Network for Studies on Pensions, Aging and Retirement

NETSPAR ACADEMIC SERIES

Pension Scheme with Bounds on Returns

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MSc 08/2018-018

Master's Thesis

Pension Scheme with Bounds on Returns

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August 12, 2018

Abstract

In the Netherlands, reforms to the pension system are negotiated. One of the alternatives to the current system is establishing an explicit buffer fund through setting bounds on the returns of the individual pension schemes. This Master's Thesis proposes a solution to imposing a self-financing property on this buffer fund in order to establish the viability of this new system. The payoffs of call and put options are exploited to form this buffer fund and an analytical approximation is derived to determine the bounds, which need to be set on the returns in the pension scheme in order to ensure a self-financing buffer fund.

Keywords: Dutch pension reforms, Monte Carlo simulation, put-call symmetry, implicit function theorem, T-forward measure, shifted log-normal distribution. [1]

Contents

1 Introduction

In 2014, the pension debate in the Netherlands started after problems in the current pension system arose during and after the financial crisis. Alternatives are being sought for the defined benefit contract, which currently is the dominant pension contract in the Netherlands. The Dutch Social Economic Council (SER) has been leading negotiations aimed at finding an alternative system. These negotiations led to four different alternatives. In the years following 2014, these alternatives have been discussed further and were elaborated. For a long time, the alternative known as variant IV-C-R has been seen as the most likely to be implemented. In this alternative, an explicit buffer fund is introduced, which is supplemented through the returns made on the personal pension capital of the participants in the pension scheme. The buffer fund is a collective buffer fund and will be financed by all the participants in the pension scheme, but all participants will also be able to finance their pension in certain scenarios from this buffer fund. A lower threshold and an upper threshold are set. The returns above the upper threshold finance the buffer fund; in case the returns are below the lower threshold, the buffer fund will supplement the personal pension capital from the individual participant through the reduction of the funds in the buffer fund. This alternative combines a transition from defined benefit contracts to defined contribution contracts with the intergenerational risk-sharing component, which is the basis of the current system. However, to maintain a viable pension system, the buffer fund should not be contributed to by outside parties, for example the government. This means that the buffer fund should be self-financing. In this Master's Thesis it is investigated how the bounds on this buffer fund should be determined in order to incorporate the self-financing property.

In section 2, the problems of the current pension system are discussed and the different alternatives, which are currently being evaluated by the negotiators, will be further elaborated on, focusing on variant IV. Section 3 will contain the mechanism of variant IV-C-R under the Black-Scholes framework. The assumptions underlying this model will be discussed and a multi-year analysis of the variant will be conducted under these assumptions. The return pattern of the variant will be analyzed and a method will be developed to incorporate the self-financing property of the buffer fund. A short call option must be written to finance a long position in a put option with strike price equal to the minimum required value of the personal pension capital in the variant determined by the lower threshold. However, to ensure the self-financing property of the buffer fund the prices of these two options need to be equal, thus in section 4, the numerical results are shown for the strike price of this call option. Since an exact analytical solution cannot be found to the problem, the put-call symmetry is derived in section 5 to retrieve a global idea on the expression for the strike price of the call option. In section 6, this idea is exploited to derive an analytical approximation of the strike price of the call option for a general distribution of the underlying investment portfolio. Therefore, leaving the Black-Scholes framework and generalizing the results to the cases where both the risk-free interest rates and the volatilities of the underlying investment portfolio can be stochastic as well as deterministic. To determine the effectiveness of the approximation, the analytical approximation is compared to the numerically derived results. However, since this approximation holds for distributions other than the log-normal distribution a second distribution is tested, namely the shifted log-normal distribution. This distribution enables the analytical approximation to be tested when the volatility smile is present. In section 7 a final conclusion will be provided.

2 Dutch Pension Reform

The Dutch pension system is built on three pillars: the Dutch State Pension Act (AOW), employee and occupational pensions, and private individual pension products. The sustainability of the second pillar has been questioned over the years due to the low interest rates and changing demographics. Therefore, this pillar has been the primary focus of the reforms to ensure the stability of the Dutch pension system.

In the Netherlands, the defined benefit (DB) pensions are the dominant contracts. However, these schemes have been constantly changing during the past decades. (Deferred) annuities are used to define the pension entitlements during both the accumulation phase and the de-accumulation phase. Since employers have progressively withdrawn as the risk sponsors of the DB schemes, the participants have to bear more systematic investment risk of the investments in the pension scheme, but also longevity risks associated with the pension scheme. A result of this is that cost of living adjustments have been dependent on the funding ratio of the pension fund. The declining funding ratios have resulted in underfunding of the pension funds and even lead to cuts in the pension payments. (Bovenberg & Nijman, 2017)

Defined contribution (DC) contracts are far less common. In 2016, however, new legislation was introduced to make changes to the DC pension schemes. The obligation to provide a guaranteed fixed nominal lifetime income during the de-accumulation phase was no longer required. This led to the offering of variable annuities during the de-accumulation phase. However, the obligation to offer lifelong payments by pooling longevity risks was maintained in this new legislation. (Bovenberg & Nijman, 2017)

The Dutch pension system has doubled its assets since the financial crisis in 2008. However, the funding ratios of the pension funds have not recovered to the levels at which they were before the crisis, due to the low market rates at which they have to discount. The funding ratio is calculated using the following equation:

$$\text{Funding Ratio} = \frac{\text{Market value of assets}}{\text{Discounted liabilities}}$$

The low discount rates are due to the large stimulus packages from the European Central Bank. The assets have, therefore, risen slower than the capital requirements, which forced the pension funds below the minimum funding ratio of 105. This in turn led to an inability to fully index the pension schemes, both the pensions in accumulation and the payout pensions. To fulfill the requirement of a funding ratio of 105, some pension funds had to cut pension payments, as already mentioned above. These cuts and inability to fully index the pensions led to a general loss of confidence in the pension system. (ATP, 2017)

Another problem is a more fundamental one. Due to the intergenerational solidarity, the volatility of the balance sheet of the pension funds would need to be absorbed by the working participants, but because of the aging population, the working participants are unable to absorb all these shocks and the low market rates. Hence a stable, real income stream to all the retirees cannot be guaranteed.

The unclarity about who has ownership rights to the assets in the pension funds arising from this solidarity also leads to friction between the generations. While younger working participants would like to ensure a high funding ratio, which leads to certainty for their pension, retirees see no benefit in this. The current system is build on the one-size-fits-all principle, in which the individual participants have no freedom of choice. In addition it is no longer fit for the current labor market, which includes a significant number of self-employed persons, who are currently not included in the second pillar. (SER, 2016b)

Due to the above mentioned problems, a reform in the Dutch pension system is required. The pension reform is negotiated by the social partners and the pension funds. The term social partners is the collective name for the employers and employees, united in employer's organization and labor unions, respectively. The negotiations are supervised by the SER (the Dutch Social Economic Council) and the government will only become involved after an agreement has been reached by the negotiating parties. At this moment, the negotiations led to four different alternatives ("*varianten*" in Dutch), which have been examined by the SER. These four *varianten* are:

I. Administration agreement with declining accrual;

II. National pension scheme;

III. Personal pension capital with voluntary risk-sharing;

IV. Personal pension capital with collective risk-sharing.

Variant I and IV have two and three subcategories, respectively (SER, 2016b). Each of these alternatives will be elaborated in the next subsections.

2.1 Variant I - Administration Agreement with Declining Accrual

In variant I, the current administration agreement is taken as a base and adjusted to overcome some of the problems. This agreement is characterized by broad intergenerational risk sharing. As a consequence, the limitation on the freedom of choice within the pension contract will still exist. In this alternative the level premium system ("*doorsneesystematiek*" in Dutch) will be removed from the contract and replaced by a declining pension accrual. Hence age differentiation can be introduced on indexation allocation or on investment policies to make the fund more effective, resulting in an increase in pension accrual for younger participants. However, interest rate sensitivity will still play a crucial role in the indexation ambitions of the pension funds. Due to this interest rate sensitivity, variant I was divided into two different subcategories:

A *Administration agreement with declining accrual*: A nominal pension commitment is made with an assurance requirement of 97.5%.

B *Conditional benefit scheme*: No nominal assurance requirement will be stated. Shocks on the financial markets and increases in longevity will be spread out over several years.

Both these subcategories, however, do not provide a solution to the risk of discontinuity due to the shorter lifespan of companies and business sectors, and the increase of self-employed persons in the labor force, who are not included in these pension contracts. (SER, 2016a)

2.2 Variant II - National Pension Scheme

For variant II the current administration agreement is again taken as a starting point on which improvements will be implemented. The intergenerational risk-sharing shall remain intact and will become one of the key characteristics. Hence freedom of choice is still limited in this alternative. Reducing the increasing risk of discontinuity, however, is one of the priorities of this alternative. Through a national pension scheme, applicable to the entire working force, the participating rate will rise, since the self-employed persons will take part in this scheme as well, and the possibility to share risk with future generations will be extended. Hence the negative effects of the *doorsneesystematiek* in the labor market will be mitigated and the *doorsneesystematiek* will stay intact. (SER, 2015)

The benefit scheme will be based on a conditional administration agreement applicable to the entire working force. Even workers whom are currently not participating in a pension scheme or are currently participating in a different kind of pension scheme have to participate in the national pension scheme. The role of the employers is reduced in this alternative from being responsible to set up the pension scheme for their employees to only collecting the premiums. (SER, 2015)

2.3 Variant III - Personal Pension Capital with Voluntary Risk-Sharing

In the following two alternatives (variant III and variant IV), the pension accrual is no longer based on collectively accumulated capital in a collective fund, but based on personal pension capital. The participants accumulate their "own pension fund", which can be managed either collectively or individually. The transition in these alternatives is made from a defined benefit plan, which is currently the dominant pension scheme, to a defined contribution plan. The advantage of this transition is the high level of transparency and flexibility for the individual participants.

In variant III, the freedom of choice for the individuals regarding the pension accrual increases significantly compared to the other alternatives and the current system. Participants can choose who will manage their pension capital, how much premium they want to invest to accrue their pension capital, they can determine the risk profile of the investments made with the pension capital, after retirement the pension capital can be converted into an annuity or paid out as a lump sum, and the participants can determine whether or not they desire an insurance associated with their pension, like a surviving dependant's pension. (SER, 2015)

Participants will become responsible for their own pension accrual, and they can be supported on a voluntary basis by the social partners. Employers can offer in this alternative a standard pension scheme in which their employees can participate voluntarily. The government will play a facilitating role through fiscal measures to prevent improper use. From a tax perspective, it must be made possible to supplement the personal pension capital with private savings. (SER, 2015)

Due to the large freedom of choice in this alternative, the possibility to share risks between generations will be limited. The investment risks will not be shared among participants and neither will the longevity risks be shared. The longevity risk can be reduced by the individuals themselves through buying an annuity, in which an external party will take the longevity risk. Furthermore, individuals can enter into individual insurance policies that cover a surviving dependant's pension and an occupational disability pension. However, due to the absence of imposition, these insurances can become less effective because of selection criteria issued by insurance companies. (SER, 2015)

The investment policy is influenced by the transition to the personal pension capital fund, since there will be a lower incentive for young participants to invest large amounts in long term fixed income derivatives with low interest rates to reduce the investment risk. Due to the lack of intergenerational risk-sharing life-cycle funds will become more common and pension funds are stimulated to adopt a more risk-adverse investment strategy in the last years before retirement, since the increased volatilities in the fund can not be covered by later generations. Individuals enjoy the benefit to change pension provider, through this choice of pension provider individuals can choose their level of risk-aversion. Individuals also have the possibility to change their personal investment policy and withdraw their pension capital at an earlier moment in time than retirement. (SER, 2015)

2.4 Variant IV - Personal Pension Capital with Collective Risk-Sharing

Variant IV is seen as the most interesting to replace the current pension system. A contract with personal pension capital with collective risk-sharing will prevent conflicts arising between the younger participants' desire for higher-risk investments, possibly leading to higher returns, and the necessity older participants have for certain returns as their payout period approaches. The collective risk-sharing is used to supplement the personal pension capital to cover some of the risks associated with the loss of intergenerational risk-sharing. (SER, 2016b)

The pension scheme is executed collectively and can be divided into three different subcategories based on the level of risk-sharing:

A *Baseline option*: Variant IV-A consists of a personal pension capital in which collective risk-sharing is integrated only for biometric risks (risks related to death, occupational disability and/or longevity). The compulsory sharing of the longevity risk between participants can be organized through either a personal pension capital, in which participants will still share biometric risks even after retirement or through annuitisation. Shocks in the investment returns during the de-accumulation phase of the pension scheme will be spread out over a maximum of five years. During the accumulation phase the investment risk will be borne by the individual participants. The investments are made according to the life-cycle principle (portfolio adjustments are made during the course of the fund's time horizon, transitioning a position of higher risk to a position of lower risk as the retirement age approaches) and individual risk preferences.

B *Hybrid option*: During the accumulation phase, variant IV-B does not differ from variant IV-A, however during the de-accumulation phase investments risks can be shared collectively between all retirees. Occurring shocks are again spread out over several years. Starting the moment of retirement the individual loses his/her freedom of choice over the investment policy.

C *Option with extensive risk-sharing*: Variant IV-C incorporates risk-sharing between generations and hence is the most extensive in terms of risk-sharing. However, it should be transparent for all participants whether a generation is sharing benefits or losses. The life-cycle principle is again used for all participants to invest. This subcategory in the alternative consists of three different options on how the buffer fund is composed and the risk is shared between generations:

1) An implicit buffer fund can be used through the allocation of an average investment return to the participant's personal pension capital. Through the use of the average investment return shocks can be mitigated. The average investment return can either be determined by a uniform investment allocation or by a specific fund.
2) An explicit buffer fund can be used combined with the personal pension capital to incorporate the risk-sharing aspect. This buffer fund can be either financed directly through the premiums paid or through stabilizing the investment returns. Stabilizing the investment returns can be realized through the following two options:
   - either by financing the buffer fund if the returns on the personal pension capital exceed a certain threshold and allocating from the buffer fund to the personal pension capital if the investment returns are lower than a certain threshold;
   - or by setting a certain monetary target for the personal pension capital and if this target is exceeded, the exceedance flows into the buffer fund and if the personal pension capital is below a certain threshold the personal pension capital is supplemented using the buffer fund.
3) Using swaps between active participants and retirees in which the active participants bear the risks of increasing longevity and inflation from retirees to share the risks. In exchange for bearing these risks the active participants will receive compensation. This compensation is done through the internal swap. As opposed to the previous two options here the risk sharing is done between several groups of participants within the overall collective. The only risks which are shared are the ones which cannot be traded on financial markets and hence are unhedgeable. The internal swap functions as a hedge against these risks.

Of these three options, the second one, in which there is an explicit buffer fund financed through the stabilization of investment returns, named variant IV-C-R, is assumed to be most likely to be implemented. (SER, 2016a)

Contracts with personal pension capital accrual and collective risk-sharing will make it possible to implement long-term investment strategy, but with the advantage that the need to hedge interest rate risks in order to guarantee nominal security disappears. Financial shocks will no longer affect the contributions made to the pension scheme, but the amounts of capital will be adjusted.

The focus of this Master's Thesis will be on pension scheme with bounds on returns and hence on the variant IV-C-R.

3 Variant IV-C-R in Black-Scholes Framework

The pension scheme proposed in variant IV-C-R, is based on a buffer fund system in which explicit bounds are set where the returns on the investment portfolio of the individual participants can vary within. If the returns exceed the upper threshold the exceedance should be transferred to the collective buffer fund and if the returns are lower than the lower threshold the personal pension capital is supplemented through the buffer fund, such that the return is equal to the lower threshold. To investigate this alternative, the Black-Scholes economy is exploited.

3.1 Black-Scholes Economy

In the Black-Scholes economy several assumptions are made (Black & Scholes, 1973):

1) The short-term interest rate is known, risk-free and constant.
2) The portfolio pays no dividends or other distributions.
3) No arbitrage opportunities are available.
4) No transaction costs are charged for buying and selling securities.
5) Continuous trading is possible.
6) The stock price follows a random walk in continuous time with a constant variance. Thus the distribution of the stock prices is log-normal.

More specifically, it is assumed that the investment portfolio follows a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{3.1.1}$$

where W_t is a Wiener process, and drift rate, μ, and volatility, σ, are assumed to be known. Solving this stochastic differential equation for arbitrary S_0 leads to (Mikosch, 1998):

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right) \tag{3.1.2}$$

The investment return made in year $t+1$ is equal to $\frac{S_{t+1}}{S_t} - 1$. Without loss of generality this is rewritten to: $r_{t+1} = \frac{S_{t+1}}{S_t}$. Resulting in the following investment return in year $t+1$:

$$r_{t+1} = \frac{S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)(t+1) + \sigma W_{t+1}\right)}{S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right)} = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta_t + \sigma \Delta_{W_t}\right) \tag{3.1.3}$$

where $\Delta_t = 1$ and $\Delta_{W_t} = W_{t+1} - W_t$ in the case of year on year returns, which is the case in the current framework, since the input and output of the buffer fund is evaluated at the end of every year. The drift term, μ, is in the Black-Scholes framework equal to r, due to the assumption that no dividend is paid.

The Black-Scholes formula for a European call option on the underlying asset S paying no dividend is

$$c(S_t, t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)} \tag{3.1.4}$$

and the Black-Scholes formula for a European put option on the underlying asset S paying no dividend is

$$p(S_t, t) = N(-d_2)Ke^{-r(T-t)} - N(-d_1)S_t \tag{3.1.5}$$

with

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = d_1 - \sigma\sqrt{T-t} \tag{3.1.6}$$

where K is the strike price of the option, S_t the value of the underlying asset at time t, r the risk-free interest rate, σ the volatility of the underlying asset, $(T-t)$ the remaining time to maturity of the option, and $N(\cdot)$ the cumulative distribution function of the standard normal distribution. (Black & Scholes, 1973)

3.2 Variant IV-C-R in a Multi-year Setting

The return on the investment portfolio, in which the personal pension capital is invested, is analyzed, as described above, at the end of every year in variant IV-C-R. To evaluate the buffer fund a geometric Brownian motion is simulated over a period of 40 years. This lifetime is chosen since a pension capital is invested for a long period of time. In other words, it is assumed that the number of years until retirement is 40 years, which implies that the accumulation phase is 40 years. In this simulation the risk-free interest rate is assumed to be 3% and no dividend is paid on the investment portfolio. The volatility of the the investment portfolio, S_t, is assumed to be 6%. This volatility is taken since the average volatility of the investment portfolio of an investment fund is equal to 6% (Sundaresan, 2009). These assumptions lead to the following simulated geometric Brownian motion for the investment portfolio:

$$dS_t = 0.03 S_t dt + 0.06 S_t dW_t \tag{3.2.1}$$

In the simulation dt is taken to be 1/52, hence conducting a simulation based on weekly prices. However, the returns are only evaluated at the end of every year. Therefore, only the yearly prices are used to determine the value of the personal pension capital in variant IV-C-R. The yearly simulation is shown in figure 1 .

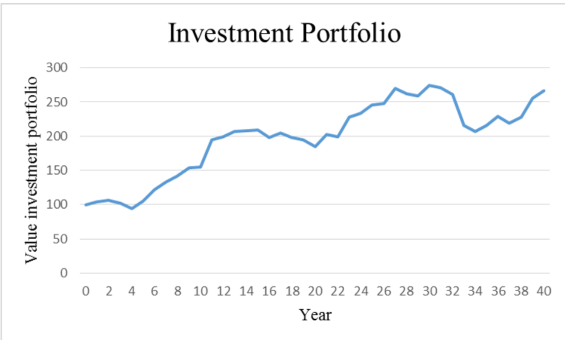

Figure 1: Simulated path for the value of the investment portfolio in which the personal pension capital is invested

The value of the investment portfolio is assumed to be €100 at time $t = 0$. Another assumption is that at the beginning of every year a premium of €1000 is paid, which is invested in investment portfolio, S_t. Hence at time $t = 0$, 10 units of this particular investment portfolio are bought. At time $t = 1$, $1000/S_1$ units of the investment portfolio are bought, and so on. The upper and lower threshold are determined using the values stated in *Analyse subvarianten IV-C* written by SER (2016a), which is 6% for the upper threshold and the lower threshold is stated to be -2%. When buying the units of the investment portfolio the capped return in variant IV-C-R does not play a role, since the actual value of the investment portfolio is not influenced by these returns. Only the amount which flows into the personal pension capital is capped.

Figure 2 shows the value of the personal pension capital with and without the restrictions under variant IV-C-R and the size of the buffer fund. Since the returns under IV-C-R can only range between -2% and 6%, the curve of the personal pension capital is smoother for the personal pension capital under IV-C-R. However, since in both scenarios the returns can become negative, the directions of the movements are the same. The size of the buffer fund is the cumulated difference between the value of the personal pension capital under the two scenarios.

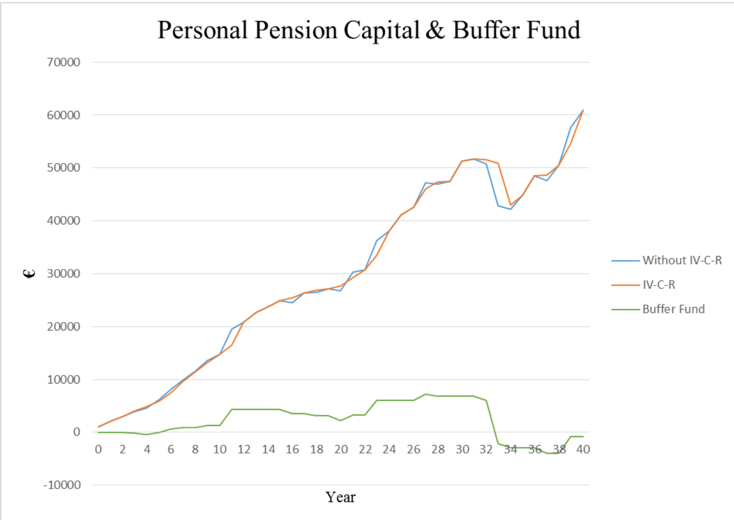

Figure 2: Value of the personal pension capital with and without the restrictions determined in variant IV-C-R and the size of the buffer fund

3.3 Returns

In variant IV-C-R the return on the investment portfolio of the individual participant is capped above and below by a certain threshold. However, this buffer fund should be self-financing, since a negative buffer fund could lead eventually to a non-viable system. Let $\alpha_h\%$ be the upper threshold, so returns above $\alpha_h\%$ are transferred to the buffer fund. $\alpha_l\%$ is the lower threshold, which implies that returns below $\alpha_l\%$ are supplemented from the buffer fund to the personal pension capital up to a return of $\alpha_l\%$. Hence the realized return in the personal pension scheme is not equal to the realized return of the investment portfolio. The following relationship holds between the realized return of the personal pension scheme, R_{t+1}, and the realized return of the investment portfolio, r_{t+1}:

$$R_{t+1} = \max\{\alpha_l, \min\{r_{t+1}, \alpha_h\}\} \tag{3.3.1}$$

A graphical representation is made to evaluate this relationship further.

In figure 3, a graph is shown of the relationship between the realized return of the investment portfolio, r_{t+1}, and the realized return of the personal pension scheme, R_{t+1}. This relationship can be replicated using call and put options. The sharp upturn in the line of the realized return of the personal pension scheme at α_l can be replicated by taking a long position in a one-year put option with strike price K equal to $S_t(1+\alpha_l)$, which results in a payoff of $\max\{S_t(1+\alpha_l) - S_{t+1}, 0\}$. The point at α_h where the the line of the realized return instantly stagnates can be replicated using a short position in a one-year call option with strike price K equal to $S_t(1+\alpha_h)$. However, since the buffer fund should be self-financing, the price of the long position in the put option should be equal to the price of the short position in the call option. If the upper threshold α_h is determined beforehand this cannot occur. Therefore the upper threshold should be determined by setting the prices of the put and call option equal.

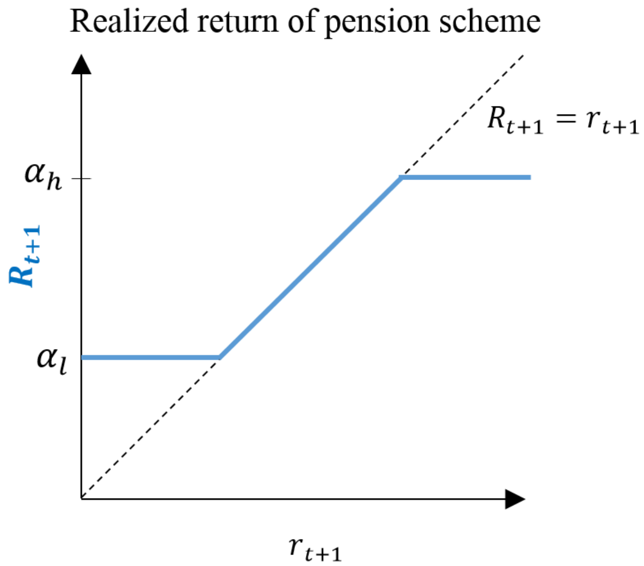

Figure 3: Graph of the realized returns in the personal pension scheme

Including the options in the portfolio of the participant, leads to the same outcome as capping the returns at the lower and upper threshold point. The same simulated investment portfolio as shown in figure 1 is shown in figure 4, in addition a portfolio is shown which includes the investment portfolio and the short position in the call option and the long position in the put option. The payoffs of the positions in the options are also shown. The lower and upper threshold are the same as before, $\alpha_l = -2\%$ and $\alpha_h = 6\%$. If the price received for selling the short position in the call option is equal to the price paid for the long position in the put option these options will only affect the value of the investment portfolio through their payoffs, which results in the exact payoff for required for variant IV-C-R. However, the thresholds determined by the SER do not result in an equal price for the call and put option.

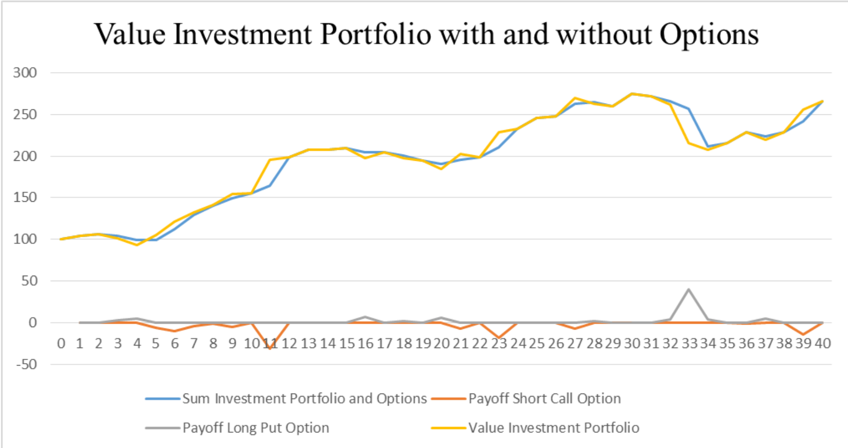

Figure 4: Graph of the simulated investment portfolio over a period of 40 years, with and without the requirements of variant IV-C-R implemented through options

By including the long position in the put option with $K = S_t(1 + \alpha_l)$ and the short position in the call option with $K = (1 + \alpha_h)S_t$, the probability density function of the overall investment portfolio changes. Figure 5 is compiled after running a Monte Carlo simulation using the Geometric Brownian motion described in equation 3.2.1 with $S_0 = 100$. 1000 simulations were run on this stochastic process with lower threshold, α_l, is -2% and upper threshold, α_h, is 6%. The resulted histogram shows the shape of the probability density function, which is the log-normal distribution. To compile figure 6 the same method is used, however, instead of simulating the value of the investment portfolio, the value of the investment portfolio including the payoffs of the long position in the put option and the short position in the call option is simulated. By including these options, the probability density function becomes denser around its peak, which results in a reduction of the tails of the probability density function. The options remove the tails of the distribution, since they ensure that the returns remain within certain bounds. Hence, the values within these bounds are achieved with a higher frequency.

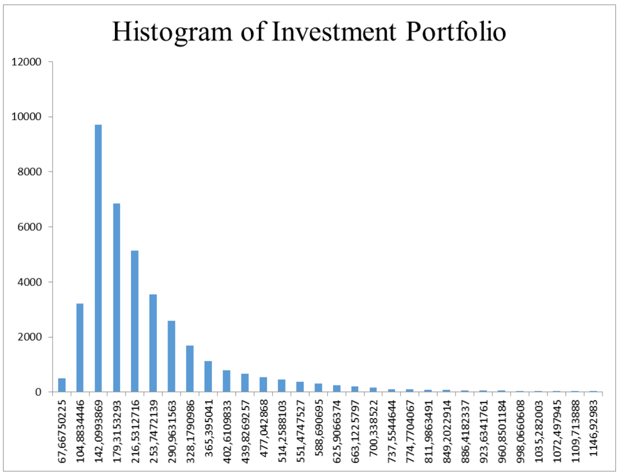

Figure 5: Histogram of the Monte Carlo simulation of the investment portfolio without the options

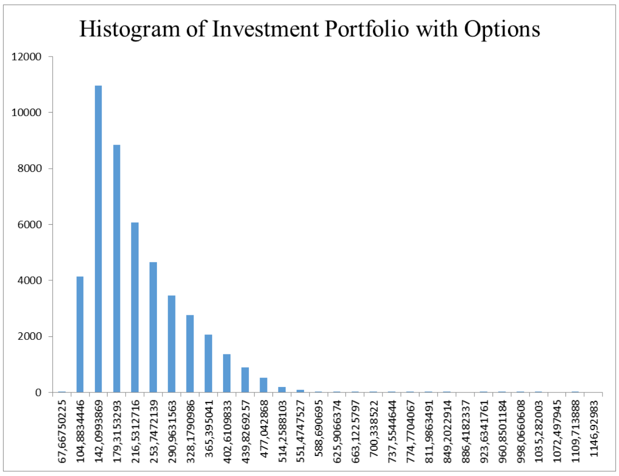

Figure 6: Histogram of the Monte Carlo simulation of the investment portfolio with the options

3.4 Determining Strike Price of Short Call Option

To replicate the lower bound at time t, as mentioned before, a long position in a put option with strike price equal to $K = S_t(1+\alpha_l)$ is taken. At the same time t a short position needs to be taken in a call option to incorporate the self-financing property of the buffer fund. Hence the strike price of the short call option can be determined by setting the price of the call option equal to the price of the long position in the put option with $K = S_t(1+\alpha_l)$. The value of the strike price of the call option should be equal to $(1+\alpha_h)S_t$, where α_h is the upper threshold determined in variant IV-C-R. However, since the strike price of the call option should be determined such that the prices are equal, this strike price is determined hereinafter as βS_t. Then α_h can be set equal to $\beta - 1$.

However, a problem arises, since the value of the investment portfolio at time t, S_t, is unknown beforehand. Hence the strike price is calculated relative to S_t. Therefore, the strike price for the call option is set to be βS_t. Then at the beginning of every year the options needed to replicate the buffer fund can be calculated by multiplying the strike price factor times the value of investment portfolio at that time. At the beginning of the year the risk-free interest rate r and the volatility σ of the investment portfolio are also known. Hence the relative price of the put option can be calculated, and then the relative price of the call option should be set equal to this.

The self-financing buffer fund will be replicated with one year options resulting in a time to maturity, $T - t$, of 1. One year options are exploited to construct the self-financing buffer fund in variant IV-C-R, since the returns on the investment portfolio in which the personal pension capital is invested are evaluated at the end of each year. At that point in time actions are taken to either finance the buffer fund or supplement the personal pension capital through the buffer fund. The strike price of the put option is equal to $S_t(1+\alpha_l)$ and the risk-free interest rate r and volatility σ are assumed to be known at time t. Then the relative price of the put option with $K = S_t(1+\alpha_l)$ equals

$$\frac{p(S_t,t)}{S_t} = N(-d_2)(1+\alpha_l)e^{-r} - N(-d_1) \tag{3.4.1}$$

with

$$d_1 = \frac{\ln\left(\frac{1}{1+\alpha_l}\right) + r + \frac{\sigma^2}{2}}{\sigma} \text{ and } d_2 = d_1 - \sigma \tag{3.4.2}$$

Hence, d_1 and d_2 are no longer dependent on the value of the investment portfolio. Therefore, the relative price of the put option can be calculated since there are no unknown factors influencing the price.

The strike price of the call option needs to be determined such that the relative price of the put option is equal to the relative price of the call option. Since the prices are relative to the value of the underlying investment portfolio, the strike price of the call option is also determined relative to this value. Therefore, even if the strike price K is unknown, it can still be written as a factor times the value of the underlying investment portfolio: $K = \beta S_t$, where β is unknown.

The call option is also an option maturing in one year, so $T - t = 1$. Here volatility σ and the risk-free interest rate r are assumed to be known, and the strike price is equal to $K = \beta S_t$. Then the relative price of the call option equals

$$\frac{c(S_t, t)}{S_t} = N(d_1) - N(d_2)\beta e^{-r} \tag{3.4.3}$$

with

$$d_1 = \frac{\ln\left(\frac{1}{\beta}\right) + r + \frac{\sigma^2}{2}}{\sigma} \text{ and } d_2 = d_1 - \sigma \tag{3.4.4}$$

The dependency on the value of the investment portfolio disappears from the equations of d_1 and d_2 and hence the only unknown parameter is β, which can be determined by setting:

$$\frac{p(S_t, t)}{S_t} = \frac{c(S_t, t)}{S_t} \tag{3.4.5}$$

To ensure the self-financing property of the buffer fund, the policymakers cannot determine both the lower threshold α_l and the upper threshold α_h, which is the idea behind variant IV-C-R discussed in section 2.4. This is due to the fact that α_h should be equal to $\beta - 1$ and β is determined by setting the relative price of the call option and put option equal which ensures the self-financing requirement. Hence β should be free from any requirements beforehand.

4 Numerical Results

In this section one long position in the put option should be financed using one short position in the call option. The lower threshold α_l is assumed to be known and determined to be $\alpha_l = -2\%$ and $\alpha_l = -7\%$ for the numerical results.

Since the pension scheme with the bounded returns is invested in a portfolio which should provide the pension for the individual participant, the investment portfolio will be invested in both risky and riskless assets. Therefore, the volatility of this portfolio is assumed to be equal to or below 13%. Hence the strike price factor on the call option, β, is calculated for an investment portfolio with volatilities ranging from 1% to 13%. For younger participants the volatility will probably be around 10%, since they can make riskier investments due the longer time till retirement. The participants closer to their retirement age will probably have an investment portfolio with a lower volatility. The volatility is assumed to be known at the beginning of each year, as well as the risk-free interest rate, which ranges from 0% to 6% in the calculations.

4.1 Graphical Representation

In figure 7, a graph is shown of the value of β, calculated using equation 3.4.5, for different combinations of volatilities and risk-free interest rates, for the case where the lower threshold has been set at -2%. From this figure it appears that the strike price factor β is increasing as the risk-free interest rate increases, but appears to only have a small dependence on volatility. The strike price factor, β, is 1.020 for risk-free interest rate, r, equal to 0% with a volatility, σ, of 1% and increases to 1.151 when the risk-free interest rate equals 6%, keeping volatility constant at 1%. Increasing the volatility to 13%, leads to a value of β of 1.022 when $r = 0\%$ and increases to 1.158 in the case of $r = 6\%$. When the risk-free interest rate is 6% the largest deviation between the strike price factors for different volatilities occurs. The difference between the strike price factor for this risk-free interest rate at volatility equal to 1% and volatility equal to 13% is 0.007. Hence, a small dependence of volatility on the strike price factor is seen. The strike price factor appears to be slightly increasing in volatility, keeping the risk-free interest rate constant.

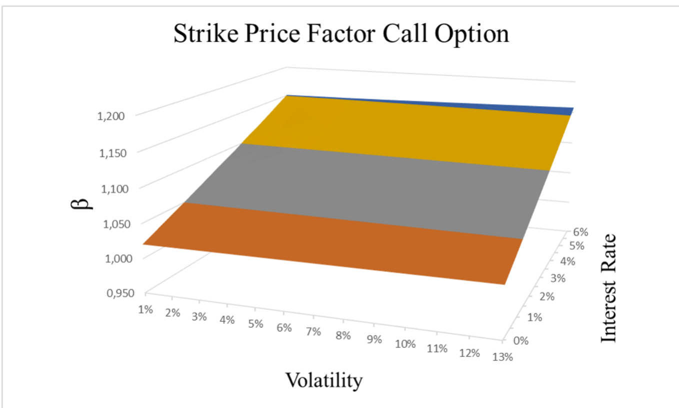

Figure 7: Graph of the value of strike price factor, β, for a value of α_l of -2% against the risk-free interest rate, r, and the volatility of the underlying investment portfolio, σ

As a robustness check for the small dependence of the strike price factor, β, on the volatility the same graph is made but with a different lower threshold on the returns of the pension scheme. In figure 8, the graph is shown with a lower threshold set at -7%. Here the same pattern in relation to the interest rate is seen, and the same small dependence of the strike price factor β on volatility appears. The strike price factor ranges from 1.075 when the interest rate equals 0% and volatility equals 1% and 1.211 when the interest rate is equal to 6% at the same volatility. The largest value for β occurs when the volatility is at 13% and the risk-free interest rate is equal to 6%: $\beta = 1.224$. The deviation between the lowest value for the strike price factor and the highest keeping interest rate constant, is again the highest when the interest rate is equal to 6%. Here the difference between having a volatility of 1% and a volatility of 13% is equal to 0.012. Hence, the effect of the volatility of the strike price factor of the call option seems to increase the further the lower threshold decreases away from 0%.

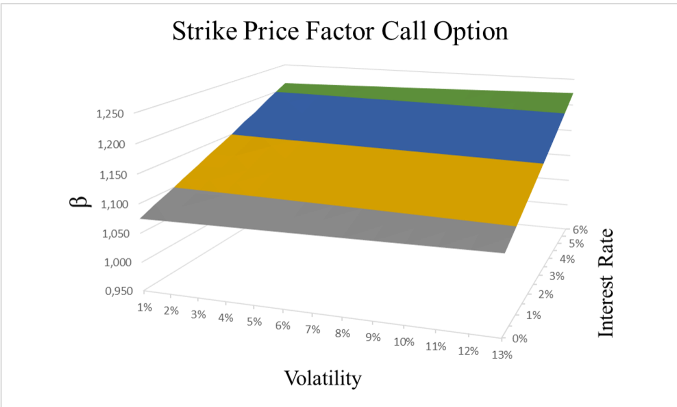

Figure 8: Graph of the value of strike price factor, β, for a value of α_l of -7% against the risk-free interest rate, r, and the volatility of the underlying investment portfolio, σ

When graphing different values of the lower threshold α_l, the relatively small dependency of volatility on the strike price factor appears to stay intact. Hence, the suspicion is confirmed that the volatility of the investment portfolio does not play a particularly significant role in determining the strike price factor, β, compared to the influence of the risk-free interest rate. Hence, a simple rule of thumb should be possible to be derived where the dependence on the volatility does not appear. However, for an exact analytical result the volatility will become apparent in the expression for the strike price factor, β. The small dependence of the volatility is due to the symmetry between put and call options in the Black-Scholes model, which relies on the assumption of log-normally distributed stock prices.

4.2 Values of α_h

As mentioned before, the value of the upper threshold α_h in variant IV-C-R should be set equal to $\beta - 1$. Since β is derived numerically, the value of α_h can be determined. In this section, the values of the upper threshold, α_h, are given for several combinations of risk-free interest rate, r, and volatility of the underlying investment portfolio, σ, for lower threshold, α_l, equal to -2% and -7%.

In table 1, the numerically derived values of α_h are shown. The risk-free interest rate, r, ranges from 0% to 6%, with steps of 0.5%. The volatility of the underlying investment portfolio in which the personal pension capital is invested, σ, ranges from 3% to 13%. This range for the volatility is selected, since the investment portfolio will always include some risky assets and therefore a volatility below 3% will be unlikely. Therefore the maximum value of the volatility is 13%.

Since α_h is equal to $\beta - 1$, the interest rate and volatility have the same effect on α_h as on β, described in section 4.1, which is also visible in table 1. If the lower threshold is set at -2%, then if the risk-free interest rate is 0%, the upper threshold has to be slightly above 2%. The value of α_h increases on average by 1.11% when the risk-free interest rate increases by 0.5%. The effect the risk-free interest rate has on the upper threshold seems to be increasing when the volatility increases. This increasing effect, however, is small. When the volatility increases by 1% the upper threshold, α_h, increases on average by 0.04%.

$r, \downarrow \sigma \rightarrow$	**3%**	**4%**	**5%**	**6%**	**7%**	**8%**	**9%**	**10%**	**11%**	**12%**	**13%**
0%	2,08%	2,09%	2,11%	2,13%	2,14%	2,16%	2,18%	2,20%	2,21%	2,23%	2,25%
0,50%	3,11%	3,13%	3,15%	3,17%	3,19%	3,21%	3,24%	3,26%	3,28%	3,30%	3,32%
1%	4,15%	4,18%	4,20%	4,22%	4,25%	4,28%	4,30%	4,33%	4,35%	4,38%	4,41%
1,50%	5,20%	5,23%	5,26%	5,29%	5,32%	5,35%	5,38%	5,41%	5,44%	5,47%	5,50%
2%	6,26%	6,30%	6,33%	6,36%	6,39%	6,43%	6,46%	6,50%	6,53%	6,57%	6,61%
2,50%	7,34%	7,37%	7,41%	7,44%	7,48%	7,52%	7,56%	7,60%	7,64%	7,68%	7,72%
3%	8,42%	8,46%	8,49%	8,54%	8,58%	8,62%	8,66%	8,71%	8,75%	8,80%	8,85%
3,50%	9,51%	9,55%	9,59%	9,64%	9,68%	9,73%	9,78%	9,83%	9,88%	9,93%	9,98%
4%	10,62%	10,66%	10,70%	10,75%	10,80%	10,85%	10,91%	10,96%	11,01%	11,07%	11,12%
4,50%	11,72%	11,77%	11,82%	11,88%	11,93%	11,98%	12,04%	12,10%	12,16%	12,22%	12,28%
5%	12,86%	12,90%	12,95%	13,01%	13,07%	13,13%	13,19%	13,25%	13,31%	13,38%	13,44%
5,50%	14,00%	14,04%	14,10%	14,15%	14,22%	14,28%	14,35%	14,41%	14,48%	14,55%	14,62%
6%	15,16%	15,19%	15,25%	15,31%	15,38%	15,44%	15,51%	15,59%	15,66%	15,73%	15,81%

Table 1: Value of the upper threshold, α_h, for a value of α_l of -2% for different values of the risk-free interest rate, r, and the volatility of the underlying investment portfolio, σ

If the lower threshold, α_l, decreases to -7% the values of the upper threshold α_h are shown in table 2. The value of α_h is equal to 7.59% if the risk-free interest rate is equal to 0% and volatility is equal to 3%. If the risk-free interest rate increases by 0.5%, the value of α_h increases on average by 1.16%, which is a slightly higher increase than for $\alpha_l = -2\%$. Hence the effect of the risk-free interest rate is influenced by the lower threshold, α_l. Similar to the previous case, in which $\alpha_l = -2\%$, the influence of the risk-free interest rate seems to be increasing when volatility increases. An increases of 1% in volatility leads on average to an increase of 0.08% in the value of α_h.

$r, \downarrow \sigma \rightarrow$	**3%**	**4%**	**5%**	**6%**	**7%**	**8%**	**9%**	**10%**	**11%**	**12%**	**13%**
0%	7,59%	7,65%	7,70%	7,76%	7,81%	7,87%	7,93%	7,99%	8,06%	8,12%	8,18%
0,50%	8,71%	8,74%	8,79%	8,85%	8,91%	8,97%	9,04%	9,10%	9,17%	9,24%	9,30%
1%	9,77%	9,83%	9,89%	9,95%	10,01%	10,08%	10,15%	10,22%	10,29%	10,36%	10,44%
1,50%	10,93%	10,94%	11,00%	11,06%	11,13%	11,20%	11,27%	11,35%	11,42%	11,50%	11,58%
2%	12,00%	12,06%	12,12%	12,18%	12,26%	12,33%	12,41%	12,49%	12,57%	12,65%	12,73%
2,50%	13,15%	13,21%	13,26%	13,32%	13,39%	13,47%	13,55%	13,64%	13,72%	13,81%	13,89%
3%	14,28%	14,31%	14,39%	14,46%	14,54%	14,62%	14,71%	14,80%	14,89%	14,98%	15,07%
3,50%	15,39%	15,47%	15,55%	15,62%	15,70%	15,79%	15,88%	15,97%	16,06%	16,16%	16,25%
4%	16,59%	16,66%	16,72%	16,79%	16,87%	16,96%	17,06%	17,15%	17,25%	17,35%	17,45%
4,50%	17,70%	17,81%	17,90%	17,97%	18,06%	18,15%	18,25%	18,34%	18,45%	18,55%	18,66%
5%	18,92%	18,97%	19,09%	19,16%	19,25%	19,35%	19,45%	19,55%	19,66%	19,77%	19,88%
5,50%	20,10%	20,20%	20,30%	20,36%	20,45%	20,56%	20,66%	20,77%	20,88%	20,99%	21,11%
6%	21,29%	21,39%	21,50%	21,57%	21,67%	21,78%	21,88%	22,00%	22,11%	22,23%	22,35%

Table 2: Value of the upper threshold, α_h, for a value of α_l of -7% for different values of the risk-free interest rate, r, and the volatility of the underlying investment portfolio, σ

Since the increase due to the volatility is small, it is investigated in the next section whether or not it is possible to derive an analytical expression for the value of β, and in turn α_h, not dependent on the volatility. Conducting this analysis will allow for a general direction for the value of β.

5 Put-Call Symmetry

In the above sections, it was assumed that one long position in the put option with strike price equal to $S_t(1+\alpha_l)$ should be financed with one short position in the call option with strike price equal to βS_t. If this assumption is loosened, a general idea about the value of β can be acquired. In this section it is assumed that it is possible to finance one long position in the put option with a quantity unequal to one of short positions in the call option with strike price βS_t: $n \cdot c(S_t, \beta S_t) = p(S_t, (1+\alpha_l)S_t)$. This β is then unequal to the strike price factor, β calculated numerically in section 4.

By making this adjustment, the problem can be solved analytically. However, in doing so, because of the factor n, figure 3 is no longer replicated, since the slope of the straight line at α_h is no longer equal to 0, but to $1-n$. This is due to the fact that n call options are shorted instead of one. Hence this result does not solve the bounds on the returns exactly, however, the expression derived for β can be used as a rule of thumb. Two cases are distinguished to solve this problem analytically: (1) $r=0$, and (2) $r \geq 0$. The first case of the risk-free rate, r, equal to zero is used to acquire the general idea behind the analytical solution. The second case is the general case, where $r \geq 0$, in which the insights obtained in the first case are extended to the general situation.

5.1 Specific Case: $r = 0$

The case of risk-free interest rate, r, equal to zero is analyzed to acquire the general idea, as mentioned before. In the Black-Scholes model defined in section 3.1, the portfolio investment S_t is assumed to follow a geometric Brownian motion. When $r=0$ the process S_t becomes a martingale under the risk-neutral measure, because the drift parameter $\mu = r = 0$ since it is assumed that no dividend is paid. Hence, let $S = (S_t)_{t\geq 0}$ be a positive local martingale for a filtration $\mathscr{F}_t$. Then if S is geometrically symmetric, the following holds

$$\mathbb{E}\left[f\left(\frac{S_T}{S_t}\right)\Big|\mathscr{F}_t\right] = \mathbb{E}\left[\frac{S_T}{S_t} f\left(\frac{S_t}{S_T}\right)\Big|\mathscr{F}_t\right] \tag{5.1.1}$$

almost surely, for all $0 \leq t \leq T$ and bounded measurable f (Tehranchi, 2009). Rewriting equation 5.1.1, leads to:

$$\mathbb{E}\left[f\left(S_T\right)\Big|\mathscr{F}_t\right] = \mathbb{E}\left[\frac{S_T}{S_t} f\left(\frac{S_t^2}{S_T}\right)\Big|\mathscr{F}_t\right] \tag{5.1.2}$$

Since S_t is geometrically symmetric, equation 5.1.2 holds for any bounded measurable function f. The max-function is a bounded measurable function, therefore the concept can be used for the payoff of a European call option, which is equal to $(S_t - K)^+$. Under the risk-neutral probability measure, $\mathbb{Q}$, the price of a European call option is equal to:

$$\mathbb{E}_{\mathbb{Q}}\left[(S_t-K)^+|\mathscr{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{S_T}{S_t}\left(\frac{S_t^2}{S_T}-K\right)^+\Big|\mathscr{F}_t\right] = \frac{K}{S_t}\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{S_t^2}{K}-S_T\right)^+\Big|\mathscr{F}_t\right] \tag{5.1.3}$$

Here $\mathbb{E}_Q\left[(S_t-K)^+|\mathscr{F}_t\right]$ is the expected payoff of a call option on S with strike price K, which should be equal to the price of the call option, assuming that no-arbitrage opportunities exist. The price of the call option is denoted by $c(S_t, K)$ and is equal to the Black-Scholes price for a call option. $\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{S_t^2}{K} - S_T\right)^+ \middle| \mathscr{F}_t\right]$ is the expected payoff of a put option with strike price equal to $\frac{S_t^2}{K}$, where K is the strike price of the call option. This should be equal to the price of the put option, $p\left(S_t, \frac{S_t^2}{K}\right)$, assuming again that there exist no arbitrage opportunities. Changing these expected values by the prices leads to

$$c(S_t, K) = \frac{K}{S_t} p\left(S_t, \frac{S_t^2}{K}\right) \tag{5.1.4}$$

Equation 5.1.4 is known as the *put-call symmetry formula* (Carr & Lee, 2009).

However, the strike price of the put option is known, which is $S_t(1 + \alpha_l)$. Therefore, rewrite the put-call symmetry formula to

$$p\left(S_t, \frac{S_t^2}{K}\right) = \frac{S_t}{K} c(S_t, K) \tag{5.1.5}$$

The strike price of the call option is equal to βS_t. Hence in equation 5.1.5 $K = \beta S_t$. Since $\frac{S_t^2}{K} = S_t(1 + \alpha_l)$, the value of β can be determined in terms of $(1 + \alpha_l)$.

$$\frac{S_t^2}{K} = S_t(1 + \alpha_l) \Leftrightarrow \frac{S_t^2}{\beta S_t} = S_t(1 + \alpha_l) \Leftrightarrow \beta = \frac{1}{1 + \alpha_l} \tag{5.1.6}$$

Therefore the call option should have a strike price of $\frac{1}{1+\alpha_l} S_t$. Next the amount of call options with $K = \frac{1}{1+\alpha_l} S_t$ that need to be shorted in order to be able to buy the long put option with strike price $S_t(1 + \alpha_l)$ needs to be determined. According to equation 5.1.5 this should be equal to

$$\frac{S_t}{K} = \frac{S_t}{\frac{1}{1+\alpha_l} S_t} = 1 + \alpha_l$$

To ensure a self-financing buffer fund when $r = 0$, one long position in a put option with strike price equal to $S_t(1 + \alpha_l)$ should be financed by shorting $1 + \alpha_l$ call options with strike price equal to $\frac{1}{1+\alpha_l} S_t$. From the strike price of the call option it becomes apparent that the strike price factor, β, does not depend on volatility, since it only depends on the known factor α_l. In table 3 the values for α_l, α_h, which is $(\beta - 1)$, and the corresponding amount of the call option which needs to be shorted are shown for values of α_l between -10% and -1%.

$\boldsymbol{\alpha_l}$	-10%	-9%	-8%	-7%	-6%	-5%	-4%	-3%	-2%	-1%
$\boldsymbol{\alpha_h}$	11.11%	9.89%	8.70%	7.53%	6.38%	5.26%	4.17%	3.09%	2.04%	1.01%
Amount	0.90	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99

Table 3: Lower and upper threshold with corresponding amount of call option which needs to be shorted, when $r = 0$

5.2 General Case: $r \geq 0$

Now, the general case where the risk-free interest rate is larger or equal to 0 is analyzed. Using the general result on pricing in a complete market without arbitrage, the price of a call option with strike price K is equal to the expected value of the payoff under the risk-neutral probability measure $\mathbb{Q}$ (Mikosch, 1998):

$$c(S_t, K) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K)^+ | \mathscr{F}_t\right] \tag{5.2.1}$$

The price of a call option depends on the volatility of the underlying asset, σ, and the risk-free interest rate, r.

The same principles are applied as in the case of $r = 0$. However to be able to do this, it first needs to be proven that a risk-neutral probability measure $\mathbb{Q}$ exists such that $e^{-rt}S_t$ is a martingale under this measure $\mathbb{Q}$. For this purpose the *Girsanov theorem* is introduced.

Theorem 1 (Girsanov Theorem, (Benth, 2003))**.** *Let $\{W(t,w), t \in [0,T], w \in \Omega\}$ be a Brownian motion defined on the probability space $(\Omega, \mathscr{F}_t, \mathbb{P})$. Let W^* be a process defined by:*

$$W_t^* = W_t + \theta t, (\theta \in \mathbb{R})$$

Then the process W^* is a Brownian motion under the probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ whose density is defined by:

$$Y_T = exp\left(-\theta W_T - \frac{\theta^2}{2}T\right)$$

Rewrite dS_t defined in section 3.1 to

$$dS_t = rS_t dt + \sigma S_t dW_t^*$$

with $W_t^* = W_t + \frac{\mu - r}{\sigma}t$, which is a Brownian motion with drift under the probability measure $\mathbb{P}$. By applying the Girsanov theorem with $\theta = \frac{\mu-r}{\sigma}$, there exists a measure $\mathbb{Q} \sim \mathbb{P}$ such that W_t^* is a standard Brownian motion under the risk-neutral probability measure $\mathbb{Q}$. Under this probability measure, $X_t = e^{-rt}S_t$ satisfies, using Ito's lemma:

$$dX_t = -rX_t dt + e^{-rt}dS_t = \sigma X_t dW_t^*$$

Hence $X_t = X_0 exp\left(-\frac{\sigma^2}{2}t + \sigma W_t^*\right)$, which implies that $X_t = e^{-rt}S_t$ is a martingale under the risk-neutral probability measure $\mathbb{Q}$, since X_t is a geometric Brownian motion without drift under $\mathbb{Q}$.

Then, due to the fact that $e^{-rt}S_t$ is a martingale and geometrically symmetric, equation 5.1.2 is used to derive the following result:

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K)^+ | \mathscr{F}_t\right] &= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\frac{S_T}{S_t}\left(e^{-r(T-t)}\left(\frac{S_t^2}{S_T} - K\right)^+\right) | \mathscr{F}_t\right] \\ &= \frac{K}{F_t}\mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\left(\frac{F_t^2}{K} - S_T\right)^+ | \mathscr{F}_t\right]\end{aligned} \tag{5.2.2}$$

where $F_t = S_t e^{r(T-t)}$. Using the fact that $c(S_t, K) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K)^+ | \mathscr{F}_t\right]$ and that $\mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\left(\frac{F_t^2}{K} - S_T\right)^+ | \mathscr{F}_t\right]$ is the expected payoff for a put option under the risk-neutral probability measure $\mathbb{Q}$ and therefore equal to the price of the put option with strike price equal to $\frac{F_t^2}{K}$, denoted by $p(S_t, \frac{F_t^2}{K})$, where K is the strike price of the call option, equation 5.2.2 can be rewritten as

$$\begin{aligned} c(S_t, K) &= \frac{K}{F_t} p\left(S_t, \frac{F_t^2}{K}\right) \\ \Leftrightarrow p\left(S_t, \frac{F_t^2}{K}\right) &= \frac{F_t}{K} c(S_t, K) \end{aligned} \tag{5.2.3}$$

The strike price of the put option is known, and equal to $S_t(1 + \alpha_l)$. The strike price of the call option is equal to βS_t. Hence K in equation 5.2.3 is equal to βS_t. Therefore, the value of β can be determined in terms of $(1 + \alpha_l)$ and r:

$$\frac{F_t^2}{K} = S_t(1 + \alpha_l) \Leftrightarrow \frac{(S_t e^{r(T-t)})^2}{\beta S_t} = S_t(1 + \alpha_l) \Leftrightarrow \beta = \frac{e^{2r(T-t)}}{1 + \alpha_l} \tag{5.2.4}$$

Then the strike price of the call option should be equal to $\frac{e^{2r(T-t)}}{1+\alpha_l} S_t$. The amount that needs to be shorted of this call option has to be equal to $\frac{F_t}{K}$:

$$\frac{F_t}{K} = \frac{S_t e^{r(T-t)}}{\frac{e^{2r(T-t)}}{1+\alpha_l} S_t} = (1 + \alpha_l) e^{-r(T-t)}$$

Therefore, to ensure a self-financing buffer fund in the general case, in which $r \geq 0$, one long position in a put option with a strike price of $K = (1 + \alpha_l)S_t$ should be financed through $(1 + \alpha_l)e^{-r(T-t)}$ short positions in a call option with a strike price equal to $\frac{e^{2r(T-t)}}{1+\alpha_l} S_t$. Since the strike price factor of the call option is no longer dependent on the value of the underlying investment portfolio, the volatility of the investment portfolio does not influence the strike price factor. Since α_l is determined beforehand and r is the one-year risk-free interest rate, because the options have a time to maturity of one year ($T - t = 1$), the strike price factor, β, can be easily calculated at the beginning of each year. Since $\alpha_h = \beta - 1$, the upper threshold can be determined as well at the beginning of the year and the buffer fund's self-financing property can be imposed.

In table 4, the values of the upper threshold α_h are given corresponding to the associated combination of the risk-free interest rate, r, and the lower threshold, α_l. If the risk-free interest rate is kept constant, the upper threshold is decreasing if the lower threshold is increasing. Hence the higher the lower threshold becomes, the lower the upper threshold has to be. If the lower threshold is kept constant, the upper threshold is increasing in the risk-free interest rate. The lowest value for the upper threshold is 1.01% if the risk-free interest rate is 0% and the lower threshold is determined at -1%. When the risk-free interest rate is equal to 6% and the lower threshold is -10%, the highest value for the upper threshold is achieved at 25.28%.

$r \downarrow$, $\alpha_l \rightarrow$	**-10%**	**-9%**	**-8%**	**-7%**	**-6%**	**-5%**	**-4%**	**-3%**	**-2%**	**-1%**
0%	11.11%	9.89%	8.70%	7.53%	6.38%	5.26%	4.17%	3.09%	2.04%	1.01%
0.50%	12.23%	10.99%	9.79%	8.61%	7.45%	6.32%	5.21%	4.13%	3.07%	2.03%
1%	13.36%	12.11%	10.89%	9.70%	8.53%	7.39%	6.27%	5.18%	4.10%	3.05%
1.50%	14.49%	13.24%	12.01%	10.80%	9.62%	8.47%	7.34%	6.23%	5.15%	4.09%
2%	15.65%	14.37%	13.13%	11.92%	10.72%	9.56%	8.42%	7.30%	6.21%	5.13%
2.50%	16.81%	15.52%	14.27%	13.04%	11.84%	10.66%	9.51%	8.38%	7.27%	6.19%
3%	17.98%	16.69%	15.42%	14.18%	12.96%	11.77%	10.61%	9.47%	8.35%	7.26%
3.50%	19.17%	17.86%	16.58%	15.32%	14.10%	12.90%	11.72%	10.57%	9.44%	8.33%
4%	20.37%	19.04%	17.75%	16.48%	15.24%	14.03%	12.84%	11.68%	10.54%	9.42%
4.50%	21.57%	20.24%	18.93%	17.65%	16.40%	15.18%	13.98%	12.80%	11.65%	10.52%
5%	22.80%	21.45%	20.13%	18.84%	17.57%	16.33%	15.12%	13.94%	12.77%	11.63%
5.50%	24.03%	22.67%	21.33%	20.03%	18.75%	17.50%	16.28%	15.08%	13.91%	12.76%
6%	25.28%	23.90%	22.55%	21.24%	19.95%	18.68%	17.45%	16.24%	15.05%	13.89%

Table 4: Upper threshold, α_h, corresponding to the associated combination of risk-free interest rate, r, and lower threshold, α_l

.

In table 5, the amounts of call options with strike price equal to $(1 + \alpha_h)S_t$, which are needed to make the buffer fund self-financing, corresponding to the associated combination of risk free interest rate, r, and lower threshold, α_l, are given. Keeping the risk-free interest rate constant, then it can be seen that the amount needed is increasing if the lower threshold is increasing. If the lower threshold is kept constant, the amount needed is decreasing if the risk-free interest rate is increasing. The lowest value of 0.848 occurs if the risk-free interest rate is equal to 6% and the lower threshold equals -10%. When the risk-free interest rate is 0% and the lower threshold is determined to be -1%, the amount needed is the highest, namely 0.990. It can be seen that the amount needed is never exceeds the value of one, this is due to the fact that the price of the call option with strike price equal to $(1+\alpha_h)S_t$ is higher than the price of the put option with strike price equal to $(1+\alpha_l)S_t$.

Hence, from table 4 and 5 the necessary information can be exerted to ensure the self-financing property of the buffer fund under the assumption that the put option is not financed one-to-one with a short call option. Since the upper threshold, α_h, and the lower threshold, α_l, cannot be set independently from each other, only one of the two can be determined, either α_h or α_l can be determined beforehand. The most logical option of the two is setting the lower threshold, α_l. This is due to the fact that the most critical concern of investors is the losses endured on their investment portfolio. After α_l is set, the value of α_h can be determined using table 4 or by the following relation:

$$\alpha_h = \frac{e^{2r(T-t)}}{1 + \alpha_l} - 1 \tag{5.2.5}$$

The amount of $(1 + \alpha_l)e^{-r(T-t)}$ call options need to be shorted of the call option with strike price $(1 + \alpha_h)S_t$ in order to raise the price of the put option with strike price $(1 + \alpha_l)S_t$ exactly.

$r \downarrow$, $\alpha_l \rightarrow$	**-10%**	**-9%**	**-8%**	**-7%**	**-6%**	**-5%**	**-4%**	**-3%**	**-2%**	**-1%**
0%	0.900	0.910	0.920	0.930	0.940	0.950	0.960	0.970	0.980	0.990
0.50%	0.896	0.905	0.915	0.925	0.935	0.945	0.955	0.965	0.975	0.985
1%	0.891	0.901	0.911	0.921	0.931	0.941	0.950	0.960	0.970	0.980
1.50%	0.887	0.896	0.906	0.916	0.926	0.936	0.946	0.956	0.965	0.975
2%	0.882	0.892	0.902	0.912	0.921	0.931	0.941	0.951	0.961	0.970
2.50%	0.878	0.888	0.897	0.907	0.917	0.927	0.936	0.946	0.956	0.966
3%	0.873	0.883	0.893	0.903	0.912	0.922	0.932	0.941	0.951	0.961
3.50%	0.869	0.879	0.888	0.898	0.908	0.917	0.927	0.937	0.946	0.956
4%	0.865	0.874	0.884	0.894	0.903	0.913	0.922	0.932	0.942	0.951
4.50%	0.860	0.870	0.880	0.889	0.899	0.908	0.918	0.927	0.937	0.946
5%	0.856	0.866	0.875	0.885	0.894	0.904	0.913	0.923	0.932	0.942
5.50%	0.852	0.861	0.871	0.880	0.890	0.899	0.909	0.918	0.928	0.937
6%	0.848	0.857	0.866	0.876	0.885	0.895	0.904	0.914	0.923	0.932

Table 5: Amount of shorted call option with strike price equal to $(1 + \alpha_h)S_t$ needed to make the buffer fund self-financing corresponding to the associated combination of risk-free rate, r, and lower threshold, α_l

.

5.3 Conclusion

As mentioned at the beginning of this section, the above results cannot be used to replicate the self-financing buffer fund required in variance IV-C-R. Due to the compensation needed in the number of short positions in the call option, the slope of the payoff will be slightly increasing with a factor $1 - (1 + \alpha_l)e^{-r(T-t)}$, instead of being equal to 0. The result derived for the strike price for the call option can be used as a simple rule of thumb, since it approximates the value of β which would be needed to replicate the buffer fund in variant IV-C-R. However, to replicate the pay-off of the investment portfolio with a buffer fund one such call option should be shorted. However by doing so the position becomes slightly over-hedged. The pension fund is selling a call option which has a strike price which is too low to hedge their position. Hence the upper threshold would then be lower, than strictly necessary to finance the long position in the put option.

In the next section, an analytical approximation for β will be derived, for which one long position in a put option with strike price $(1 + \alpha_l)S_t$ is financed using one short position in a call option with strike price βS_t. From the result using the put-call symmetry it can be concluded that β can be expressed in terms of α_l, which will be exploited in the next section.

6 Analytical Approximation for a General Distribution

In section 5, a strong assumption had to be made in order to derive the put-call symmetry, namely the geometric symmetry of the distribution of the value of the investment portfolio, which is true in the Black-Scholes framework. Due to the symmetry the dependence on the volatility can be eliminated through slightly overhedging the upper threshold. However, an analytical approximation can be derived also outside the Black-Scholes framework. The only assumption necessary in this section is that there are no arbitrage opportunities in the market. Obviously it is also assumed that the call and put options are sold in the market. Therefore, this derivation can be used for any distribution underlying the value of the investment portfolio, S_t, and not only the log-normal distribution assumed in the Black-Scholes framework.

In this section the implicit function theorem is used to derive an analytical result for the expression of the strike price factor β. The returns in variant IV-C-R are replicated in this section. Hence, one long position in a one-year put option with strike price equal to $(1+\alpha_l)S_t$ is replicated through the use of one short position in a one-year call option with strike price equal to βS_t. This β is equal to the β in section 4.

6.1 Price of a European Call and Put Option

As already mentioned in section 5, the price of a European call option can be written as the expected value of the payoff of the option. The underlying is the value of an investment portfolio, S_t. However, the forward value of the underlying investment portfolio can be exploited to derive an expression for the price of the call option. The price of the European call option is, therefore, denoted by $c(F_t, K)$ and is equal to:

$$c(F_t, K) = e^{-r(T-t)}\mathbb{E}\left[(S_T - K)^+ \middle| \mathscr{F}_t\right] \tag{6.1.1}$$

The price of a European put option is equal to:

$$p(F_t, K) = e^{-r(T-t)}\mathbb{E}\left[(K - S_T)^+ \middle| \mathscr{F}_t\right] \tag{6.1.2}$$

The expectation will be taken under a carefully chosen probability measure.

Defining the price of the European call option and put option as in equation 6.1.1 and 6.1.2, respectively, opens up the possibility to analyze not only the strike prices under a deterministic interest rate and volatility but also under stochastic risk-free interest rates and stochastic volatilities. Stochastic volatilities are already included when the expectation is taken under the risk-neutral measure $\mathbb{Q}$, however, the risk-free interest rate is assumed to be deterministic in this case. Changing the probability measure to the T-forward measure, $\mathbb{Q}^T$ also includes the case of stochastic risk-free interest rates. This change of probability measure can be exploited since the only assumption made is that no arbitrage opportunities exist in the market. Therefore, when the expectation under the T-forward measure is taken all cases possible in the market are included in the analytical approximation of the strike price factor, β, and a general analytical approximation is derived.

The T-forward measure is a probability measure which is absolutely continuous with respect to the risk-neutral measure $\mathbb{Q}$. However, instead of taking the money market account as numéraire, a zero-coupon bond with maturity at time T is taken as the numéraire. Let $M(T)$ be the money market account numéraire, defined by

$$M(T) = \exp\left(\int_0^T r(s)ds\right) \tag{6.1.3}$$

The price process of the zero-coupon bond is denoted by $B_T(t)$ and given by

$$B_T(t) = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)ds\right)\Big|\mathscr{F}_t\right] \tag{6.1.4}$$

where the expectation is taken under the risk-neutral measure $\mathbb{Q}$ which takes the money market account as numéraire. Through the risk-neutral measure $\mathbb{Q}$, the T-forward measure, $\mathbb{Q}^T$ is defined using the Radon-Nikodym derivative given by

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{1}{M(T)}\frac{M(0)}{B_T(0)} = \frac{1}{B_T(0)}\exp\left(-\int_0^T r(s)ds\right) \tag{6.1.5}$$

(Geman, El Karoui, & Rochet, 1995). When the risk-free interest rates are deterministic the T-forward measure and the risk-neutral measure coincide. According to Geman et al. (1995), the forward price, relative to time T, of a non dividend paying security, S, with maturity T is equal to the expectation of the value at time T of this security under the T-forward measure: $F(t) = \mathbb{E}_{\mathbb{Q}^T}\left[\frac{S_T}{B_T(T)}\Big|\mathscr{F}_t\right]$. Hence the forward price is a martingale under the T-forward measure, $\mathbb{Q}$.

Taking the expectation in equation 6.1.1 under the T-forward measure, $\mathbb{Q}^T$, leads to the following expression for the price of a European call option:

$$\begin{aligned} c(F_t, K) &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}^T}\left[(S_T - K)^+\Big|\mathscr{F}_t\right] = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}^T}\left[(S_T - K)\mathbb{1}_{[S_T>K]}\Big|\mathscr{F}_t\right] \\ &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}^T}\left[S_T\mathbb{1}_{[S_T>K]}\Big|\mathscr{F}_t\right] - e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}^T}\left[K\mathbb{1}_{[S_T>K]}\Big|\mathscr{F}_t\right] \\ &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}^T}\left[S_T\mathbb{1}_{[S_T>K]}\Big|\mathscr{F}_t\right] - e^{-r(T-t)}K\mathbb{Q}^T\left[S_T > K\right] \end{aligned} \tag{6.1.6}$$

where $\mathbb{Q}^T\left[S_T > K\right]$ is the probability that S_T exceeds K under the T-forward measure. The first term in equation 6.1.6 requires further investigation. S_T is a random variable and can therefore not be taken out of the expectation like K. However, $\frac{S_T}{\mathbb{E}_{\mathbb{Q}^T}[S_T]}$ is by definition a positive martingale with mean 1 and, therefore, it is a Radon-Nikodym derivative process. The probability measure $\mathbb{S}$ is the probability measure induced by this Radon-Nikodym derivative process. Then

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}^T}\left[S_T\mathbb{1}_{[S_T>K]}\Big|\mathscr{F}_t\right] &= \mathbb{E}_{\mathbb{Q}^T}\left[S_T\Big|\mathscr{F}_t\right]\mathbb{E}_{\mathbb{Q}^T}\left[\frac{S_T}{\mathbb{E}_{\mathbb{Q}^T}[S_T]}\mathbb{1}_{[S_T>K]}\Big|\mathscr{F}_t\right] \\ &= \mathbb{E}_{\mathbb{Q}^T}\left[S_T\Big|\mathscr{F}_t\right]\mathbb{E}_{\mathbb{S}}\left[\mathbb{1}_{[S_T>K]}\right] \\ &= \mathbb{E}_{\mathbb{Q}^T}\left[S_T\Big|\mathscr{F}_t\right]\mathbb{S}\left[S_T > K\right] \\ &= F_t\mathbb{S}\left[S_T > K\Big|\mathscr{F}_t\right] \end{aligned} \tag{6.1.7}$$

where $F_t = S_t e^{r(T-t)}$ and $\mathbb{S}\left[S_T > K\Big|\mathscr{F}_t\right]$ is the probability that $S_T > K$ under the probability measure $\mathbb{S}$. Replacing $\mathbb{E}_{\mathbb{Q}^T}\left[S_T\mathbb{1}_{[S_T>K]}\right]$ in equation 6.1.6 by the expression derived in equation 6.1.7 results in the following expression for the price of a European call option:

$$c(F_t, K) = e^{-r(T-t)}F_t\mathbb{S}\left[S_T > K\right] - e^{-r(T-t)}K\mathbb{Q}^T\left[S_T > K\right] \tag{6.1.8}$$

Similarly, the price of a European put option is derived, resulting in the following expression:

$$\begin{aligned} p(F_t, K) &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}^T} \left[(K - S_T)^+ \middle| \mathscr{F}_t \right] \\ &= e^{-r(T-t)} K \mathbb{Q}^T [S_T < K] - e^{-r(T-t)} F_t \mathbb{S} [S_T < K] \end{aligned} \tag{6.1.9}$$

where $\mathbb{Q}^T[S_T < K]$ is the $\mathbb{Q}^T$-probability that S_T is smaller than K and $\mathbb{S}[S_T < K]$ is the probability that $S_T < K$ under the probability measure $\mathbb{S}$.

6.2 Analytical Approximation

To conduct the analysis using the implicit function theorem, the strike price factor of the long position in the put option is rewritten to $e^a F_t$. Setting this equal to the original expression of the strike price leads to the following relationship between a and the known term α_l:

$$(1 + \alpha_l) S_t = e^a F_t = e^a e^{r(T-t)} S_t \Leftrightarrow a = \ln(1 + \alpha_l) - r(T - t)$$

Similarly, the strike price of the short position in the call option is adjusted to be $e^b F_t$. Since β is the unknown term for which an expression is derived, the new strike price is rewritten such that an expression for β is obtained:

$$\beta S_t = e^b F_t = e^b e^{r(T-t)} S_t \Leftrightarrow \beta = e^{b + r(T-t)}$$

The relation between β and α_h remains intact and therefore the following expression for α_h holds:

$$\alpha_h = \beta - 1 = e^{b + r(T-t)} - 1 \tag{6.2.1}$$

These adjustments to the strike prices of the call and put option are made to ensure that the T-forward measure described in section 6.1 can be exploited.

The following equation has to be solved with respect to the unknown variable b:

$$c(F_t, K_c) = p(F_t, K_p) \tag{6.2.2}$$

where $K_c = e^b F_t$ and $K_p = e^a F_t$ This equation can not be solved analytically. However, an analytical approximation can be derived through the implicit function theorem. The analysis done in section 5 implies that b is implicitly defined by the term a. In other words, b is a function of a: $b = b(a)$. Hence equation 6.2.2 is rewritten to the following to explicitly state this implicit relation:

$$c(F_t, e^{b(a)} F_t) = p(F_t, e^a F_t) \tag{6.2.3}$$

Differentiating equation 6.2.3 on both sides with respect to a leads to the following relationship between the derivatives:

$$\frac{\partial c(F_t, K_c)}{\partial K_c} \cdot \frac{\partial K_c}{\partial b(a)} \cdot \frac{\partial b(a)}{\partial a} = \frac{\partial p(F_t, K_p)}{\partial K_p} \cdot \frac{\delta K_p}{\delta a} \tag{6.2.4}$$

where $\frac{\partial K_c}{\partial b(a)} = e^{b(a)} F_t$ and $\frac{\delta K_p}{\delta a} = e^a F_t$.

To derive $\frac{\partial c(F_t, K_c)}{\partial K_c}$ and $\frac{\partial p(F_t, K_p)}{\partial K_p}$, *Euler's Theorem for a homogeneous function of order one* is introduced.

Theorem 2 (Euler's Theorem for a homogeneous function of order one, (Elghribi, Othman, & Al-Nashri, 2017))**.** *Let $f(x, y)$ be a homogeneous function of order one, such that*

$$f(\lambda x, \lambda y) = \lambda f(x, y)$$

Then

$$x\frac{\partial f(x,y)}{\partial x} + y\frac{\partial f(x,y)}{\partial y} = f(x,y)$$

Proof. Differentiating both sides of $f(\lambda x, \lambda y) = \lambda f(x, y)$ with respect to λ and applying the chain rule to the left-hand side, leads to:

$$\frac{\partial f(\lambda x, \lambda y)}{\partial \lambda x}\frac{\partial \lambda x}{\partial \lambda} + \frac{\partial f(\lambda x, \lambda y)}{\partial \lambda y}\frac{\partial \lambda y}{\partial \lambda} = f(x,y)$$

$$\Leftrightarrow \frac{\partial f(\lambda x, \lambda y)}{\partial \lambda x}x + \frac{\partial f(\lambda x, \lambda y)}{\partial \lambda y}y = f(x,y)$$

Since the expression holds for arbitrary λ, λ can be set equal to 1, resulting in the expression in the theorem:

$$x\frac{\partial f(x,y)}{\partial x} + y\frac{\partial f(x,y)}{\partial y} = f(x,y)$$

□

First, it is proven that $c(F_t, K_c)$ and $p(F_t, K_p)$ are homogeneous function of order one, which is also called linearly homogeneous. The functions $(S_T - K)^+$ and $(K - S_T)^+$ are linearly homogeneous functions with respect to the forward price and the strike price, since

$$(\lambda F_T - \lambda K)^+ = \lambda(F_T - K)^+ \text{ and } (\lambda K - \lambda F_T)^+ = \lambda(K - F_T)^+$$

Determining an expected value is a linear mathematical operation. Performing a linear operation on a linearly homogeneous function preserves the property of linear homogeneity. The multiplication of the expectation by $e^{-r(T-t)}$ is discounting the expected value and therefore also preserves the linear homogeneity of the function. Consequently, the price of a European call option and the price of a European put option are linearly homogeneous with respect to the forward price and the strike price.

Exploiting *Euler's Theorem for a homogeneous function of order one* on the price of a European call option can be written as:

$$c(F_t, K_c) = \frac{\partial c(F_t, K_c)}{\partial F_t}F_t + \frac{c(F_t, K_c)}{\partial K_c}K_c \tag{6.2.5}$$

Comparing this equation to equation 6.1.1, leads to the following expression for $\frac{\partial c(F_t,K_c)}{\partial K_c}$:

$$\frac{\partial c(F_t, K_c)}{\partial K_c} = -e^{-r(T-t)}\mathbb{Q}^T[S_T > K_c] \tag{6.2.6}$$

Similarly, for the price of a European put option:

$$p(F_t, K_p) = \frac{\partial p(F_t, K_p)}{\partial F_t}F_t + \frac{p(F_t, K_p)}{\partial K_p}K_p \tag{6.2.7}$$

Comparing this equation to equation 6.1.2, the expression for $\frac{\partial p(F_t,K_p)}{\partial K_p}$ can be derived:

$$\frac{\partial p(F_t, K_p)}{\partial K_p} = e^{-r(T-t)}\mathbb{Q}^T[S_T < K_p] \tag{6.2.8}$$

Filling equations 6.2.6 and 6.2.8 into equation 6.2.4 leads to

$$\begin{aligned} &-e^{-r(T-t)}\mathbb{Q}^T[S_T > K_c]e^{b(a)}F_t\frac{\partial b(a)}{\partial a} = e^{-r(T-t)}\mathbb{Q}^T[S_T < K_p]e^aF_t \\ &\Leftrightarrow -\mathbb{Q}^T[S_T > e^{b(a)}F_t]e^{b(a)}\frac{\partial b(a)}{\partial a} = \mathbb{Q}^T[S_T < e^aF_t]e^a \end{aligned} \tag{6.2.9}$$

Equation 6.2.9 is a non-linear differential equation of the function $b(a)$, which cannot be solved for arbitrary a. However, using the put-call parity it is known that call and put options with the same time to maturity and written with the same underlying are at-the-money forward at the same strike price. If the option is at-the-money forward, the strike price is at the same level as the prevailing market price of the underlying forward contract: $K = F_t$. Hence at the at-the-money forward point:

$$K_p = e^aF_t = F_t \Leftrightarrow e^a = 1 \Leftrightarrow a = 0$$

and since $a = 0$

$$K_c = e^{b(0)}F_t = F_t \Leftrightarrow e^{b(0)} = 1 \Leftrightarrow b(0) = 0$$

Therefore, at the point $a = 0$ and $b(0) = 0$ the probabilities can be rewritten to:

$$\mathbb{Q}^T[S_T > e^{b(a)}F_t] = \mathbb{Q}^T[S_T > F_t] \tag{6.2.10}$$

$$\mathbb{Q}^T[S_T < e^aF_t] = \mathbb{Q}^T[S_T < F_t] \tag{6.2.11}$$

Now the probabilities no longer depend on the unknown term $b(a)$ and therefore at the point $a = 0$ and $b(0) = 0$, an expression for $\frac{\partial b(a)}{\partial a}$ can be derived. Equation 6.2.9 boils at this point down to:

$$-\mathbb{Q}^T[S_T > F_t]\frac{\partial b(0)}{\partial a} = \mathbb{Q}^T[S_T < F_t]$$

Leading to the following expression for $\frac{\partial b(0)}{\partial a}$:

$$\frac{\partial b(0)}{\partial a} = -\frac{\mathbb{Q}^T[S_T < F_t]}{\mathbb{Q}^T[S_T > F_t]} \tag{6.2.12}$$

Since the value of $\frac{\delta b(a)}{\delta a}$ can be derived at the point $a = 0$, an approximation for $b(a)$ can be derived using the first-order Taylor expansion. The first-order Taylor expansion is performed around the point $a = 0$, and leads to the following expression for $b(a)$:

$$b(a) = -a\frac{\mathbb{Q}^T[S_T < F_t]}{\mathbb{Q}^T[S_T > F_t]} \tag{6.2.13}$$

The expression for β is then equal to:

$$\begin{aligned} \beta &= e^{b(a)+r(T-t)} \\ &= \exp\left(-(\ln(1+\alpha_l) - r(T-t))\frac{\mathbb{Q}^T[S_T < F_t]}{\mathbb{Q}^T[S_T > F_t]} + r(T-t)\right) \end{aligned} \tag{6.2.14}$$

However, the self-financing buffer fund is evaluated at the end of every year, therefore, the call and put option have a time to maturity of one ($T - t = 1$). Hence the expression for β can be simplified to:

$$\begin{aligned} \beta &= e^{b(a)+r} \\ &= \exp\left(-(\ln(1+\alpha_l) - r)\frac{\mathbb{Q}^T[S_T < F_t]}{\mathbb{Q}^T[S_T > F_t]} + r\right) \end{aligned} \tag{6.2.15}$$

The value for α_h can be easily derived from this expression, since $\alpha_h = \beta - 1$. In the remainder of this section only values for β will be shown due to the simple relation between β and α_h.

6.3 Log-Normally Distributed Investment Portfolio

In this section the values for β and consequently α_h are derived under the assumption that the value of the investment portfolio is log-normally distributed. In this setting, the same assumptions are made as in the Black-Scholes framework. Under these assumptions, the risk-free interest rates are deterministic and therefore is the T-forward measure $\mathbb{Q}^T$ equal to the risk-neutral measure $\mathbb{Q}$.

The probabilities required to derive the analytical approximation for β are based on the value of the investment portfolio at maturity, S_T. Therefore, the distribution and dynamics of S_T are defined. Under the risk-neutral measure $\mathbb{Q}$, the value of the investment portfolio, S_t, satisfies the following stochastic differential equation (Mikosch, 1998):

$$dS_t = rS_t dt + \sigma S_t dW_t$$

where, W_t is a Wiener process, r is the risk-free interest rate and σ is the volatility of the underlying asset. Solving this stochastic differential equation, leads to the following expression for S_t:

$$S_T|S_t = S_t e^{\left(r-0.5\sigma^2\right)(T-t)+\sigma(W_T-W_t)}$$

Hence $S_T|S_t$ is log-normally distributed with mean equal to $\ln(S_t) + \left(r - 0.5\sigma^2\right)(T-t)$ and variance $\sigma^2(T-t)$:

$$S_T|S_t \sim logN\left(\ln(S_t) + \left(r - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right)$$

6.3.1 Analytical Derivation

To derive the expression for β, the probabilities $\mathbb{Q}[S_T < F_t]$ and $\mathbb{Q}[S_T > F_t]$ are calculated first taking into account that $F_t = S_t e^{r(T-t)}$.

$$\begin{aligned}
\mathbb{Q}[S_T < F_t] &= \mathbb{Q}[S_T < S_t e^{r(T-t)}] \\
&= \mathbb{Q}\left[\ln(S_T) < \ln(S_t) + r(T-t)\right] \\
&= \mathbb{Q}\left[\ln(S_T) < \frac{\ln(S_t) + r(T-t) - \ln(S_t) - r(T-t) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right] \\
&= \mathbb{Q}\left[\frac{1}{2}\sigma\sqrt{T-t}\right] \\
&= N\left(\frac{1}{2}\sigma\sqrt{T-t}\right)
\end{aligned} \tag{6.3.1}$$

$$\begin{aligned}
\mathbb{Q}[S_T > F_t] &= 1 - \mathbb{Q}[S_T < F_t] \\
&= 1 - N(\frac{1}{2}\sigma\sqrt{T-t}) \\
&= N(-\frac{1}{2}\sigma\sqrt{T-t})
\end{aligned} \tag{6.3.2}$$

Note that, the probability $\mathbb{Q}[S_T < F_t]$ is exactly equal to $N(d_2)$ in Black's formula for the price of a European call option at the at-the-money forward point, which is the probability that the call option will be exercised. Similarly for $\mathbb{Q}[S_T > F_t]$, however this is $N(-d_2)$, which is the probability that a European put option will be exercised. Then

$$b(a) = -a\frac{N(\frac{1}{2}\sigma\sqrt{T-t})}{N(-\frac{1}{2}\sigma\sqrt{T-t})} \tag{6.3.3}$$

where $a = \ln(1+\alpha_l) - r(T-t)$. Leading to the following expression for β:

$$\beta = e^{b(a)+r(T-t)} = \exp\left(-\left(\ln(1+\alpha_l) - r(T-t)\right)\frac{N\left(\frac{1}{2}\sigma\sqrt{T-t}\right)}{N\left(-\frac{1}{2}\sigma\sqrt{T-t}\right)} + r(T-t)\right) \tag{6.3.4}$$

Since the self-financing buffer fund is evaluated at the end of every year, the call and put option have a time to maturity of one year, $T - t = 1$. Hence the equation for β boils down to:

$$\beta = e^{b(a)+r} = \exp\left(-\left(\ln(1+\alpha_l) - r\right)\frac{N\left(\frac{1}{2}\sigma\right)}{N\left(-\frac{1}{2}\sigma\right)} + r\right) \tag{6.3.5}$$

The dependence of the strike price factor of the call option, β, on the volatility has been quantified in equation 6.3.5. The volatility, σ, only appears within the cumulative distribution function of the standard normal distribution. Since volatility is always positive and a property of the cumulative distribution function of the standard normal distribution is that $N(-x) = 1 - N(x)$, the part of β including the volatility can be rewritten to:

$$\frac{N\left(\frac{1}{2}\sigma\right)}{N\left(-\frac{1}{2}\sigma\right)} = \frac{N\left(\frac{1}{2}\sigma\right)}{1 - N\left(\frac{1}{2}\sigma\right)} > 1$$

The inequality is due to the fact that $N\left(\frac{1}{2}\sigma\right) > 0.5$, which is another property of the cumulative distribution function of the standard normal distribution ($N(x) > 0.5, \forall x$). Since $(1 + \alpha_l) < 1$, $\ln(1 + \alpha_l) < 0$, which becomes positive due to the negative sign in front. Hence the volatility has an increasing effect on the strike price factor for the call option.

The increasing effect of the risk-free interest rate on the strike price factor of the call option also has become more apparent. Since $\frac{N\left(\frac{1}{2}\sigma\right)}{N\left(-\frac{1}{2}\sigma\right)} > 1$ and the sign in front of the risk-free interest rate becomes positive, the risk-free interest rate has an increasing effect in the first term within the exponential. Obviously, the second term r in the expression for β has an increasing effect due to the positive sign in front.

6.3.2 Graphical Representation

To evaluate the effectiveness of the analytical approximation derived in equation 6.3.3 for the value of $b(a)$ and in effect the value of β, the values of β are graphed in the same matter as done for the numerical results in section 4. The same lower threshold are evaluated, namely $\alpha_l = -2\%$ and $\alpha_l = -7\%$.

In figure 9, the value of the strike price factor for the call option, β, derived using equation 6.3.5, is shown for a lower threshold determined at -2% against the risk-free interest rate, r, and the volatility of the underlying investment portfolio, σ. The strike price factor is increasing in the risk-free interest rate and the dependency on the volatility is small. The graph in figure 9 appears similar to 7, in section 4, which is a graph of the strike price factor of the call option using numerical results.

The lowest value of the strike price factor for the call option is at the point where the risk-free interest rate is equal to 0% and the volatility is equal to 1%, which is equal to 1.021. The largest value of β is achieved when $r = 6\%$ and $\sigma = 13\%$, namely $\beta = 1.161$. If the graph would be extended further with higher values for the risk-free interest rate and higher values for the volatility, the strike price factor for the call option would increase further, as can be seen from equation 6.3.5.

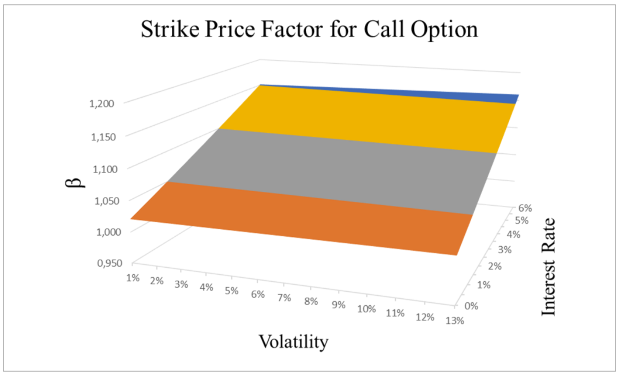

Figure 9: Graph of the value of strike price factor, β, derived through the analytical derivation, for a value of α_l of -2% against the risk-free interest rate, r, and the volatility of the underlying investment portfolio, σ

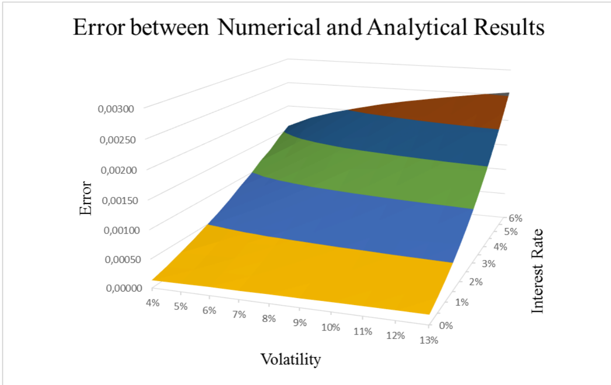

Figure 10: Graph of the value of the error between the numerical results for $\alpha_l = -2\%$ and analytical approximation in this section against the risk-free interest rate, r, and the volatility of the underlying investment portfolio, σ

In figure 10, the value of the error between the numerically calculated results and the analytical approximation of equation 6.3.5 is graphed for several combinations of risk-free interest rate and volatility. It can be seen that the error in the approximation increases when the risk-free interest rate increases and volatility is kept constant. The error also increases when the volatility increases and the risk-free interest rate is kept constant, but this error increase is smaller. The value of the error ranges between 0.00014 and 0.00256, which are small approximation errors.

In figure 11, the value of the strike price factor for the call option, β, is graphed for a lower threshold of -7% against both the volatility of the investment portfolio and the risk-free interest rate. The graph is similar to figure 8 in section 4, in both value and shape. The lowest value for the strike price factor of the call option is again achieved when the risk-free interest rate is equal 0% and the volatility is equal to 1%, namely $\beta = 1.076$. The largest value for the strike price factor for the call option is 1.230, when the risk-free interest rate is equal to 6% and the volatility is equal to 13%.

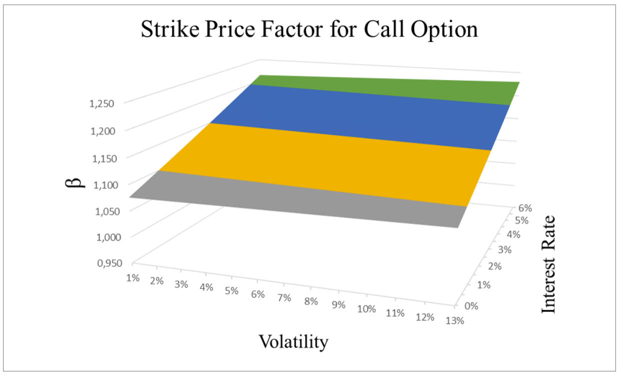

Figure 11: Graph of the value of strike price factor, β, derived through the analytical derivation, for a value of α_l of -7% against the risk-free interest rate, r, and the volatility of the underlying investment portfolio, σ

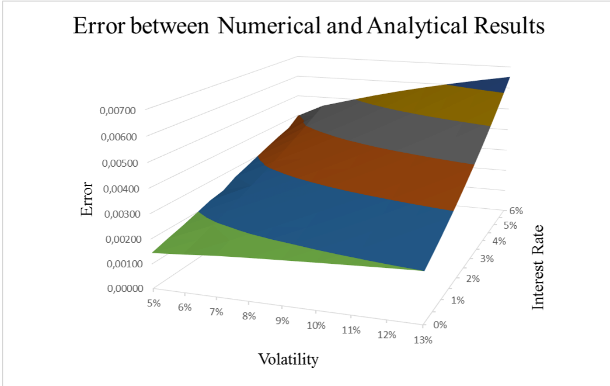

Figure 12: Graph of the value of the error between the numerical results for $\alpha_l = -7\%$ and analytical approximation in this section against the risk-free interest rate, r, and the volatility of the underlying investment portfolio, σ

Figure 12 shows the values of the error between the numerical results derived in section 4 and the analytical approximation derived in this section. The value of the error ranges between 0.00124 and 0.00653. These values are higher than the error values for $\alpha_l = -2\%$. This is due to the larger role α_l plays in the expression for the analytical approximation. The trends, however, related to the risk-free interest rate and the volatility of the underlying investment portfolio are the same.

Since the value of the lower threshold α_l in figure 11 is lower than the lower threshold in figure 9, the influence of the volatility is more significant in this case. This can be seen in equation 6.3.5. The lower threshold α_l appears in equation 6.3.5 only within the ln-term. Therefore when taking the absolute value of this term, since $\ln(1 + \alpha_l) < 1$ and $\ln(x)$ is an increasing function, the effect of the lower threshold, α_l will increase. Hence the suspicion arising at the end of section 4, that the effect of the volatility increases the further the lower threshold is from 0% to the downside is confirmed. This phenomenon manifests itself in the graph through the increasing line at the color transition. In figure 9, these lines at the color transition are not as steep as these same lines at the same color transitions in figure 11.

6.4 Shifted Log-Normally Distributed Investment Portfolio

The analytical approximation derived in this section does not only apply to the Black-Scholes framework for option pricing. The investment portfolio can have any distribution. Therefore, to check the robustness of the analytical approximation, in this section the value of the investment portfolio is assumed to have a shifted log-normal distribution. The risk-free interest rates are assumed to be deterministic and therefore equals the T-forward measure, $\mathbb{Q}^T$, the risk-neutral measure, $\mathbb{Q}$.

A basic process Y is first considered, which follows the geometric Brownian motion under the risk-neutral measure, $\mathbb{Q}$:

$$dY_t = rY_t dt + \sigma Y_t dW_t$$

The value of the investment portfolio S_t is defined as $S_t = Y_t + \gamma e^{rt}$, where γ denotes the shift or location parameter, which is a real constant. This process evolves under the risk-neutral measure according to

$$dS_t = rS_t dt + \sigma_S(S_t - \gamma e^{rt})dW_t$$

where σ_S is the volatility associated with the shifted log-normal distributed process S_t. Then the investment portfolio, S_t, is said to have a shifted log-normal distribution and can be written as

$$S_t = \gamma e^{rt} + (S_0 - \gamma)e^{(r-0.5\sigma_S^2)t+\sigma_S dW_t}$$

Moreover, the value of the investment portfolio at maturity, S_T, conditional on S_t also has a shifted log-normal distribution, which implies that $S_T - \gamma e^{r(T-t)}$ conditional on S_t, is log-normally distributed.

$$S_T - \gamma e^{r(T-t)} | S_t \sim logN\left(\ln(S_t - \gamma) + \left(r - \frac{1}{2}\sigma_S^2\right)(T-t), \sigma_S^2(T-t)\right)$$

Through constructing the dynamics of the value of the investment portfolio using the shifted log-normal distribution a non-flat volatility structure is achieved. Hence, the volatility smile is included in the model. When $\gamma = 0$, S_t is log-normally distributed and the general Black-Scholes formula suffices to calculate put and call option. If $\gamma \neq 0$ the shift parameter, γ, has two effects on the implied volatility of the options. If $\gamma < 0$ the volatility curve is strictly decreasing and moves upwards and if $\gamma > 0$ it is strictly increasing and moves downwards. (Brigo & Mercurio, 2007)

6.4.1 Analytical Derivation

The probabilities $\mathbb{Q}^T[S_T < F_t]$ and $\mathbb{Q}^T[S_T > F_t]$ are determined by conducting an analysis using the above dynamics, the fact that the T-forward measure $\mathbb{Q}^T$ is equal to the risk-neutral measure $\mathbb{Q}$ and $F_t = S_t e^{r(T-t)}$.

$$\begin{aligned}
\mathbb{Q}[S_T < F_t] &= \mathbb{Q}\left[\ln\left(S_T - \gamma e^{r(T-t)}\right) < \ln\left(F_t - \gamma e^{r(T-t)}\right)\right] \\
&= \mathbb{Q}\left[\ln\left(S_T - \gamma e^{r(T-t)}\right) < \ln(S_t - \gamma) + r(T-t)\right] \\
&= \mathbb{Q}\left[\ln\left(S_T - \gamma e^{r(T-t)}\right) < \frac{\ln(S_t - \gamma) + r(T-t) - \ln(S_t-\gamma) - r(T-t) + \frac{1}{2}\sigma_S^2(T-t)}{\sigma_S\sqrt{T-t}}\right] \\
&= \mathbb{Q}\left[\ln\left(S_T - \gamma e^{r(T-t)}\right) < \frac{1}{2}\sigma_S\sqrt{T-t}\right] \\
&= N\left(\frac{1}{2}\sigma_S\sqrt{T-t}\right)
\end{aligned} \tag{6.4.1}$$

$$\begin{aligned}
\mathbb{Q}[S_T > F_t] &= 1 - \mathbb{Q}[S_T < F_t] \\
&= 1 - N\left(\frac{1}{2}\sigma_S\sqrt{T-t}\right) \\
&= N\left(-\frac{1}{2}\sigma_S\sqrt{T-t}\right)
\end{aligned} \tag{6.4.2}$$

Note that, $\mathbb{Q}[S_T < F_t]$ is equal to $N(d_2)$ in the shifted Black formula for the price of a European call option with the strike price equal to the forward price, F_t. Similarly for $\mathbb{Q}[S_T > F_t]$, which is $N(-d_2)$ in the shifted Black formula for the price of a European put option at the at-the-money forward point.

Then

$$b(a) = -a\frac{N(\frac{1}{2}\sigma_S\sqrt{T-t})}{N(-\frac{1}{2}\sigma_S\sqrt{T-t})} \tag{6.4.3}$$

where $a = \ln(1+\alpha_l) - r(T-t)$. Resulting in the following expression for β:

$$\beta = e^{b(a)+r(T-t)} = \exp\left(-\left(\ln(1+\alpha_l) - r(T-t)\right)\frac{N\left(\frac{1}{2}\sigma_S\sqrt{T-t}\right)}{N\left(-\frac{1}{2}\sigma_S\sqrt{T-t}\right)} + r(T-t)\right) \tag{6.4.4}$$

The self-financing buffer fund is evaluated at the end of every year, therefore $T-t=1$ as mentioned before. Hence the equation for β reduces to:

$$\beta = e^{b(a)+r} = \exp\left(-\left(\ln(1+\alpha_l) - r\right)\frac{N\left(\frac{1}{2}\sigma_S\right)}{N\left(-\frac{1}{2}\sigma_S\right)} + r\right) \tag{6.4.5}$$

Hence, the analytical approximation of the value of β does not dependent on the shift parameter, γ. The effects of the risk-free interest rate and the volatility of the underlying investment portfolio, σ_S, are the same as in the case in which the investment portfolio is log-normally distributed. The only difference between equation 6.3.5 and equation 6.4.5 is the volatility taken. In equation 6.4.5, the volatility is the volatility of the investment portfolio under the shifted log-normal distribution and in equation 6.3.5 it is the log-normally distributed investment portfolio's volatility.

6.4.2 Graphical Representation

Since the analytical approximation of β does not dependent on γ, figure 9 and figure 11 in section 6.3.2 also show the values of the approximation of β when the investment portfolio is shifted log-normally distributed. However, to make a comparison between the approximation and the actual value, numerical results are calculated for the value of β if the investment portfolio follows a shifted log-normal distribution.

To compute the numerical results the shifted Black Formula is used. The price of a European call option under the shift log-normal distribution, denoted by $c_S(F_t, K)$, is given by the following expression:

$$c_S(F_t, K) = e^{-r(T-t)}[(F_t - \Gamma)N(d_1) - (K - \Gamma)N(d_2)] \tag{6.4.6}$$

The price of a European put option, denoted by $p_S(F_t, K)$, is given by:

$$p_S(F_t, K) = e^{-r(T-t)}[(K - \Gamma)N(-d_2) - (F_t - \Gamma)N(-d_1)] \tag{6.4.7}$$

with

$$d_1 = \frac{\ln\left(\frac{F_t-\Gamma}{K-\Gamma}\right) + \frac{1}{2}\sigma_S^2(T-t)}{\sigma_S\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma_S\sqrt{T-t}$$

Here σ_S is the volatility of the underlying investment portfolio, which follows the shifted log-normal distribution. Γ is the shift parameter associated with the forward price. Two restrictions are placed on the value of Γ in this model: (1) $F_t > \Gamma$ and (2) $\Gamma < 0$. (Brigo & Mercurio, 2000)

The strike price for the long position in the put option is equal to $K_p = (1 + \alpha_l)S_t$ and the strike price for the call option in which a short position is taken equals $K_c = \beta S_t$. Through analyzing several equity options, two different values of Γ are chosen to compute the strike price factor β numerically, namely $\Gamma = -15$ and $\Gamma = -30$. The volatilities range from 4% to 15% and the risk-free interest rate have the same range as before, from 0% to 6%. S_t is chosen to be 100. Then the only unknown variable is the strike price factor for the call option, β, which can be computed by setting the prices of the two options equal:

$$c_S(F_t, \beta S_t) = p_S(F_t, (1 + \alpha_l)S_t) \tag{6.4.8}$$

In this section only results are computed for a lower threshold equal to -2%.

In figure 13, the value of the strike price factor, β, is shown for several combinations of volatility and risk-free interest rate when $\Gamma = -15$. The same pattern is visible as for the log-normally distributed investment portfolio: β is increasing when the risk-free interest rate increases and volatility is kept constant, and β is slightly increasing when volatility increases and the risk-free interest rate is kept constant. The value of β range from 1.021 when $\sigma_S = 4\%$ and $r = 0\%$ to 1.159 for $\sigma_S = 15\%$ and $r = 6\%$. When the risk-free interest rate is kept constant at 0%, the value of β increases to 1.023 if the volatility increases to 15%. Keeping volatility constant at 4% β increases to 1.151 if $r = 6\%$.

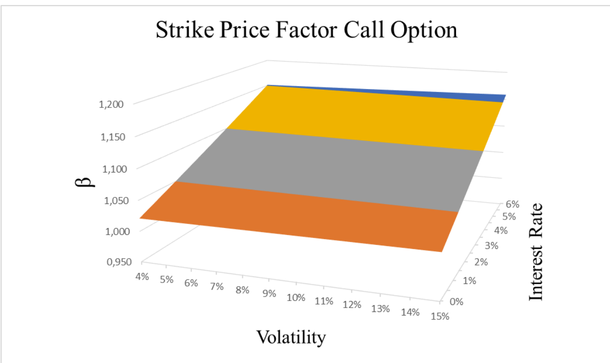

Figure 13: Graph of the value of strike price factor, β, numerically derived, for a value of α_l of -2% against the risk-free interest rate, r, and the volatility of the underlying investment portfolio, σ, with $\Gamma = -15$

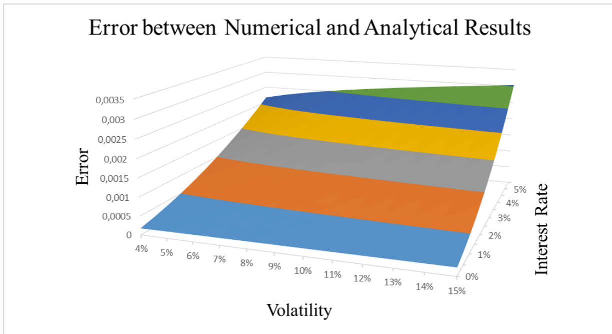

Figure 14: Graph of the value of the error between the numerical results for $\alpha_l = -2\%$ and analytical approximation against the risk-free interest rate, r, and the volatility of the underlying investment portfolio, σ, with $\Gamma = -15$

Figure 14 shows the difference between the numerical value of β and the analytical approximation of β against the risk-free interest rate and the volatility. The error is increasing in the risk-free interest rate keeping volatility constant. The error is also increasing in the volatility if the risk-free interest rate is kept constant. However, this increase is smaller and less significant. Hence the influence of the volatility is captured rather well. For values of the volatility between 4% and 15% and values of the risk-free interest rate between 0% and 6%, the value of the error ranges between 0.00018 and 0.00303.

Since the analytical approximation does not include Γ a different Γ of -30 is taken to investigate whether or not it has an effect on the value of β or not. In figure 15, the same graph is shown as in figure 13, the only difference is that the value of Γ is now equal to -30. The effect of the risk-free interest rate and the volatility is the same as if $\Gamma = -15$. However, the values of β are slightly lower, ranging from 1.021 if $\sigma_S = 4\%$ and $r = 0\%$ to 1.158 for $\sigma_S = 15\%$ and $r = 6\%$.

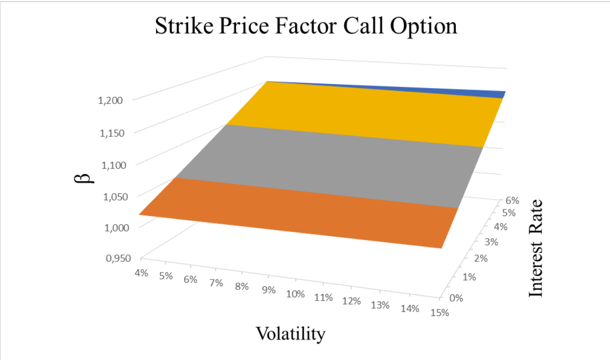

Figure 15: Graph of the value of strike price factor, β, numerically derived, for a value of α_l of -2% against the risk-free interest rate, r, and the volatility of the underlying investment portfolio, σ, with $\Gamma = -30$

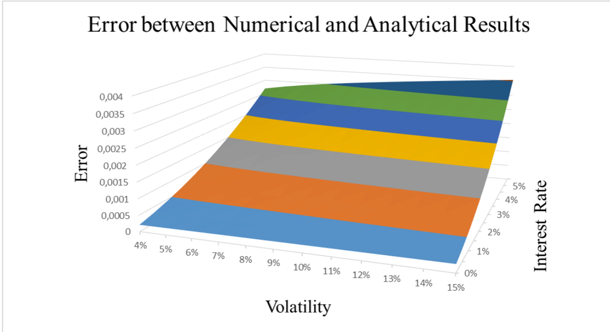

Figure 16: Graph of the value of the error between the numerical results for $\alpha_l = -2\%$ and analytical approximation against the risk-free interest rate, r, and the volatility of the underlying investment portfolio, σ, with $\Gamma = -30$

The value of β calculated from the derived analytical approximation are the same for all Γ. Therefore, since the value of β is slightly lower, the value of the error is larger. These errors are graphically shown in figure 16. The value of the error ranges from 0.000208 for $r = 0\%$ and $\sigma_S = 4\%$ to 0.003538 if $r = 6\%$ and $\sigma_S = 15\%$. The errors in both cases ($\Gamma = -15$ & $\Gamma = -30$) are relatively small. Therefore, it can be concluded that the analytical approximation using the first-order Taylor approximation captures the movements well. Hence, the second-order effect is small.

7 Conclusion

The Dutch pension system is in need of a reform and at the moment these reforms are negotiated by the social partners supervised by the SER. Several alternatives have been discussed. When major changes are made to the current system, the variant IV-C-R is the most likely to be implemented. In variant IV-C-R an explicit buffer fund is established through setting lower and upper thresholds on the returns made on the investments in the personal pension capital. However, in the variant no measures are taken to ensure that the buffer fund is self-financing. In this Master's Thesis it was investigated how this self-financing property can be implemented.

In order to establish a self-financing buffer fund in variant IV-C-R, the lower and upper threshold cannot be set independently of each other. Either the lower threshold is determined in the variant or the upper threshold is determined. In this thesis it was assumed that the lower threshold is set, since the participants in the pension scheme are more concerned about the downturns in their investments and minimizing these negative returns. The upper threshold is then determined by equating the price of a long position in a European put option with strike price $K = (1+\alpha_l)S_t$, which ensures that the investment returns are $\alpha_l\%$ or higher, and the price of a European call option with strike price equal to $(1+\alpha_h)S_t$, which is shorted to ensure that the returns are equal to or below $\alpha_h\%$.

The buffer fund is evaluated at the end of every year, therefore, the options are one year options. Hence, the time to maturity of the options is known and at the beginning of every year, when the options are required to be bought and sold. The one-year risk-free interest rate is known and the distribution of the investment portfolio can be evaluated. Since all these components are known a value for α_h can be derived. Throughout this Master's Thesis $(1+\alpha_h)$ was set equal to β. Numerical results for β were derived for an investment portfolio which is log-normally distributed. Through exploitation of the patterns shown in these numerical results the put-call symmetry was derived to approximate the value of β without a dependence on volatility, which is the most uncertain variable in the model. The put-call symmetry led to an overhedging of the upside returns. However, it provided an overview in which direction an analytical approximation of β was. The put-call symmetry led to the following approximation of β:

$$\beta = \frac{e^{2r(T-t)}}{1+\alpha_l}$$

However, here a strong assumption about the symmetry of the distribution of the investment portfolio had to be made and the result was derived in the Black-Scholes economy.

An analytical approximation for β was derived without assuming a specific distribution of the investment portfolio. The only necessary assumption for this approximation is that there is no-arbitrage in the market. The T-forward measure was exploited to price both the European call and put option. Through this choice of probability measure the approximation also holds for stochastic risk-free interest rates and stochastic volatilities. The implicit function theorem was used to derive a relationship between the derivatives of the prices of the call and put option. Through a first-order Taylor expansion around the at-the-money forward point, the following approximation for β could be derived:

$$\beta = \exp\left(-(\ln(1+\alpha_l) - r)\frac{\mathbb{Q}^T[S_T < F_t]}{\mathbb{Q}^T[S_T > F_t]} + r\right)$$

This approximation was tested against the numerical results derived when the investment portfolio is log-normally distributed. The differences between the approximation and the numerical result are small. As a robustness check, an investment portfolio following a shifted log-normal distribution was analyzed. The shifted log-normal has as an advantage that the volatility smile of the options is included in the model. In this case the error values slightly increased, however, these values were still small. Therefore it can be concluded that the analytical approximation for β, which only uses the first order effect, replicates the numerical values well. Therefore, at the beginning of each year, the buffer fund can impose the self-financing property through calculating the value of α_h using the analytical approximation for β and the fact that $\alpha_h = \beta - 1$.

In order to implement variant IV-C-R, the self-financing property must be imposed on the buffer fund making the system viable. The lower threshold can be set in the variant, however, the upper threshold needs to be determined at the beginning of every year exploiting the distribution of the investment portfolio and the analytical approximation provided in this Master's Thesis.

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