous forward rate r_t^u acts as a discount rate in this setting.

I Derivation of wealth dynamics

Note that wealth at time t can be expressed as the sum of initial wealth and wealth accrual up until time t:

$$W_t = W_0 + \int_0^t \mathrm{d}W_u. \tag{I.1}$$

We can also express wealth a time t as the sum of initial wealth, the accumulated investment returns (using expression (3.3.1)) and the negative cumulative consumption up until time t:

$$\begin{aligned} W_t &= W_0 + \int_0^t \mathrm{d}(\eta_u^S S_u + \eta_u^P P_u^h + \eta_u^B B_u) - \int_0^t c_u \,\mathrm{d}u \\ &= W_0 + \int_0^t \eta_u^S \,\mathrm{d}S_u + \int_0^t \eta_u^P \,\mathrm{d}P_u^h + \int_0^t \eta_u^B \,\mathrm{d}B_u - \int_0^t c_u \,\mathrm{d}u. \end{aligned} \tag{I.2}$$

The second equality is based upon application of integration by parts and the assumption that the sum of the rebalancing costs (defined below as Ξ_u) equals zero, which is shown below:

$$\begin{aligned} \mathrm{d}(\eta_u^S S_u + \eta_u^P P_u^h + \eta_u^B B_u) &= \eta_u^S \,\mathrm{d}S_u + \eta_u^P \,\mathrm{d}P_u^h + \eta_u^B \,\mathrm{d}B_u + \\ &\quad S_u \,\mathrm{d}\eta_u^S + P_u^h \,\mathrm{d}\eta_u^P + B_u \,\mathrm{d}\eta_u^B + (\mathrm{d}\eta_u^S)(\mathrm{d}S_u) + (\mathrm{d}\eta_u^P)(\mathrm{d}P_u^h) + (\mathrm{d}\eta_u^B)(\mathrm{d}B_u) \\ &\equiv \eta_u^S \,\mathrm{d}S_u + \eta_u^P \,\mathrm{d}P_u^h + \eta_u^B \,\mathrm{d}B_u + \Xi_u \\ &= \eta_u^S \,\mathrm{d}S_u + \eta_u^P \,\mathrm{d}P_u^h + \eta_u^B \,\mathrm{d}B_u. \end{aligned} \tag{I.3}$$

These zero rebalancing costs at all times is a result from assuming a financial market with no transaction costs and considering total wealth to be a self-financing portfolio (ceteris paribus increasing the bond position must result in decreasing another position). From (I.1) and (I.2), we can observe that

$$\mathrm{d}W_t = \eta_t^S \,\mathrm{d}S_t + \eta_t^P \,\mathrm{d}P_t^h + \eta_t^B \,\mathrm{d}B_t - c_t \,\mathrm{d}t. \tag{I.4}$$

We define $(\omega_t^S, \omega_t^P, \omega_t^B)$ to be the fractions of total wealth invested in the three assets at time t. Then, the time t asset amounts can be expressed as:

$$\begin{aligned} \eta_t^S &= \frac{\omega_t^S W_t}{S_t}, \\ \eta_t^P &= \frac{\omega_t^P W_t}{P_t^h}, \\ \eta_t^B &= \frac{(1 - \omega_t^S - \omega_t^P) W_t}{B_t}, \end{aligned} \tag{I.5}$$

and substituting these expressions alongside with $\mathrm{d}S_t$ (3.2.4), $\mathrm{d}P_t^h$ (3.2.14) and $\mathrm{d}B_t$ (3.2.16) in (I.4) yields:

$$\begin{aligned} \mathrm{d}W_t &= W_t \left(\left(r_t + \omega_t^S \sigma_S \lambda_S - \omega_t^P \sigma_r D(h) \lambda_r \right) \mathrm{d}t + \omega_t^S \sigma_S \,\mathrm{d}Z_t^S - \omega_t^P \sigma_r D(h) \,\mathrm{d}Z_t^r \right) - c_t \,\mathrm{d}t \\ &= \left(r_t + \boldsymbol{\omega}_t' \boldsymbol{\Sigma} \boldsymbol{\lambda} \right) W_t \,\mathrm{d}t + W_t \boldsymbol{\omega}_t' \boldsymbol{\Sigma} \,\mathrm{d}\boldsymbol{Z}_t - c_t \,\mathrm{d}t, \end{aligned} \tag{I.6}$$

with

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_S & 0 \\ 0 & -\sigma_r D(h) \end{pmatrix}, \quad \boldsymbol{\omega}_t = \begin{pmatrix} \omega_t^S \\ \omega_t^P \end{pmatrix}.$$

J Derivation of the optimal consumption strategy

We start with the Lagrangian function that corresponds to the maximization problem stated in Section 3.5:

$$\begin{aligned}\mathcal{L}(c_s, \varphi) &= \mathbb{E}_0\left[\int_{T_R}^{T_D} e^{-\delta s}\frac{c_s^{1-\gamma}}{1-\gamma}\,\mathrm{d}s\right] - \varphi\left(\mathbb{E}_0\left[\int_{T_R}^{T_D} M_s c_s\,\mathrm{d}s\right] - W_0\right)\\ &= \mathbb{E}_0\left[\int_{T_R}^{T_D} e^{-\delta s}\frac{c_s^{1-\gamma}}{1-\gamma} - \varphi M_s c_s\,\mathrm{d}s\right] + \varphi W_0,\end{aligned} \tag{J.1}$$

where φ is the Lagrange multiplier corresponding to the static budget constraint. The first-order derivative of $\mathcal{L}$ with respect to c_s for each state of the world and each time s, results in:

$$\frac{\partial}{\partial c_s}\left(e^{-\delta s}\frac{c_s^{1-\gamma}}{1-\gamma} - \varphi M_s c_s\right) = e^{-\delta s}c_s^{-\gamma} - \varphi M_s. \tag{J.2}$$

The first-order derivative of $\mathcal{L}$ with respect to φ results in:

$$\mathbb{E}_0\left[\int_{T_R}^{T_D} M_s c_s\,\mathrm{d}s\right] - W_0. \tag{J.3}$$

Therefore, the first-order optimality conditions can be expressed as:

$$\begin{cases} e^{-\delta s}c_s^{-\gamma} - \varphi M_s = 0 \\ \mathbb{E}_0\left[\int_{T_R}^{T_D} M_s c_s\,\mathrm{d}s\right] = W_0 \end{cases}. \tag{J.4}$$

Solving the former first-order condition for c_s yields:

$$c_s = e^{-\delta s/\gamma}\varphi^{-1/\gamma}M_s^{-1/\gamma}, \qquad T_R < s < T_D. \tag{J.5}$$

Substituting this expression in the latter first-order optimality condition (which equals the static budget constraint that holds with equality) results in the following expression for the Lagrange multiplier:

$$\varphi^{-1/\gamma} = \frac{W_0}{\int_{T_R}^{T_D} e^{-\delta s/\gamma}\mathbb{E}_0\left[M_s^{\frac{\gamma-1}{\gamma}}\right]\mathrm{d}s}. \tag{J.6}$$

Substituting the expression for the Lagrange multiplier back in (J.5), yields the optimal consumption strategy. So, the optimal consumption decision for each state of the world and $T_R < s < T_D$ can be written as

$$\begin{aligned}c_s^* &= e^{-\delta s/\gamma}M_s^{-1/\gamma}\left(\int_{T_R}^{T_D} e^{-\delta s/\gamma}\mathbb{E}_0\left[M_s^{\frac{\gamma-1}{\gamma}}\right]\mathrm{d}s\right)^{-1} W_0\\ &= e^{-\delta s/\gamma}M_s^{-1/\gamma}\left(\int_{T_R}^{T_D} e^{-\delta s/\gamma}\Theta\left(0, s, 1-1/\gamma\right)\mathrm{d}s\right)^{-1} W_0,\end{aligned} \tag{J.7}$$

where the function Θ is defined in appendix G.

K Derivation of investment strategy total wealth

The value of the optimal total liabilities over the individual's lifetime can be expressed as follows:

$$
\begin{aligned}
V_t^* &= \mathbb{E}_t\left[\int_{\max\{t,T_R\}}^{T_D} c_s^* \frac{M_s}{M_t}\,\mathrm{d}s\right] \\
&= \int_{\max\{t,T_R\}}^{T_D} \varphi^{-\frac{1}{\gamma}} e^{-\delta s/\gamma}\mathbb{E}_t\left[\frac{M_s^{\frac{\gamma-1}{\gamma}}}{M_t}\right]\mathrm{d}s \\
&= M_t^{-\frac{1}{\gamma}}\varphi^{-\frac{1}{\gamma}}\int_{\max\{t,T_R\}}^{T_D} e^{-\delta s/\gamma}\Theta(t,s,1-1/\gamma)\,\mathrm{d}s,
\end{aligned}
\tag{K.1}
$$

where $\Theta(t,s,1-1/\gamma)$ is defined as function of r_t in Appendix G. Therefore, we can define

$$
V_t^* = f(t, r_t, M_t), \tag{K.2}
$$

which is a function of two random variables such that the standard Itô's formula falls short here. We need a multivariate application and derive this result next, using simply a second-order Taylor series expansion of the stochastic differential equation for the multivariate function. This derivation is structured along the lines of Etheridge (2002). Suppose we have the following system of stochastic differential equations

$$
\mathrm{d}X_t^i = \mu_i(t)dt + \sum_{j=1}^n \sigma_{ij}(t)\,\mathrm{d}Z_t^j, \qquad i = 1,2,\dots,n, \tag{K.3}
$$

where $\{Z_t^i\}_{t\geq 0}$, $i = 1,\dots,n$ are independent $\mathbb{P}$-Brownian motions. Let $\{\mathbf{X}_t\}_{t\geq 0} = \{X_t^1, X_t^2, \dots, X_t^n\}_{t\geq 0}$ be the vector of solution processes that satisfy the system mentioned above. Now, we define a new stochastic process $V_t = f(t, \boldsymbol{X}_t)$. Suppose $f(t,x)$: $\mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, which is sufficiently differentiable to apply Taylor's theorem. Then, Taylor's theorem gives the following expression for $\mathrm{d}V_t$:

$$
\mathrm{d}V_t = \frac{\partial f}{\partial t}(t,X_t)\,\mathrm{d}t + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t,X_t)\,\mathrm{d}X_t^i + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(t,X_t)\,\mathrm{d}X_t^i\,\mathrm{d}X_t^j + \dots \tag{K.4}
$$

where $x = (x_1,\dots,x_n)$. Using the known multiplication results $(\mathrm{d}t)^2 = 0$, $(\mathrm{d}t)(\mathrm{d}Z_t^i) = 0$ and $(\mathrm{d}Z_t^i)(\mathrm{d}Z_t^j) = \mathrm{d}t$ for $i,j = 1,\dots,n$, we obtain the following product definition: $\mathrm{d}X_t^i\,\mathrm{d}X_t^j = \sum_{k=1}^n \sigma_{ik}(t)\sigma_{jk}(t)$ as well as the insight that $(\mathrm{d}X_t^i)(\mathrm{d}X_t^j)(\mathrm{d}X_t^k)$ is of $\mathcal{O}(\mathrm{d}t)$. Therefore, (K.4) can be written as

$$
\mathrm{d}f(t,X_t) = \frac{\partial f}{\partial t}\,\mathrm{d}t + \sum_{i=1}^n \frac{\partial f}{\partial x_i}\,\mathrm{d}X_t^i + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} Q_{ij}(t)\,\mathrm{d}t, \tag{K.5}
$$

where $Q_{ij}(t) = \sum_{k=1}^n \sigma_{ik}(t)\sigma_{jk}(t)$. This generalizes the result of the one-factor Itô's formula. Now, we are able to derive the dynamics of the funding ratio V_t^*.

Applying this result to V_t^* yields the following dynamics:

$$
\mathrm{d}V_t^* = \mu_t^{V^*}\,\mathrm{d}t + \frac{\partial V_t^*}{\partial M_t}(-M_t\boldsymbol{\lambda}'\,\mathrm{d}\mathbf{Z}_t) + \frac{\partial V_t^*}{\partial r_t}\sigma_r\,\mathrm{d}Z_t^r, \tag{K.6}
$$

where

$$\mu_t^{V^*} = \frac{\partial V_t^*}{\partial t} + \frac{\partial V_t^*}{\partial r_t}\kappa(\bar{r}-r_t) - \frac{\partial V_t^*}{\partial M_t}M_t r_t - \frac{\partial^2 V_t^*}{\partial r_t \partial M_t}\sigma_r\lambda_r M_t + \frac{1}{2}\left(\frac{\partial^2 V_t^*}{\partial r_t^2}\sigma_r^2 + \frac{\partial^2 V_t^*}{\partial M_t^2}M_t^2\|\boldsymbol{\lambda}\|^2\right). \quad \text{(K.7)}$$

Determining the first-order partial derivatives of V_t^* with respect to the pricing kernel and the interest rate results in:

$$\begin{aligned}
\frac{\partial V_t^*}{\partial M_t} &= -\frac{1}{\gamma}M_t^{-1-\frac{1}{\gamma}}\varphi^{-\frac{1}{\gamma}}\int_{\max\{t,T_R\}}^{T_D} e^{-\delta s/\gamma}\Theta(t,s,1-1/\gamma)\,\mathrm{d}s \\
&= -\frac{1}{\gamma}M_t^{-1}V_t^*, \\
\frac{\partial V_t^*}{\partial r_t} &= (M_t\varphi)^{-\frac{1}{\gamma}}\int_{\max\{t,T_R\}}^{T_D} e^{-\delta s/\gamma}\,\frac{\partial\Theta}{\partial r_t}(t,s,1-1/\gamma)\,\mathrm{d}s \\
&= -\frac{\gamma-1}{\gamma}(M_t\varphi)^{-\frac{1}{\gamma}}\int_{\max\{t,T_R\}}^{T_D} e^{-\delta s/\gamma}D(s-t)\Theta(t,s,1-1/\gamma)\,\mathrm{d}s.
\end{aligned} \tag{K.8}$$

Substituting these first-order partial derivatives in the dynamics for the optimal liabilities (K.6) yields:

$$\begin{aligned}
\mathrm{d}V_t^* = \mu^{V^*}(t,r_t,M_t)\,\mathrm{d}t - \frac{1}{\gamma}M_t^{-1}V_t^*(-M_t\lambda_r\,\mathrm{d}Z_t^r - M_t\lambda_S\,\mathrm{d}Z_t^S) + \\
\frac{1-\gamma}{\gamma}(M_t\varphi)^{-\frac{1}{\gamma}}\int_{\max\{t,T_R\}}^{T_D} e^{-\delta s/\gamma}D(s-t)\Theta(t,s,1-1/\gamma)\,\mathrm{d}s\;\sigma_r\,\mathrm{d}Z_t^r.
\end{aligned} \tag{K.9}$$

Then, dividing both sides by V_t^* results in the following geometric Brownian motion expression with drift and volatility of the process both time-varying:

$$\frac{\mathrm{d}V_t^*}{V_t^*} = \bar{\mu}^{V^*}(t,r_t,M_t)\,\mathrm{d}t + \frac{\lambda_S}{\gamma}\,\mathrm{d}Z_t^S + \left(\frac{\lambda_r}{\gamma} + \frac{1-\gamma}{\gamma}\sigma_r D_t^{V^*}\right)\mathrm{d}Z_t^r, \tag{K.10}$$

with

$$D_t^{V^*} \equiv \frac{\int_{\max\{t,T_R\}}^{T_D} e^{-\delta s/\gamma}D(s-t)\Theta(t,s,1-1/\gamma)\,\mathrm{d}s}{\int_{\max\{t,T_R\}}^{T_D} e^{-\delta s/\gamma}\Theta(t,s,1-1/\gamma)\,\mathrm{d}s}, \tag{K.11}$$

and

$$\bar{\mu}^{V^*}(t,r_t,M_t) \equiv \mu_t^{V^*}(t,r_t,M_t)/V_t^*. \tag{K.12}$$

The dynamics of total wealth for the stated problem can be expressed as:

$$\mathrm{d}W_t = \begin{cases} (r_t + \boldsymbol{\omega}_t'\boldsymbol{\Sigma}\boldsymbol{\lambda})\,W_t\,\mathrm{d}t + W_t\boldsymbol{\omega}_t'\boldsymbol{\Sigma}\,\mathrm{d}\boldsymbol{Z}_t & \text{for } \; 0 < t \leq T_R \\ (r_t + \boldsymbol{\omega}_t'\boldsymbol{\Sigma}\boldsymbol{\lambda})\,W_t\,\mathrm{d}t + W_t\boldsymbol{\omega}_t'\boldsymbol{\Sigma}\,\mathrm{d}\boldsymbol{Z}_t - c_t\,\mathrm{d}t & \text{for } T_R < t < T_D \end{cases}, \tag{K.13}$$

where $\boldsymbol{\Sigma}$, $\boldsymbol{\lambda}$ and $\boldsymbol{\omega}_t$ are defined in (3.3.2). The budget equation of the individual states that at all times (also in optimality):

$$W_t^* = V_t^*, \tag{K.14}$$

for which the optimality of W_t is a result of choosing certain values for the embedded decision variable $\boldsymbol{\omega}_t$. Applying Itô's lemma to the budget condition implies

$$\mathrm{d}W_t^* = \mathrm{d}V_t^*, \quad \text{or, equivalently} \quad \mathrm{d}\log W_t^* = \mathrm{d}\log V_t^*. \tag{K.15}$$

First deriving the log dynamics of wealth and optimal liabilities in a similar fashion as in Appendix D, and then equating the diffusion terms of the processes yields:

$$\begin{cases} \frac{\lambda_S}{\gamma} = \omega_t^{S*}\sigma_S \\ \frac{\lambda_r}{\gamma} + \left(\frac{1-\gamma}{\gamma}\right)\sigma_r D_t^{V^*} = -\omega_t^{P*}\sigma_r D(h) \end{cases}. \tag{K.16}$$

Solving for $\boldsymbol{\omega}_t^*$ results in the following optimal portfolio fractions:

$$\begin{aligned} \boldsymbol{\omega}_t^* &= \frac{1}{\gamma}\begin{pmatrix} \frac{\lambda_S}{\sigma_S} \\ -\frac{\lambda_r}{\sigma_r D(h)} \end{pmatrix} + \left(1 - \frac{1}{\gamma}\right)\begin{pmatrix} 0 \\ \frac{D_t^{V^*}}{D(h)} \end{pmatrix} \\ &= \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda} + \left(1 - \frac{1}{\gamma}\right)\begin{pmatrix} 0 \\ \frac{D_t^{V^*}}{D(h)} \end{pmatrix}. \end{aligned} \tag{K.17}$$

L Derivation of human capital dynamics

According to (3.3.4), human capital is defined as

$$H_t = \int_0^{T_D - t} P_t^h K(t+h)\,\mathrm{d}h = \int_t^{T_D} P_t^{h-t} K(h)\,\mathrm{d}h, \tag{L.1}$$

with a deterministic quantity $K(h)$ as defined in (3.3.5). In this case, human capital only consists of a deterministic labor income component and therefore an equivalent expression is

$$H_t = \int_0^{[T_R - t]_+} P_t^h Y_{t+h} p_{t+h}\,\mathrm{d}h. \tag{L.2}$$

However, we keep intact the generic definition of human capital (L.1), such that the results can also be applied when adjustments are made in the components of human capital. We can define human capital as a function of the bond price $H_t = f(t, P_t^h)$. Then, the dynamics become

$$\begin{aligned} \mathrm{d}H_t &= \frac{\partial H_t}{\partial t}\,\mathrm{d}t + \frac{\partial H_t}{\partial P_t^h}\,\mathrm{d}P_t^h + \frac{1}{2}\frac{\partial^2 H_t}{\partial (P_t^h)^2}(\mathrm{d}P_t^h)^2, \quad \text{with} \\ \frac{\partial H_t}{\partial t} &= \frac{\partial}{\partial t}\int_t^{T_D} P_t^{h-t} K(h)\,\mathrm{d}h = -P_t^{t-t}K(t)\frac{\partial}{\partial t}t = -K(t), \\ \frac{\partial H_t}{\partial P_t^h} &= \int_0^{T_D - t} \frac{\partial}{\partial P_t^h} P_t^h K(t+h)\,\mathrm{d}h = \int_0^{T_D - t} K(t+h)\,\mathrm{d}h, \\ \frac{\partial^2 H_t}{\partial (P_t^h)^2} &= 0, \end{aligned} \tag{L.3}$$

where for the partial derivative of H_t with respect to t Leibniz rule is applied. When substituting the partial derivatives, the dynamics of human capital can be expressed as:

$$\begin{aligned}
\mathrm{d}H_t &= -K(t)\,\mathrm{d}t + \int_0^{T_D-t} K(t+h)\,\mathrm{d}h(\mathrm{d}P_t^h) \\
&= -K(t)\,\mathrm{d}t + \int_0^{T_D-t} K(t+h)P_t^h\left((r_t - \lambda_r\sigma_r D(h))\,\mathrm{d}t - \sigma_r D(h)\,\mathrm{d}Z_t^r\right)\mathrm{d}h \\
&= -K(t)\,\mathrm{d}t + \left(r_t H_t - \lambda_r\sigma_r \int_0^{T_D-t} D(h)P_t^h K(t+h)\,\mathrm{d}h\right)\mathrm{d}t \\
&\quad - \sigma_r\,\mathrm{d}Z_t^r \int_0^{T_D-t} D(h)P_t^h K(t+h)\,\mathrm{d}h \\
&= -K(t)\,\mathrm{d}t + (r_t - \sigma_r D_t^H \lambda_r)H_t\,\mathrm{d}t - \sigma_r D_t^H H_t\,\mathrm{d}Z_t^r,
\end{aligned} \tag{L.4}$$

with

$$D_t^H \equiv \frac{\int_0^{T_D-t} D(h)P_t^h K(t+h)\,\mathrm{d}h}{\int_0^{T_D-t} P_t^h K(t+h)\,\mathrm{d}h}, \tag{L.5}$$

which can be interpreted as the interest rate 'duration' of human capital at time t. This expression is equivalent to (3.5.14).

M Derivation of investment strategy financial wealth

First of all, we use the fact that total wealth can be decomposed as:

$$W_t = F_t + H_t, \tag{M.1}$$

on which an application of Itô's lemma yields

$$\mathrm{d}F_t = \mathrm{d}W_t - \mathrm{d}H_t. \tag{M.2}$$

The dynamics of human capital is derived in Appendix L and the dynamics of total wealth is derived in Appendix I. Therefore, we can uniquely identify the dynamics of financial wealth. In a similar fashion as total wealth, the functional form of the dynamics of financial wealth can be expressed as:

$$\mathrm{d}F_t = \left(r_t + \boldsymbol{\theta}_t'\boldsymbol{\Sigma\lambda}\right)F_t\,\mathrm{d}t + F_t\boldsymbol{\theta}_t'\boldsymbol{\Sigma}\,\mathrm{d}\boldsymbol{Z}_t + p_t Y_t\,\mathrm{d}t, \tag{M.3}$$

with

$$\boldsymbol{\theta}_t^\top = \begin{pmatrix}\theta_t^S, & \theta_t^P\end{pmatrix},$$

which denotes the fractions of financial wealth invested in the available assets. Once again, the fraction of financial wealth invested in the risk-free bank account is residually determined as $\theta_t^B = 1 - \boldsymbol{\theta}_t'\boldsymbol{\iota}$. In the above equation, p_t is again the fixed pension premium percentage at time t and Y_t the labor income at time t. Note that financial wealth starts at zero and gradually increases because of investment returns and premium contributions. Total wealth however decreases over time due to consumption.

Then, according to (M.2), when matching diffusion terms and constructing a replicating portfolio for financial wealth out of human capital and total wealth, the following equality must hold (also in optimality):

$$F_t \boldsymbol{\theta}_t^{*'} \boldsymbol{\Sigma} = W_t \boldsymbol{\omega}_t^{*'} \boldsymbol{\Sigma} + H_t \begin{pmatrix} 0 \\ \sigma_r D_t^H \end{pmatrix}'. \tag{M.4}$$

Solving for $\boldsymbol{\theta}_t^*$ results in the optimal fractions of financial wealth invested in the asset menu:

$$\boldsymbol{\theta}_t^* = \left(1 + \frac{H_t}{F_t}\right) \boldsymbol{\omega}_t^* - \frac{H_t}{F_t} \begin{pmatrix} 0 \\ D_t^H / D(h) \end{pmatrix}. \tag{M.5}$$

N Derivation of funding ratio dynamics

The asset value of the pension fund at time t is equal to

$$A_t \equiv \eta_t^S S_t + \eta_t^P P_t^h + \eta_t^B B_t, \tag{N.1}$$

where $\hat{\boldsymbol{\eta}}_t^\top = (\hat{\eta}_t^S, \hat{\eta}_t^P, \hat{\eta}_t^B)$ denote the number of the different assets held by the pension fund at time t. The asset value at time t can also be described by an initial value A_0 plus gains from trading and received premiums minus payed out benefits:

$$A_t = A_0 + \int_0^t \eta_t^S \, \mathrm{d}S_s + \int_0^t \eta_t^P \, \mathrm{d}P_s^h + \int_0^t \eta_s^B \, \mathrm{d}B_s + \int_0^t p_s \, \mathrm{d}s - \int_0^t u_s \, \mathrm{d}s \tag{N.2}$$

where p_t denotes the rate at time t at which new premiums are received and invested and u_t represents the rate at time t at which (total) benefits are payed out. Note that p_t and u_t are dependent on the number of participants at time t and their age.

Assuming that at all times the outflow of total benefit payment equals the total inflow of new premiums, results in

$$\int_0^t p_s \, \mathrm{d}s - \int_0^t u_s \, \mathrm{d}s = 0 \quad \text{for all } t. \tag{N.3}$$

Assuming that A is a self-financing portfolio and following steps similar to Appendix I, the dynamics of A_t can be expressed as:

$$\begin{aligned} \mathrm{d}A_t &= \hat{\eta}_t^S \, \mathrm{d}S_t + \hat{\eta}_t^P \, \mathrm{d}P_t^h + \hat{\eta}_t^B \, \mathrm{d}B_t \\ &= \hat{\omega}^S A_t \frac{\mathrm{d}S_t}{S_t} + \hat{\omega}^P A_t \frac{\mathrm{d}P_t^h}{P_t^h} + (1 - \hat{\omega}^S - \hat{\omega}^P) A_t r_t \, \mathrm{d}t \\ &= \left(r_t + \hat{\boldsymbol{\omega}}' \boldsymbol{\Sigma} \boldsymbol{\lambda}\right) A_t \, \mathrm{d}t + A_t \hat{\boldsymbol{\omega}}' \boldsymbol{\Sigma} \, \mathrm{d}\boldsymbol{Z}_t, \end{aligned} \tag{N.4}$$

with

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_S & 0 \\ 0 & -\sigma_r D(h) \end{pmatrix}, \quad \hat{\boldsymbol{\omega}} = \begin{pmatrix} \omega^S \\ \omega^P \end{pmatrix}, \quad \boldsymbol{\lambda} = \begin{pmatrix} \lambda_S \\ \lambda_r \end{pmatrix}, \tag{N.5}$$

where ω^S (ω^P) denotes the constant fraction of the pension fund assets invested in the stock price index (nominal zero-coupon bond).

Since we assume the liabilities L_t to replicate a nominal bond with duration $D(l)$, the dynamics for the fund's liabilities are as follows:

$$\frac{\mathrm{d}L_t}{L_t} = (r_t - \lambda_r \sigma_r D(l))\,\mathrm{d}t - \sigma_r D(l)\,\mathrm{d}Z_t^r. \tag{N.6}$$

Now that the dynamics of both the numerator and the denominator are known, we can derive the dynamics of the funding ratio itself. Defining $G_t = f(t, A_t, L_t) = \frac{A_t}{L_t}$, the dynamics of G_t can be determined by applying the multivariate version of Itô's lemma derived in Appendix K (Note that for notational convenience, the time subscript is suppressed):

$$\begin{aligned} \mathrm{d}G &= \frac{1}{L}\,\mathrm{d}A - \frac{A}{L^2}\,\mathrm{d}L + \frac{1}{2}\left(0 + \frac{2A}{L^3}(\mathrm{d}L)^2 - \frac{2}{L^2}(\mathrm{d}A)(\mathrm{d}L)\right) \\ &= \frac{1}{L}\,\mathrm{d}A - \frac{A}{L^2}\,\mathrm{d}L + \frac{A}{L^3}(\mathrm{d}L)^2 - \frac{1}{L^2}(\mathrm{d}A)(\mathrm{d}L). \end{aligned} \tag{N.7}$$

Now dividing both sides by $\frac{A}{L}$, results in

$$\frac{\mathrm{d}G}{G} = \frac{1}{A}\,\mathrm{d}A - \frac{1}{L}\,\mathrm{d}L + \frac{1}{L^2}(\mathrm{d}L)^2 - \frac{1}{AL}(\mathrm{d}A)(\mathrm{d}L), \tag{N.8}$$

where $(\mathrm{d}L)^2$ represents the quadratic variation of the liabilities. Substituting (N.4) and (N.6) in (N.8), yields the following expression:

$$\begin{aligned} \frac{\mathrm{d}G_t}{G_t} &= (r_t + \hat{\omega}^S \lambda_S \sigma_S - \hat{\omega}^P \lambda_r \sigma_r D(h))\,\mathrm{d}t + \hat{\omega}^S \sigma_S\,\mathrm{d}Z_t^S - \hat{\omega}^P \sigma_r D(h)\,\mathrm{d}Z_t^r\ - \\ &\quad (r_t - \lambda_r \sigma_r D(l))\,\mathrm{d}t + \sigma_r D(l)\,\mathrm{d}Z_t^r\ + \\ &\quad \sigma_r^2 D(l)^2\,\mathrm{d}t\ - \\ &\quad \hat{\omega}^P \sigma_r^2 D(h) D(l)\,\mathrm{d}t \\ &= \left(\hat{\omega}^S \lambda_S \sigma_S - \hat{\omega}^P \left(\lambda_r \sigma_r D(h) + \sigma_r^2 D(h) D(l)\right) + \lambda_r \sigma_r D(l) + \sigma_r^2 D(l)^2\right)\mathrm{d}t\ + \\ &\quad \hat{\omega}^S \sigma_S\,\mathrm{d}Z_t^S + \left(\sigma_r D(l) - \hat{\omega}^P \sigma_r D(h)\right)\mathrm{d}Z_t^r \\ &= \mu_g\,\mathrm{d}t + \boldsymbol{\sigma}_g'\,\mathrm{d}\boldsymbol{Z}_t, \end{aligned} \tag{N.9}$$

with

$$\mu_g \equiv \hat{\omega}^S \lambda_S \sigma_S - \hat{\omega}^P \left(\lambda_r \sigma_r D(h) + \sigma_r^2 D(h) D(l)\right) + \lambda_r \sigma_r D(l) + \sigma_r^2 D^2(l), \tag{N.10}$$

$$\boldsymbol{\sigma}_g^\top \equiv \begin{pmatrix} \hat{\omega}^S \sigma_S \\ \sigma_r D(l) - \hat{\omega}^P \sigma_r D(h) \end{pmatrix}. \tag{N.11}$$

Notice that also for the funding ratio, just like for the assets and the liabilities, the relative change is level-independent. Another desirable feature is that this relative change of G_t is also independent of the risk-free interest rate level r_t.

O Derivation of pricing kernel under new numeraire $\widetilde{M}_t$

We first derive the dynamics of the log transformation of the annuity factor $\bar{a}_t$, write these dynamics in integral form and take the exponent, similar to the steps in Appendix D. This yields the following closed-form expression for $\bar{a}_t$ as function of the current value of $\bar{a}_0$ and the Brownian short rate increment:

$$\bar{a}_t = \bar{a}_0 \exp\left\{ -\psi\kappa\bar{r}t + \psi\kappa \int_0^t r_u \,\mathrm{d}u - \psi\sigma_r Z_t^r \right\}. \tag{O.1}$$

The same steps result in the following closed-form expression of the funding ratio G_t:

$$G_t = G_0 \exp\left\{ \left(\mu_g - \frac{1}{2} \left\|\boldsymbol{\sigma}_g\right\|^2 \right) t + \boldsymbol{\sigma}_g' \boldsymbol{Z}_t \right\}. \tag{O.2}$$

Then, given the closed-form expression of the pricing kernel under the cash numeraire M_t in (3.2.8), the pricing kernel under the new numeraire can be expressed as follows:

$$\begin{aligned}
\widetilde{M}_t &= \bar{a}_t G_t M_t \\
&= \bar{a}_0 G_0 M_0 \exp\left\{ -\int_0^t r_u \,\mathrm{d}u + \psi\kappa \int_0^t r_u \,\mathrm{d}u - \frac{1}{2}\left\|\boldsymbol{\lambda}\right\|^2 t - \frac{1}{2}\left\|\boldsymbol{\sigma}_g\right\|^2 t - \psi\kappa\bar{r}t + \mu_g t \right\} \times \\
&\qquad \exp\left\{ -\left(\boldsymbol{\lambda} - \boldsymbol{\sigma}_g + \psi\kappa\boldsymbol{e}_2\right)' \boldsymbol{Z}_t \right\} \\
&= \bar{a}_0 G_0 M_0 \exp\left\{ -(1-\psi\kappa) \int_0^t r_u \,\mathrm{d}u + \left(\mu_g - \psi\kappa\bar{r} - \frac{1}{2}\left\|\boldsymbol{\lambda}\right\|^2 - \frac{1}{2}\left\|\boldsymbol{\sigma}_g\right\|^2 \right) t - \tilde{\boldsymbol{\nu}}' \boldsymbol{Z}_t \right\} \\
&= \widetilde{M}_0 \exp\left\{ -(1-\psi\kappa) \int_0^t r_u \,\mathrm{d}u - \left(\frac{1}{2}\left\|\tilde{\boldsymbol{\nu}}\right\|^2 + c_1 \right) t - \tilde{\boldsymbol{\nu}}' \boldsymbol{Z}_t \right\},
\end{aligned} \tag{O.3}$$

where

$$\tilde{\boldsymbol{\nu}} \equiv \boldsymbol{\lambda} - \boldsymbol{\sigma}_g + \psi\sigma_r \boldsymbol{e}_2, \tag{O.4}$$

$$c_1 \equiv \psi\kappa\bar{r} - \sigma_r^2 D(l)(D(h) + D(l)) - \frac{1}{2}\psi^2\sigma_r^2 - \lambda_r\psi\sigma_r + \sigma_g^r\psi\sigma_r. \tag{O.5}$$

Following the same derivation steps as stated in Appendix F, the equivalent dynamics of $\widetilde{M}_t$ can be expressed as:

$$\frac{\mathrm{d}\widetilde{M}_t}{\widetilde{M}_t} = ((1-\psi\kappa)r_t + c_1)\,\mathrm{d}t - \tilde{\boldsymbol{\nu}}' \,\mathrm{d}\boldsymbol{Z}_t. \tag{O.6}$$

One could also define $\widetilde{M}_t = f(t, \bar{a}_t, G_t, M_t)$ and applying the multivariate version of Itô's lemma provided in Appendix K, results in the same expression.

P Expression for adapted cond. expectation $\widetilde{M}_t$

We derive an expression for the following conditional expectation of a power function of the pricing kernel under the new numeraire:

$$\widetilde{\Theta}(t,s,x) \equiv \mathbb{E}_t\left[\left(\frac{\widetilde{M}_s}{\widetilde{M}_t}\right)^x\right], \qquad t < s < T_R.$$

Below the following derivation, brief commentary is provided on the taken steps. Note that this derivation is similar to Appendix G.

$$\begin{aligned}
\mathbb{E}_t\left[\left(\frac{\widetilde{M}_s}{\widetilde{M}_t}\right)^x\right] &= \mathbb{E}_t\left[\exp\left\{-x\int_0^{s-t}\left((1-\kappa\psi)r_{t+u}+\frac{1}{2}\left\|\boldsymbol{\lambda}\right\|^2+c_1\right)\mathrm{d}u - x\int_0^{s-t}\boldsymbol{\lambda}'\,\mathrm{d}\mathbf{Z}_{t+u}\right\}\right] \\
&= \mathbb{E}_t\left[\exp\left\{-x\int_0^{s-t}(1-\kappa\psi)\mathbb{E}_t\left[r_{t+u}\right]+\frac{1}{2}\left\|\boldsymbol{\lambda}\right\|^2+c_1\,\mathrm{d}u\right\}\times\right. \\
&\qquad \left.\exp\left\{-\sigma_r(1-\kappa\psi)x\int_0^{s-t}D(s-t-u)\,\mathrm{d}Z^r_{t+u} - x\int_0^{s-t}\boldsymbol{\lambda}'\,\mathrm{d}\mathbf{Z}_{t+u}\right\}\right] \\
&= \exp\left\{x\left((\kappa\psi-1)(r_tD(s-t)+\bar{r}(s-t-D(s-t))) - (s-t)\left(\frac{1}{2}\left\|\boldsymbol{\lambda}\right\|^2+c_1\right)\right)\right\}\times \\
&\qquad \mathbb{E}_t\left[\exp\left\{-x\int_0^{s-t}(\boldsymbol{\lambda}+\sigma_r(1-\kappa\psi)D(s-t-u)\mathbf{e}_2)'\,\mathrm{d}\mathbf{Z}_{t+u}\right\}\right] \\
&= \exp\left\{x\left((\kappa\psi-1)(r_tD(s-t)+\bar{r}(s-t-D(s-t))) - (s-t)\left(\frac{1}{2}\left\|\boldsymbol{\lambda}\right\|^2+c_1\right)\right)\right\}\times \\
&\qquad \exp\left\{\frac{1}{2}x^2\int_0^{s-t}\left\|\boldsymbol{\lambda}+\sigma_r(1-\kappa\psi)D(u)\mathbf{e}_2\right\|^2\mathrm{d}u\right\} \\
&= \exp\left\{x(\kappa\psi-1)(r_tD(s-t)+\bar{r}(s-t-D(s-t))) + (s-t)\left(\frac{1}{2}x(x-1)\left\|\boldsymbol{\lambda}\right\|^2 - xc_1\right)\right\}\times \\
&\qquad \exp\left\{\frac{1}{2}x^2\int_0^{s-t}2\sigma_r\lambda_r(1-\kappa\psi)D(u)+\sigma_r^2(1-\kappa\psi)^2D(u)^2\,\mathrm{d}u\right\} \\
&= \exp\left\{x(\kappa\psi-1)(r_tD(s-t)+\bar{r}(s-t-D(s-t))) + (s-t)\left(\frac{1}{2}x(x-1)\left\|\boldsymbol{\lambda}\right\|^2 - xc_1\right)\right\}\times \\
&\qquad \exp\left\{\frac{x^2\sigma_r\lambda_r(1-\kappa\psi)}{\kappa}(s-t-D(s-t))\right\}\times \\
&\qquad \exp\left\{\frac{x^2\sigma_r^2(1-\kappa\psi)^2}{\kappa^2}\left(\frac{1}{2}(s-t)-D(s-t)+\frac{1}{4}D(2s-2t)\right)\right\} \\
&= \exp\left\{-x\left(\tilde{a}(s-t)+D(s-t)(1-\kappa\psi)r_t\right)\right\},
\end{aligned} \tag{P.1}$$

where

$$\begin{aligned}
\tilde{a}(u) &\equiv -\frac{1}{2}(x-1)\left\|\boldsymbol{\lambda}\right\|^2u + \left(\bar{r}-\frac{x\sigma_r\lambda_r}{\kappa}\right)(1-\kappa\psi)(u-D(u)) - \\
&\qquad \frac{x\sigma_r^2(1-\kappa\psi)^2}{\kappa^2}\left(\frac{1}{2}u-D(u)+\frac{1}{4}D(2u)\right)+c_1u,
\end{aligned} \tag{P.2}$$

$$D(u) \equiv \frac{1}{\kappa}\left(1-e^{-\kappa u}\right). \tag{P.3}$$

The vector $\boldsymbol{e}_i$ used above, denotes the i-th unit vector: a zero-valued vector except for entry i, which is equal to one. In the second equality, the expression for r_t derived in appendix C (C.2,C.3) is used:

$$\begin{aligned} r_{t+s} &= r_t + (\bar{r} - r_t)(1 - e^{-\kappa s}) + \sigma_r \int_t^{t+s} e^{-\kappa(t+s-u)}\,\mathrm{d}Z_u^r \\ &= \mathbb{E}_t[r_{t+s}] + \sigma_r \int_t^{t+s} e^{-\kappa(t+s-u)}\,\mathrm{d}Z_u^r. \end{aligned}$$

For the fourth equality, the Itô isometry property and the expectation of the lognormal distribution is used, as well as the following function property of D:

$$\int_0^n D(n-u)\,\mathrm{d}u = \int_0^n D(u)\,\mathrm{d}u.$$

For the sixth equality, the integral of $D(u)^2$ is solved in the following way:

$$\begin{aligned} D(u)^2 &= \frac{1}{\kappa^2}\left(1 - 2e^{-\kappa u} + e^{-2\kappa u}\right) \\ &= \frac{1}{\kappa^2}\left(1 - e^{-\kappa u} + 1 - e^{-\kappa u} - \left(1 - e^{-2\kappa u}\right)\right) \\ &= \frac{1}{\kappa}\left(2D(u) - D(2u)\right), \end{aligned}$$

so that the integral can be written as

$$\begin{aligned} \frac{1}{2}\int_0^h D(u)^2 du &= \frac{1}{2\kappa}\int_0^h 2D(u)\,\mathrm{d}u - \frac{1}{2\kappa}D(2u)\,\mathrm{d}u \\ &= \frac{1}{\kappa}\left(\frac{1}{\kappa}(h - D(h)\right) - \frac{1}{2\kappa}\left(\frac{1}{\kappa}\left(h - \frac{1}{2}D(2h)\right)\right) \\ &= \frac{1}{\kappa^2}\left(\frac{1}{2}h - D(h) + \frac{1}{4}D(2h)\right). \end{aligned}$$

Note that for $x = 1$, we obtain $\widetilde{\Theta}(t, s, 1) = \widetilde{P}_t^{(s-t)}$ the price of a nominal zero-coupon bond with maturity $(s - t)$ under the new numeraire.

Q Derivation of (optimal) wealth dynamics $\mathrm{d}\widetilde{W}_t$

Regarding the optimal wealth path (5.4.7), the dynamics of the log process are determined in the following way. Define $\widetilde{W}_t^* = f(t, \widetilde{M}_t, r_t)$. Then, applying a multivariate version of Itô's lemma derived in Appendix K, results in the following dynamics:

$$\begin{aligned} \mathrm{d}\widetilde{W}_t^* &= \frac{\partial \widetilde{W}_t^*}{\partial \widetilde{M}_t}(\mathrm{d}\widetilde{M}_t) + \frac{\partial \widetilde{W}_t^*}{\partial r_t}(\mathrm{d}r_t) + \frac{1}{2}\left(\frac{\partial^2 \widetilde{W}_t^*}{\partial \widetilde{M}_t^2}(\mathrm{d}\widetilde{M}_t)^2 + \frac{\partial^2 \widetilde{W}_t^*}{\partial r_t^2}(\mathrm{d}r_t)^2 + 2\frac{\partial^2 \widetilde{W}_t^*}{\partial r_t \partial \widetilde{M}_t}(\mathrm{d}r_t)(\mathrm{d}\widetilde{M}_t)\right) \\ &= \mu^{\tilde{W}_t^*}(t, r_t, \widetilde{W}_t^*)\,\mathrm{d}t + \frac{1}{\gamma}\widetilde{W}_t^* \tilde{\boldsymbol{\nu}}'\,\mathrm{d}\boldsymbol{Z}_t - \left(1 - \frac{1}{\gamma}\right)\widetilde{W}_t^*(1 - \psi\kappa)D(T_R - t)\sigma_r\,\mathrm{d}Z_t^r, \end{aligned} \tag{Q.1}$$

where $\mu^{\tilde{W}_t^*}(t, r_t, \widetilde{W}_t^*)$ is the drift of the process expressed as a general function of the variables time, the interest rate and the optimal terminal wealth level at time t. The second equality follows from determining the partial derivatives and substituting the dynamics (O.6) and (3.2.1).

Then, define $\log \widetilde{W}_t^* = f(t, \widetilde{W}_t^*)$. Applying Itô's lemma results in the following stochastic differential equation:

$$\begin{aligned}
\mathrm{d}\log \widetilde{W}_t^* &= \frac{1}{\widetilde{W}_t^*}\,\mathrm{d}\widetilde{W}_t^* - \frac{1}{2}\frac{1}{(\widetilde{W}_t^*)^2}(\mathrm{d}\widetilde{W}_t^*)^2 \\
&= \mu^{\log \tilde{W}_t^*}(t, r_t)\,\mathrm{d}t + \frac{1}{\gamma}\tilde{\boldsymbol{\nu}}'\,\mathrm{d}\boldsymbol{Z}_t - \left(1 - \frac{1}{\gamma}\right)(1-\psi\kappa)D(T_R - t)\sigma_r\,\mathrm{d}Z_t^r, \qquad \text{(Q.2)}
\end{aligned}$$

where $\mu^{\log \tilde{W}_t^*}(t, r_t)$ is the drift of the process expressed as a general function of time (or investment horizon) and the short rate.

Regarding the wealth process under the new numeraire,

$$\widetilde{W}_t = \frac{W_t}{\bar{a}_t G_t}, \qquad \text{(Q.3)}$$

the dynamics are determined in the following manner. Define the function $\widetilde{W}_t = f(t, W_t, \bar{a}_t, G_t)$. Then, applying once again the multivariate version of Itô's lemma derived in Appendix K, yields the following dynamics:

$$\begin{aligned}
\mathrm{d}\widetilde{W}_t = {} & \frac{\partial \widetilde{W}_t}{\partial W_t}(\mathrm{d}W_t) + \frac{\partial \widetilde{W}_t}{\partial \bar{a}_t}(\mathrm{d}\bar{a}_t) + \frac{\partial \widetilde{W}_t}{\partial G_t}(\mathrm{d}G_t) + \frac{1}{2}\left(\frac{\partial^2 \widetilde{W}_t}{\partial \bar{a}_t^2}(\mathrm{d}\bar{a}_t)^2 + \frac{\partial^2 \widetilde{W}_t}{\partial G_t^2}(\mathrm{d}G_t)^2\right) + \\
& \frac{\partial^2 \widetilde{W}_t}{\partial W_t \partial \bar{a}_t}(\mathrm{d}W_t)(\mathrm{d}\bar{a}_t) + \frac{\partial^2 \widetilde{W}_t}{\partial W_t \partial G_t}(\mathrm{d}W_t)(\mathrm{d}G_t) + \frac{\partial^2 \widetilde{W}_t}{\partial \bar{a}_t \partial G_t}(\mathrm{d}\bar{a}_t)(\mathrm{d}G_t) \\
= {} & \mu^{\tilde{W}_t}(t, r_t, \widetilde{W}_t)\,\mathrm{d}t + \widetilde{W}_t\left(\boldsymbol{\Sigma}'\boldsymbol{\omega}_t - \boldsymbol{\sigma}_g + \psi\sigma_r\boldsymbol{e}_2\right)'\mathrm{d}\boldsymbol{Z}_t, \qquad \text{(Q.4)}
\end{aligned}$$

with

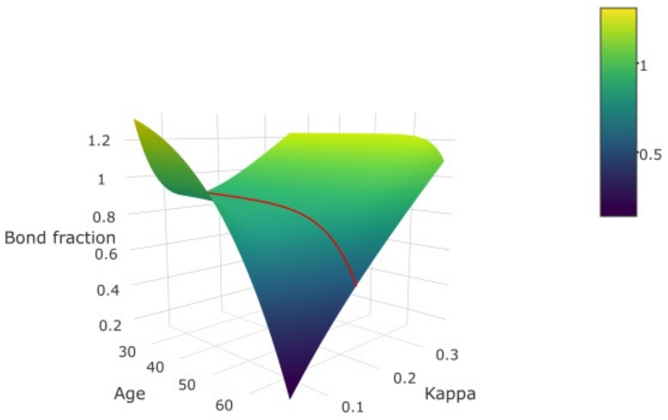
$$\mu^{\tilde{W}_t}(t, r_t, \widetilde{W}_t) \equiv \widetilde{W}_t\Big((1-\psi\kappa)r_t + \boldsymbol{\omega}_t'\boldsymbol{\Sigma}(\boldsymbol{\lambda} - \boldsymbol{\sigma}_g) + \|\boldsymbol{\sigma}_g\|^2 - \mu_g + \psi\kappa\bar{r} + \frac{1}{2}\psi^2\sigma_r + \psi\sigma_r(\boldsymbol{\omega}_t'\boldsymbol{\Sigma} - \boldsymbol{\sigma}_g')\boldsymbol{e}_2\Big).$$

The second equality follows from determining the partial derivatives and substituting the dynamics (3.3.2), (5.4.1) and (5.2.2). Then, define $\log \widetilde{W}_t = f(t, \widetilde{W}_t)$. Applying Itô's lemma results in the following stochastic differential equation:

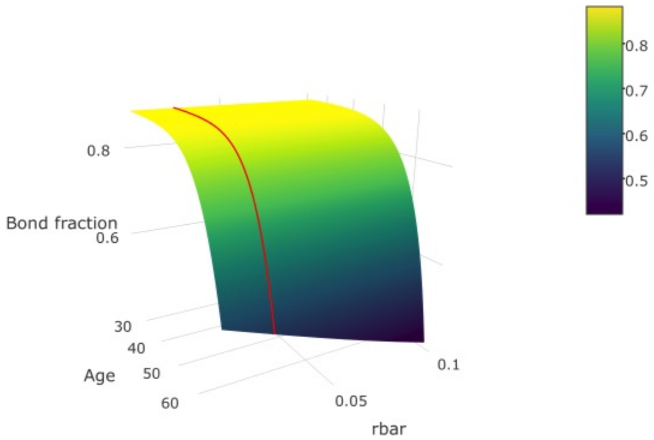
$$\begin{aligned}
\mathrm{d}\log \widetilde{W}_t &= \frac{1}{\widetilde{W}_t}\,\mathrm{d}\widetilde{W}_t - \frac{1}{2}\frac{1}{(\widetilde{W}_t)^2}(\mathrm{d}\widetilde{W}_t)^2 \\
&= \mu^{\log \tilde{W}_t}(t, r_t)\,\mathrm{d}t + \left(\boldsymbol{\Sigma}'\boldsymbol{\omega}_t - \boldsymbol{\sigma}_g + \psi\sigma_r\boldsymbol{e}_2\right)'\mathrm{d}\boldsymbol{Z}_t, \qquad \text{(Q.5)}
\end{aligned}$$

where $\mu^{\log \tilde{W}_t}(t, r_t)$ is the drift of the process expressed as a general function of the remaining investment horizon and the short rate state variable.

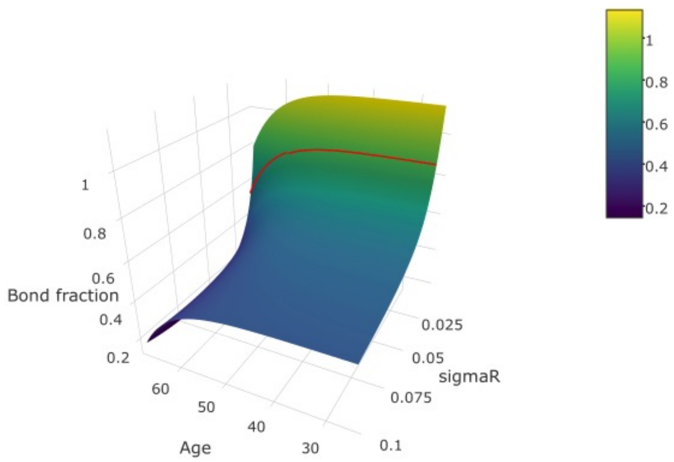
R Figures sensitivity analysis short rate parameters



(a) mean reversion κ



(b) long-run mean short rate $\bar{r}$



(c) short rate volatility σ_r

Figure R.1: The sensitivity of the optimal total wealth bond fraction ω_t^{P*} (5.4.11) w.r.t. a particular short rate process parameter is displayed ceteris paribus the benchmark parameter set of Section 5.3.1. The red glide path represents the benchmark case displayed in Figure 5.1.

S Optimal investment strategy financial wealth for a top-up plan

Total wealth under the cash numeraire can be split up in the following components:

$$W_t^{(\boldsymbol{\omega}_t^*)} = F_t^{(\boldsymbol{\theta}_t^*)} + H_t^{np} + H^{(1,2)}\bar{a}_t G_t, \quad t \in [T_E, T_R]. \tag{S.1}$$

Applying Itô's lemma results in the following dynamics with respect to this equality:

$$\mathrm{d}F_t^{(\boldsymbol{\theta}_t^*)} = \mathrm{d}W_t^{(\boldsymbol{\omega}_t^*)} - \mathrm{d}H_t^{np} - H^{(1,2)}\,\mathrm{d}(\bar{a}_t G_t). \tag{S.2}$$

From Appendix O, one can derive the dynamics of the product of the annuity factor and the funding ratio:

$$\frac{\mathrm{d}(\bar{a}_t G_t)}{\bar{a}_t G_t} = \left(\mu_g - \psi\kappa(\bar{r} - r_t) + \frac{1}{2}\psi^2\sigma_r^2 - \psi\sigma_r\sigma_g^r\right)\mathrm{d}t + (\boldsymbol{\sigma}_g - \psi\sigma_r\boldsymbol{e}_2)'\,\mathrm{d}\boldsymbol{Z}_t. \tag{S.3}$$

Regarding the dynamics of the labor income component eligible for the top-up plan, this can be expressed in the following way based on Appendix L:

$$\mathrm{d}H_t^{np} = -K(t)\,\mathrm{d}t + (r_t - \sigma_r D_t^H\lambda_r)H_t^{np}\,\mathrm{d}t - \sigma_r D_t^H H_t^{np}\,\mathrm{d}Z_t^r, \quad \text{with} \tag{S.4}$$

$$D_t^H \equiv \frac{\int_0^{T_R-t} D(h)P_t^h K(t+h)\,\mathrm{d}h}{\int_0^{T_R-t} P_t^h K(t+h)\,\mathrm{d}h} \quad \text{and} \quad K(u) = \begin{cases} 0 & \text{for } u \in [0, T_E) \\ p_u(Y_u - \overline{\Upsilon}) & \text{for } u \in [T_E, T_R) \end{cases}.$$

Recall that the dynamics of financial wealth, similar to the dynamics of total wealth, is stated as (Appendix M):

$$\mathrm{d}F_t = \left(r_t + \boldsymbol{\theta}_t'\boldsymbol{\Sigma}\boldsymbol{\lambda}\right)F_t\,\mathrm{d}t + F_t\boldsymbol{\theta}_t'\boldsymbol{\Sigma}\,\mathrm{d}\boldsymbol{Z}_t + p_t Y_t\,\mathrm{d}t, \quad \text{with} \tag{S.5}$$

$$\boldsymbol{\theta}_t^\top = \begin{pmatrix}\theta_t^S & \theta_t^P\end{pmatrix},$$

which denotes the fraction of financial wealth invested in the available assets. Based on Equation (S.2), the dynamics of financial wealth must replicate the dynamics of the three terms on the right hand side, also in optimality. Therefore, the following optimality condition with regard to the investment strategy of financial wealth can be stated:

$$F_t\boldsymbol{\theta}_t^{*\prime}\boldsymbol{\Sigma} = W_t\boldsymbol{\omega}_t^{*\prime}\boldsymbol{\Sigma} + H_t^{np}\sigma_r D_t^H\boldsymbol{e}_2' - H^{(1,2)}\bar{a}_t G_t(\boldsymbol{\sigma}_g - \psi\sigma_r\boldsymbol{e}_2)'. \tag{S.6}$$

Then, straightforward calculus results in

$$\boldsymbol{\theta}_t^* = \left(1 + \frac{H_t^{(np)}}{F_t} + H^{(1,2)}\frac{\bar{a}_t G_t}{F_t}\right)\boldsymbol{\omega}_t^* - \frac{H_t^{np}}{F_t}\begin{pmatrix}0 \\ \frac{D_t^H}{D(h)}\end{pmatrix} - H^{(1,2)}\frac{\bar{a}_t G_t}{F_t}\begin{pmatrix}\frac{\sigma_g^S}{\sigma_S} \\ \frac{\psi\sigma_r - \sigma_g^r}{\sigma_r D(h)}\end{pmatrix} \tag{S.7}$$

$$= \left(1 + \frac{H_t^{np}}{F_t} + \frac{\bar{A}_t}{F_t}\right)\boldsymbol{\omega}_t^* - \frac{H_t^{np}}{F_t}\begin{pmatrix}0 \\ \frac{D_t^H}{D(h)}\end{pmatrix} - \frac{\bar{A}_t}{F_t}\begin{pmatrix}\frac{\sigma_g^S}{\sigma_S} \\ \frac{\psi\sigma_r - \sigma_g^r}{\sigma_r D(h)}\end{pmatrix}. \tag{S.8}$$