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Evaluating the Impacts of Bond Pricing Misspecification on Forecasted Funding Ratio

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Abstract

In defined benefit pension fund investments, the funding ratio is affected by bond yields through both the asset and liability sides. This paper evaluates the impact of misspecification uncertainty on the funding ratio forecasted 10 years ahead, using a prediction interval incorporating misspecification uncertainty to account for using a wrong model. We focus on some commonly-used affine term structure models to investigate the impact of misspecification uncertainty. These models include those taking into account macroeconomic factors, and those not explicitly excluding arbitrage opportunities but performing well from an empirical point of view. As an application, we consider a stylized defined benefit pension. The empirical analysis shows that the impact of misspecification uncertainty on the forecasted funding ratio is non-negligible, especially when a simple-structured nominal model is preferred in practice.

JEL Classification: C52; C58; C68; G12

Keywords: prediction interval, misspecification uncertainty, misspecification interval, pension fund investment, funding ratio

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1 Introduction

This paper empirically investigates the impact of misspecification uncertainty on a defined benefit pension funding ratio forecasted 10 years ahead from 2016, using a prediction interval incorporating model misspecification uncertainty.

Typically, a pension fund in the second pillar, supported by both employers and employees, can be classified into three types: a defined contribution type which requires a defined amount of contribution before retirement without promising a defined amount to pay after retirement; a defined benefit type which promises to pay a defined pension every period after retirement with the contribution before retirement specified in an actuarially fair way; and a hybrid type which combines the above two types. This paper looks into the defined benefit type. We consider a set-up in which the defined pensions to be paid form the liabilities of the fund, and the initial assets are set such that their value is equal to the actuarially fair value of the initial liabilities. It is of crucial importance for the pension fund management to monitor closely the funding ratio, i.e., the ratio of assets over liabilities. It helps to perceive the changes of this ratio in the future, so as to adjust policies for preventing the impact from unfavorable outcomes.

Before the financial crisis in 2008, the Dutch pension funding ratio was about 1.40 on average.[1] This average dropped at the crisis, and has been fluctuating between 0.90 and 1.10 since then. The average funding ratio in the third quarter of 2016 is right about 1.00, and in the fourth quarter it is slightly higher. In other words, the pension funding ratio has not yet returned to pre-crisis levels.

A balanced funding ratio shall be maintained being at least one, but preferably more. In other words, assets should at least redeem liabilities. Along the time horizon, the investment return rate affects the assets, and the discount rate is used to value the liabilities. If these two rates are balanced, there is little worry about the funding ratio. Ideally, this could be realized by investing in only fixed-income securities (e.g., bonds) with duration matching the liabilities. In this case, the bond yields on the asset side correspond to the discount effects on the liability side. Because the same rates are used on both sides, the assets and the liabilities are balanced, and the funding ratio remains constant in real terms. However, most pension funds have the ambition to offset the potential loss caused by inflation. To achieve this goal, the fund might want to include stocks as a part of the asset portfolio in order to improve the investment return. As a consequence, the asset side is affected by both bond yields and stock returns. While the liability side is affected only by bond yields, the balance between assets and liabilities is no longer guaranteed. In such a case, in order to forecast the funding ratio, we do need to model stock returns and bond yields.

[1] Source: the Dutch central bank, DNB.

The needs to model stocks and bonds and make forecast raise worries about model uncertainty. We classify model uncertainty into two types. One is parameter uncertainty, and the other is misspecification uncertainty. The former type can be eliminated by increasing the data sample size, if the nominal model would be correctly specified. The estimated parameters are usually evaluated by a classical confidence interval. And a typical prediction interval for the funding ratio forecast can be constructed to indicate where the true outcome is expected to fall with a given significance level. The latter type, however, results from model misspecification. It refers to the situation when the nominal model is essentially misspecified from the true data generating process (DGP).[2] In this situation, simply increasing the sample size cannot eliminate the estimation discrepancy between the nominal model and the true DGP. Put differently, when a nominal model is statistically indistinguishable from the true DGP, only parameter uncertainty exists given a finite sample. But if the nominal model is too far from the true DGP, reasonably misspecification uncertainty appears as well, and considering only parameter uncertainty is no longer sufficient. A good forecast is expected to take into account both types of model uncertainty.

In this paper, we choose to ignore parameter uncertainty, because parameter uncertainty can be eliminated eventually by enlarging the sample size. In the big data era, taking into account model misspecification becomes comparatively more imperative, because the impact of parameter uncertainty could be mitigated with more data information, while model misspecification uncertainty could not. Moreover, in practice, simpler and easier-to-implement models are preferred, even though they might fit the data poorly. This implies that misspecification uncertainty is inevitable in many circumstances. And this is what concerns us the most.[3]

To sum up, when modelling the returns of stocks and bonds, the model uncertainty inevitably affects the assets, the liabilities, and the funding ratio. To deal with this issue, we incorporates model uncertainty in forecasting the funding ratio. In particular, we focus on the impact of misspecification uncertainty on forecasting bonds, because bonds are the fundamental ingredients affecting both the assets and liabilities, and eventually the funding ratio. The impact of model uncertainty on the funding ratio is considered as transmitted from that on bonds only. We do not consider the model uncertainty of any other elements that could be modelled, such as change in population, survival probabilities, stock returns, etc.

[2]In terms of sample, one could consider situation that the nominal model is statistically distinguishable from a pseudo true DGP. "Statistically distinguishable" is the opposite of "statistically indistinguishable" which is used to describe models performing equivalently well based on some statistical criteria, e.g., a test constructed on their log-likelihood functions for a finite sample.

[3]In small samples, parameter uncertainty might play a role. In such a case, it can be incorporated straightforwardly, although finding the (approximate) sampling distribution might be somehow complicated.

The key to construct a prediction interval incorporating misspecification uncertainty is a misspecification interval. The underlying idea of a misspecification interval is that the expectation, or the forecast, might be derived from models alternative to the nominal one, and it indicates the range of such alternative expectations. In practice, the size of the misspecification interval depends on the discrepancy between the nominal model and the so-call pseudo true DGP, a selected model performing the best empirically. If the nominal model would be far from the true one, the expectation range would be large. Therefore, incorporating a misspecification interval to form a prediction interval makes it more powerful to tell where the true outcome might fall.

In this paper, we consider a class of affine term structure models (ATSMs) to price bonds. This includes three categories: the canonical three-factor model proposed by Litterman and Scheinkman [1991], without explicitly imposing the arbitrage-free condition and modelling the evolution of the factors (e.g., VAR(1)); the dynamic Nelson-Siegel model (DNS) by Diebold et al. [2006], with the modelling of the factors but not the arbitrage-free condition; and the 3-step OLS model by Adrian et al. [2013], with both the arbitrage-free condition and modelling the evolution of the factors. For each category, we postulate some specific models different in terms of estimation approaches and factors. We take the canonical three-factor model as the benchmark nominal model, because it is simple, popular in practice, and is believed to fit the empirical data fairly well. Moreover, the simple structure also implies strong impact of model uncertainty. This could provide us with a clearer effect to analyze the impact.

The research is carried out based on monthly data of the recent 10 years, i.e., from Jan 2007 to Dec 2016. Using the historical data, all postulated models are estimated, and used to select the pseudo true DGPs. We assume that the pseudo true DGPs remain unchanged in the 10 years ahead forecast. We forecast by simulations using Monte-Carlo approach with the Kalman filter.

We set up the evolution of assets and liabilities, for the future 10 years from 2017 to 2026 and the cohort that is 65 years old in 2017 that no longer contributes and has just received its first pension payments. The liabilities are considered up to 30 years from 2017 for the surviving individuals. The survival probabilities estimates are given by the Dutch Royal Actuarial Society, and assumed unaffected by model uncertainty or future changes. Moreover, we impose indexation for inflation: the annual pensions increase yearly by the (assumed to be) constant inflation rate. The value of the initial assets is set equal to the value of the initial liabilities. The assets consist of both stocks and bonds with weights 45% and 55%, respectively. The invested bonds are zero-coupon bonds with 10-year maturity. A buy and sell operation is implemented every period to maintain the 10-year duration. For every period after paying the due pensions, all assets are reinvested. The future stock returns are simulated with mean and variance based on the historical data, while the bond yields are simulated by models. We assume no correlation between bonds and stocks. Additionally, we apply the lower bound restriction on bond yields to be -2%; any bond yield lower than -2% are set to -2%.[4] The funding ratio for each period is formed by the corresponding assets and liabilities.

We need the bond yields with maturities up to 30 years for discounting the liabilities, because the pensions are to be paid for 30 years from 2016. We specifically model the bonds with the maturities of 1-year, 5-year, 9-year, 10-year, 15-year, 19-year, 20-year, 25-year and 30-year. The bond yields with other maturities can be recovered by interpolating the yield curve formed by the modelled bond yields.

The main contribution of this paper is applying a prediction interval incorporating misspecification uncertainty to analyze the forecasts of the funding ratio. We find that this prediction interval is gradually enlarged over the forecast horizon and significantly so, as compared to the prediction interval for the nominal model. In 2026, the forecast of the funding ratio derived from the nominal model is 0.95; its 95% prediction interval predicts that the ratio could fall between 0.66 to 1.36. Yet, the one with misspecification uncertainty shows that it could be between 0.48 and 1.88. The difference between the two types of intervals is considerable and hard to ignore. In such a case, policy makers face a much larger decision interval, which suggests more caution and preparations because the funding ratio would fluctuate in a wides range.

In the sensitivity analysis, we choose another nominal model for comparison, namely, the DNS model with yield-only factors estimated by a 2-step approach. This model shows narrower intervals due to a better forecast with the extra assumption on the factors. The forecast by the nominal model is raised to 1.07, meaning that the balance is marginally maintained. The corresponding prediction interval is between 0.85 and 1.33, while the one with misspecification uncertainty is 0.81 and 1.38. The misspecification interval is substantially lessened.

Moreover, we also consider the effect of the duration of the bond portfolio. By including 1-year bonds, we reduce the duration to about 5-year, and by including 20-year bond, we increase the duration to about 15-year. In both cases, the differences between intervals without and with misspecification uncertainty are wider, as compared to the case with 10-year duration. But they are not as significant as in case of changing the benchmark nominal model to the alternative one.

The early works on model uncertainty is reviewed in the book Hansen and Sargent [2008]. They combine the ambiguity aversion concept and the max-min preferences in decision theory, and robust control theory in engineering, to study macroeconomics and finance. Decisions are made to maximize the worst case of certain alternative models. The most important approach in their framework is to construct the uncertainty set by the so-called detection error probability approach

[4] The -2% lower bound restriction on bond yields corresponds to the nowadays' negative interest rate that is not lower than -2%.

[Anderson et al., 2003]. This approach, however, by construction only includes statistically indistinguishable models. From our point of view, it addresses only parameter uncertainty, which is not sufficient to explain all uncertainties when there exists a significant discrepancy between the nominal model and the pseudo true one.

The approach in our paper is closely related to the proposal in Schneider and Schweizer [2015]. The only difference lies in the determination of the amount of model uncertainty. They apply the model confidence set (MCS) procedure [Hansen et al., 2011] to a set including a number of models, and select a subset containing indistinguishable models that fit empirically the best. The amount of model uncertainty is given by the largest divergence from the MCS. We, instead, measure the divergence straightforwardly from the best-performing one judged by a loss function. We regard this as sufficient to approximate the pseudo true DGP.

This paper is also related to Glasserman and Xu [2014], who provide an explicit routine to construct the upper and lower bound models. They apply their approach to risk management, and focus on the worst case as most studies do, while our paper takes into account both the worst and the best cases to form an interval with misspecification uncertainty. This paper contributes to the limited literature on applying such an interval.

This paper is an extension of Li [2017], in which misspecification interval is applied to analyze ATMS empirically. The extension this paper are that mainly threefold. Firstly, the interval under consideration is a prediction interval incorporating model uncertainty, rather than a simple misspecification interval.Secondly, this paper takes into account more types of ATSMs and factors, while in Li [2017] only yield-only factor models are considered. Thirdly, this paper considers a more complicated process that the impact of model uncertainty transmits, rather than only a simple-10-year annuity as an example. The transmitted impact of model uncertainty on funding ratio is of practical meanings and policy implications for pension fund management.

In terms of the pension topic, Iyengar and Ma [2010] propose a robust optimization approach for pension fund management. Unlike our paper, it looks into the problem of pension contribution of firms, and focuses more on the engineering approach to develop. Other pension-related papers considering model uncertainty include Pelsser [2011] and Shen et al. [2014], who focus on asset pricing and hedging in an incomplete market setting, respectively. Our paper pays attention to the application of a prediction interval with misspecification uncertainty, and analyzes the dynamic change for the future funding ratio.

In the following, we first discuss in Section 2 the model uncertainty in theory, the construction of a prediction interval with misspecification uncertainty. Section 3 elaborates on the evolutions of assets and liabilities. Section 4 presents the types of ATSMs taken into account, the simulation methods to forecast, and the loss function constructed for the pseudo true DGP selection. Section 5 contains the empirical analysis. In this section, we choose a benchmark nominal model and the pseudo true DGPs to construct the prediction intervals. We first analyze the bond yields, followed by an analysis of the assets, the liabilities, and the funding ratios. A sensitivity analysis is carried out for an alternative nominal model and different bond durations. Section 6 concludes.

2 The Construction of Prediction Interval with Misspecification Uncertainty

A prediction interval with misspecification uncertainty describes the range in which the true outcome might fall, with high confidence. It includes the possibility that a nominal model might be misspecified, and the outcome might be obtained from an alternative model. This section discusses our construction of this prediction interval. We firstly describe the framework to evaluate model uncertainty, and then form such a prediction interval in terms of bond yields. We extend this construction to form the prediction intervals with misspecification uncertainty, for the assets, the liabilities, and the funding ratio.

2.1 Model Uncertainty Evaluation Framework

An important step in evaluating model uncertainty is to decide a reasonable amount of model uncertainty, an amount that is sufficient to measure the discrepancy between the nominal model and the true DGP.

Anderson et al. [2003] propose to use the detection error probability approach. This approach uses empirical data to quantify the amount of model uncertainty and to form an uncertainty set, assuming that the nominal model is statistically indistinguishable from the true DGP. This assumption restricts the uncertainty set such that it accounts for parameter uncertainty only, which can be eliminated when a sufficiently large sample is available. It neglects misspecification uncertainty that cannot be eliminated only by increasing the sample size.

Relaxing the above assumption, Schneider and Schweizer [2015] propose to identify pseudo true DGP(s) first from a set of plausible models, so that we have a better idea about the reasonable amount of model uncertainty. This amount of model uncertainty can be measured by the Kullback-Leibler (KL) divergence between the nominal model and the pseudo true DGP(s), quantifying the discrepancy between them.

Given the information provided by the data, they use the model confidence set (MCS) selection procedure [Hansen et al., 2011] to identify a set of best-performing and statistically indistinguishable models from a large collection of plausible models. The MCS is regarded to represent the true DGP. Specifically, the procedure ranks the plausible models in the large set by their performances according to a loss function. The one with the largest value is considered to be the worst-performing one. Each round in the procedure tests whether it is indistinguishable from the rest, at a specified significance level. If not, it will be removed, and the procedure will be imposed on the set of the remaining models. This is repeated until no further exclusion is needed. When only one model remains, it means that the information is sufficient to distinguish a unique best performing model, and this model is simply the one with the smallest value of the loss function. When there is more than one remaining model, the amount of model uncertainty is determined by the largest divergence from the corresponding model in the set.

The MCS ensures the greatest possibility to capture the true DGP because all indistinguishable models are included. In our paper, we ignore the consideration of this greatest possibility, and assume that the information given by the available data is sufficient to rule out all models but the one with the least loss function value. The amount of model uncertainty is then measured from this unique model. This model is called the pseudo true DGP.

To sum up, we first form an uncertainty set with the KL divergence representing a proper uncertainty amount between the nominal model and the true DGP. This requires a selection from a large set of plausible models, by ranking them according to a loss function that evaluates their performances. A pseudo true DGP whose loss function value is the least is chosen to determine the KL divergence. Secondly, we form the misspecification interval, bounded by the best and the worst expectations of a quantity of interest (e.g., a bond yield), among all expectations derived from the models in the uncertainty set. The models underlying the bounds of the interval are called the “upper bound” and “lower bound” models, respectively. This constructed misspecification interval is the base to incorporate misspecification uncertainty into prediction interval.

In mathematical terms, suppose $g(X)$ is a quantity of interest being a function of a random variable X, assumed to follow the probability distribution $p_X(\Phi_X)$ with Φ_X a parameter vector. With misspecification uncertainty, one suspects this assumption of X, and claims that X may be ruled by an alternative distribution. Then we evaluate the impact of misspecification uncertainty of p_X on $g(X)$ by a misspecification interval. Here, p_X is the nominal model, whose parameter(s) in Φ_X are estimated by the nominal modelling structure and approach.

Suspecting p_X not being the true DGP, we consider an uncertainty set $\mathcal{P}_X$, which contains alternative models and tries to capture the true DGP as much as possible. Thus, constructing $\mathcal{P}_X$ needs to quantify the discrepancy between the nominal model and the pseudo true DGP given by

$$\widetilde{p}_X \in \left\{ \bar{p}_X \in \{p_{X_1}, p_{X_2}, \cdots, p_{X_k}\} \middle| l(\bar{p}_X) \geq l(p_{X_j}), j = 1, \cdots, k \right\},$$

where $l(p_{X_j})$ is the loss function of a model p_{X_j} with Φ_{X_j}, and it is likewise for $l(\bar{p}_X)$.

The discrepancy is measured by the KL divergence defined by $\kappa_j = \mathbb{E}\left(m_j \log m_j\right)$, where m_j is the Radon-Nikodym (RN) derivative defined by the ratio of the alternative model j and the nominal model, $m_j = p_{X_j}/p_X$. Specifically, for the models with a univariate normally distributed random variable, and $\Phi_X = [\mu_X, \sigma_X^2]'$ with μ_X the mean and σ_X^2 the variance of X, the KL divergence κ_j can be calculated by

$$\kappa_j = \log \frac{\sigma_X}{\sigma_{X_j}} + \frac{\sigma_{X_j}^2 + \left(\mu_{X_j} - \mu_X\right)^2}{2\sigma_X^2} - \frac{1}{2}. \tag{1}$$

Let the KL divergence κ^* denote the discrepancy between the nominal model p_X and the pseudo true DGP $\widetilde{p}_X$. The model uncertainty set $\mathcal{P}_X$ is then constructed using κ^*, given by

$$\mathcal{P}_X = \left\{ p_{X_j} | p_{X_j} = m_j \cdot p_X, \quad \text{and} \quad \mathbb{E}\left(m_j \log m_j\right) \leq \kappa^* \right\},$$

to include any model j whose KL divergence implied by the corresponding m_j is within the constraint κ^*. $\mathcal{P}_X$ can be equivalently expressed in terms of m_j by

$$\mathcal{P}_m = \left\{ m_j | \mathbb{E}\left(m_j \log m_j\right) \leq \kappa^* \right\}. \tag{2}$$

The expectation of $g(X)$ under p_X is given by $\mathbb{E}\left[g(X)\right]$. The expectation under an alternative model p_{X_j} is given by $\mathbb{E}\left[m_j \cdot g(X)\right]$. From all the expectations derived from the models in the uncertainty set, we are interested in the best and the worst cases. Subject to (2), they are obtained by the optimization problems

$$\sup_{m_j \in \mathcal{P}_m} \mathbb{E}\left[m_j \cdot g(X)\right] \qquad \text{and} \qquad \inf_{m_j \in \mathcal{P}_m} \mathbb{E}\left[m_j \cdot g(X)\right], \tag{3}$$

respectively. These two quantities form the upper bound and lower bound of the misspecification interval, providing the range within which the true expectation would fall. The corresponding models are the bound models. According to Glasserman and Xu [2014], the optimal solution of m_j is given by

$$m_j^* = \frac{\exp\left[\theta \cdot g(X)\right]}{\mathbb{E}\left\{\exp\left[\theta \cdot g(X)\right]\right\}}, \tag{4}$$

where θ is in fact the Lagrangian multiplier constraining the condition (2) and indicating the impact of misspecification uncertainty on the nominal model. When $\theta > 0$, m_j^* is used for the upper bound model, and when $\theta < 0$, it is used for the lower bound model. As seen, the parameter θ is the crucial ingredient to determine the misspecification interval. We call θ the uncertainty parameter.

Suppose $g(X) = X$ where X is a univariate random variable, and assume that $X \sim \mathcal{N}\left(\mu_X, \sigma_X^2\right)$ under the nominal model p_X. Since $p_{m_j^*} = m_j^* \cdot p_X$, substituting this in (4) shows that X under a bound model $p_{m_j^*}$ follows a normal distribution as well, with mean $\mu_X + \theta\sigma_X^2$, and variance σ_X^2, i.e.,

$$X \sim \mathcal{N}\left(\mu_X + \theta\sigma_X^2, \sigma_X^2\right). \tag{5}$$

Using the information in (5) and applying (1), the KL divergence between a bound model and the nominal model is given by,

$$\kappa_\theta = \frac{1}{2}\theta^2\sigma_X^2. \tag{6}$$

Given a nominal model with a specific variance σ_X^2, κ_θ is a function of θ.

By the construction of the problem (3), the optimal solution m_j^* exists at the boundary of the constraint (2), i.e., $\kappa^* = \mathbb{E}\left(m_j^* \log m_j^*\right)$. Thus $\kappa_\theta = \kappa^*$. After κ^* is obtained by the known distributions p_X and $\widetilde{p}_X$, we are able to solve for the uncertainty parameter θ. The bound models and expectations can then be recovered.

2.2 Prediction Interval and Misspecification Interval

In our application, the fundamental quantity of interest is a τ-maturity bond yield at time t, i.e., $g\left(y_t^{(\tau)}\right) = y_t^{(\tau)}$. The nominal model assumes that $y_t^{(\tau)} \sim \mathcal{N}\left(\mu_t, \sigma_t^2\right)$. In the absence of model uncertainty, the outcome of $y_t^{(\tau)}$ is expected to fall in the prediction interval with α significance level given by

$$PI_t\left(y_t^{(\tau)}\right) = \left[\mu_t - z_{1-\frac{\alpha}{2}}\sigma_t, \quad \mu_t + z_{1-\frac{\alpha}{2}}\sigma_t\right] \tag{7}$$

where $z_{1-\frac{\alpha}{2}}$ is the value for the $(1-\frac{\alpha}{2})$-quantile of a standard Gaussian distribution. For instance, for a 95% prediction interval, $\alpha = 5\%$, and $z_{1-\frac{\alpha}{2}} = 1.96$.

In the presence of misspecification uncertainty, we also consider the prediction intervals of the bound models, as constructed in the previous subsection, rather than that of the nominal model only.[5] The interval describing where the true outcome falls is firstly expanded by the misspecification interval accounting for misspecification uncertainty. This misspecification interval describes the range of expectations for the models in the uncertainty set. Let $\widetilde{y}_t^{(\tau)}$ denote the bond yield under the bound

[5]The construction of these bound models is based on the first moments only (as discussed in the previous subsection). The construction of these bound models, taking into account the second moments, and in case of non-normality, higher order moments, is a topic of future research.

model.[6] By (5), the distribution of a bound model is $\widetilde{y}_t^{(\tau)} \sim \mathcal{N}\left(\mu_t \pm \theta_t\sigma_t^2, \sigma_t^2\right)$, where $\theta_t > 0$ is the uncertainty parameter at time t. The misspecification interval of $y_t^{(\tau)}$ is thus bounded by the expectations of $\widetilde{y}_t^{(\tau)}$ derived from the bound models, i.e.,

$$MI_t\left(y_t^{(\tau)}\right) = \left[\mu_t - \theta_t\sigma_t^2, \;\; \mu_t + \theta_t\sigma_t^2\right], \qquad \theta_t > 0 \tag{8}$$

The prediction interval taking into account misspecification uncertainty is given by

$$MUPI_t\left(y_t^{(\tau)}\right) = \left[\mu_t - \theta_t\sigma_t^2 - z_{1-\frac{\alpha}{2}}\sigma_t, \;\; \mu_t + \theta_t\sigma_t^2 + z_{1-\frac{\alpha}{2}}\sigma_t\right], \quad \theta_t > 0. \tag{9}$$

In other words, with misspecification uncertainty, the true outcome would fall between the upper bound of the upper bound model's prediction interval, and the lower bound of the lower bound model's prediction interval, with a significance level based on the relevant intervals' construction.

2.3 The Prediction Intervals of the Funding Ratio, with Misspecification Uncertainty

The impact of misspecification uncertainty on the bond yields is transmitted to influence the assets and the liabilities, and eventually the funding ratio. Without making an assumption about the distributions for these variables, we consider their $(1-\alpha)$ prediction intervals given by the $\frac{\alpha}{2}$ and $1-\frac{\alpha}{2}$ quantiles. The prediction intervals of the assets, the liabilities, and the funding ratio based on the nominal model of $y_t^{(\tau)}$ are

$$\begin{aligned} PI_t\left(A_t\right) &= \left[Q_{\frac{\alpha}{2}}\left(A_t\right), \;\; Q_{1-\frac{\alpha}{2}}\left(A_t\right)\right], \\ PI_t\left(L_t\right) &= \left[Q_{\frac{\alpha}{2}}\left(L_t\right), \;\; Q_{1-\frac{\alpha}{2}}\left(L_t\right)\right], \\ PI_t\left(FR_t\right) &= \left[Q_{\frac{\alpha}{2}}\left(FR_t\right), \;\; Q_{1-\frac{\alpha}{2}}\left(FR_t\right)\right], \end{aligned}$$

where $Q_q(A_t)$, $Q_q(L_t)$ and $Q_q(FR_t)$ are the q-quantile of A_t, L_t, and FR_t, respectively.

When taking into account misspecification uncertainty, the prediction intervals based on the bound models of $y_t^{(\tau)}$ become

$$\begin{aligned} MUPI_t\left(A_t\right) &= \left[Q_{\frac{\alpha}{2}}\left(A_t^l\right), \;\; Q_{1-\frac{\alpha}{2}}\left(A_t^u\right)\right], \\ MUPI_t\left(L_t\right) &= \left[Q_{\frac{\alpha}{2}}\left(L_t^l\right), \;\; Q_{1-\frac{\alpha}{2}}\left(L_t^u\right)\right], \\ MUPI_t\left(FR_t\right) &= \left[Q_{\frac{\alpha}{2}}\left(FR_t^l\right), \;\; Q_{1-\frac{\alpha}{2}}\left(FR_t^u\right)\right], \end{aligned}$$

[6] It is worthy to note that the observations of $\widetilde{y}_t^{(\tau)}$ and $y_t^{(\tau)}$ are identical, but their underlying distributions are not. In this paper, a variable with $\widetilde{\cdot}$ is denoted as one that follows an alternative distribution.

where the variables with the superscript u are derived from upper bound model of $y_t^{(\tau)}$. Likewise, the superscript l is used to indicate the empirical variables derived from the lower bound model of $y_t^{(\tau)}$.

3 Assets and Liabilities

This section describes the evolutions of the assets, the liabilities, and the funding ratio. The time line starts at the end of 2016, denoted by $t = 0$. We focus on only the cohort of 65 years-old at $t = 0$. At this point, the cohort's labor salary stops, and the cohort has already received its first pension for the coming year. Thus, this first pension payment is excluded in our research.

The liabilities are considered at the end of each period after the annual pensions for the next year are paid. The liabilities sum up the discounted annual pensions to pay. The annual pensions are compensated for inflation, and paid to surviving individuals only. We ignore the condition for inflation indexation, which usually requires the funding ratio to reach a sufficiently high level so as to compensate for the inflation. We assume that the liabilities for this cohort last up to 30 years from $t = 0$.

The initial value of the assets are set to equal that of the initial liabilities, so that the funding ratio is just balanced at $t = 0$. From $t = 0$, the assets are (re)invested in bonds and stocks. The pension fund benefits only from the assets' returns[7] at $t = 1$, and pays the pension. The value of assets at $t = 1$ is considered after the payment. We follow this process for 10 years.

The funding ratio at time t, FR_t, is given by

$$FR_t = \frac{A_t}{L_t}, \quad t = 0, \cdots, 10,$$

where A_t and L_t are the assets and liabilities at time t. When $t = 0$, it is the end of the year 2016, and $t = 10$ is at the end of the year 2026. At $t = 0$, $A_0 = L_0$, and $FR_0 = 1$.

3.1 Liabilities

The liabilities at time t aggregate the pensions to be paid in the future, discounted to t. Suppose each individual alive has received the amount of pension π at $t = 0$, and he is promised to receive the same amount annually in future, in real terms. The probability for him to survive τ years and receive pensions is denoted by ${}_{\tau}p_{65,0}^{(g)}$, where g is for the gender. For instance, at $t = 0$, to receive the pension at $t = 1$, an individual has to survive at least $\tau = 1$ year, with the probability ${}_{1}p_{65,0}^{(g)}$. Suppose

[7] Because all pension members are already retired, they do not pay contribution anymore.

the population of the cohort is $N_{65,0}^{(g)}$ for the gender g. In real terms, the aggregate pensions to be paid at time t are the aggregate pensions to receive for the individuals surviving $\tau = t$ year(s), given by

$$\Pi_t = \sum_g {}_t p_{65,0}^{(g)} \cdot N_{65,0}^{(g)} \cdot \pi, \qquad t = 1, \cdots, 30.$$

Taking into account a constant inflation rate IF, the pension to pay at $t = 1$ would in fact be $(1 + IF) \cdot \Pi_1$. The same annual amount is promised for future. At $t = 2$, the amount to pay is again compensated for the inflation, actualized as $(1 + IF)^2 \cdot \Pi_2$, and this amount becomes the future promise. So on and so forth, the annual pension promised at time t to pay from next period $t + 1$ is thus $(1 + IF)^t \cdot \Pi_t$. The diagram in Figure 1 shows the flows of the liabilities for each individual alive.

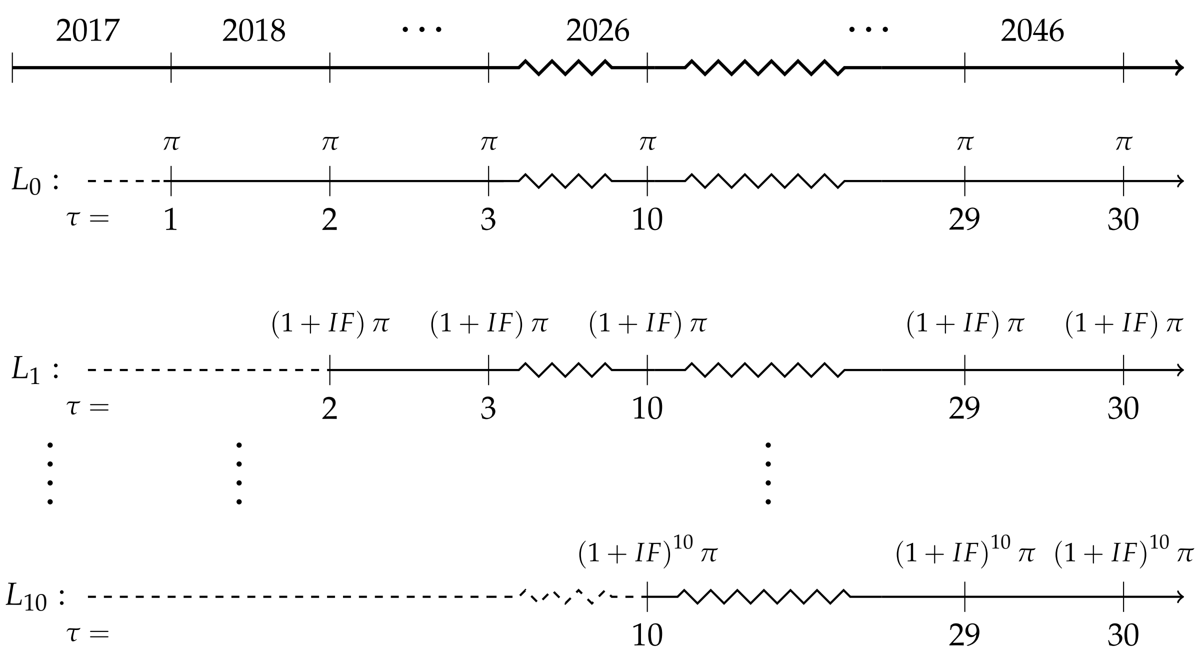

Figure 1: *Liabilities L_t for individual alive at $t = 0, \cdots, 10$*

The liabilities L_t are the liabilities after the pensions amount $(1 + IF)^t\, \Pi_t$ is paid, i.e., summing up all Π_τ for $\tau > t$ that are discounted by the τ-term interest rate $y_t^{(\tau)}$

at time t. As a consequence, it is given by

$$\begin{aligned} L_t &= (1+IF)^t \, \Pi_{t+1} \cdot \exp\left(-1 \cdot y_t^{(1)}\right) + \cdots + (1+IF)^t \, \Pi_{30} \cdot \exp\left[-(30-t) \cdot y_t^{(30-t)}\right] \\ &= (1+IF)^t \sum_{\tau=t+1}^{30} \Pi_\tau \cdot \exp\left[-\left(\tau - t\right) \cdot y_t^{(\tau-t)}\right] \\ &= (1+IF)^t \sum_{\tau=t+1}^{30} \sum_g {}_\tau p_{65,0}^{(g)} \cdot N_{65,0}^{(g)} \cdot \pi \cdot \exp\left[-\left(\tau - t\right) \cdot y_t^{(\tau-t)}\right] \\ &= (1+IF)^t \sum_{\tau=t+1}^{30} \sum_g {}_\tau p_{65,0}^{(g)} \cdot N_{65,0}^{(g)} \cdot \pi \cdot P_{b,t}^{(\tau-t)}, \qquad t = 0, \cdots 10. \end{aligned}$$

where $P_{b,t}^{(\tau-t)}$ is the price of the bond with time-to-maturity $(\tau - t)$, at time t.

In our research, we consider only the cohort of age 65 in 2016, that has already received its first year pension benefits. The reason for not looking into younger ages is because we do not need to consider the contributions on the asset side, for simplicity. And the reason for not looking into older ages is because we would like mitigate the influence, as the micro-longevity risk[8] on the survival probability ${}_\tau p_{65,t}^{(g)}$ is neglected.

We are also aware that the factor $\left({}_\tau p_{65,0}^{(g)} \cdot P_{b,t}^{(\tau-t)}\right)$ drops dramatically during the 30 years since 65 years-old, and eventually results in a small amount of liabilities afterwards. This is the reason for considering the pension to pay for only 30 years.

3.2 Assets

The value of the assets at $t = 0$, A_0, by construction satisfies the initial condition $A_0 = L_0$. It is then invested in zero-coupon bonds and stocks. In future periods, after benefiting from the investment with the return rate r_t and paying the pensions $(1+IF)^t \, \Pi_t$, the values of the assets in the next period are given by

$$A_t = A_{t-1} \left(1 + r_t\right) - (1+IF)^t \, \Pi_t, \qquad t = 1, \cdots, 10.$$

The portfolio return rate r_t is constructed as

$$r_t = w_s \cdot r_{s,t} + w_b \cdot r_{b,t}, \qquad t = 1, \cdots, 10,$$

where $r_{s,t}$ and $r_{b,t}$ are the returns of stocks and bonds, respectively, and w_s and w_b are the weights of the assets allocated to stocks and bonds, respectively. Additionally, $w_s + w_b = 1$ by assumption. The asset investment process is illustrated in the diagram in Figure 2.

[8]It refers to non-systematic causes, resulting in, for instance, a smaller and smaller population $N_{65+t,t}^{(g)}$ along the forecast horizon. See Hari et al. [2008].

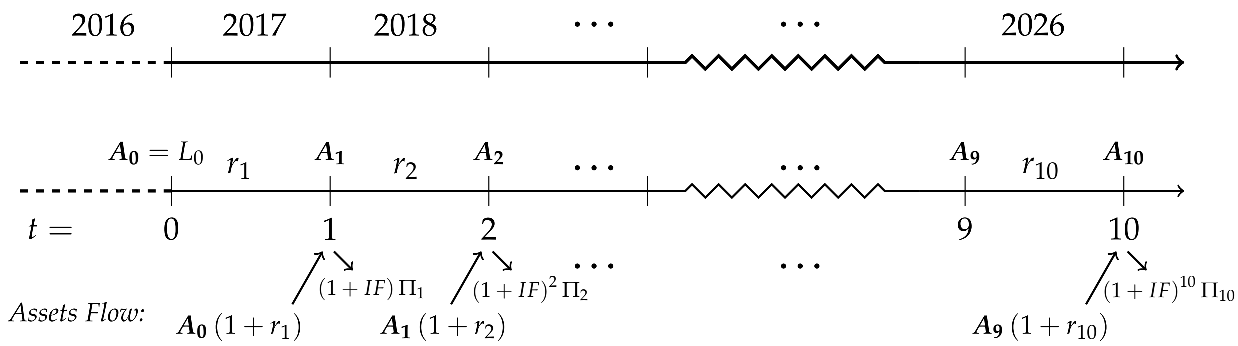

Figure 2: *Pension Assets A_t at $t = 0, \cdots, 10$*

In the Netherlands, the investment typically allocates $w_s = 45\%$ to stocks and $w_b = 55\%$ to bonds. The stock gross return $r_{s,t}$ is given by

$$r_{s,t} = \frac{I_{s,t} - I_{s,t-1}}{I_{s,t-1}}, \qquad t = 1, \cdots, 10,$$

where $I_{s,t}$ is the total return index including both the price return and the dividend return.

The investment in bonds consists of only the zero-coupon bonds with 10-year maturity at time t. It is maintained by selling them at 9-year maturity and buying new 10-year zero-coupon bonds at time $t+1$. That is to say, the duration of the bond investment is sustained as 10 years. The bonds' holding return $r_{b,t}$ is given by

$$r_{b,t} = \frac{P^{(9)}_{b,t} - P^{(10)}_{b,t-1}}{P^{(10)}_{b,t-1}}, \qquad t = 1, \cdots, 10,$$

where $P^{(9)}_{b,t}$ and $P^{(10)}_{b,t-1}$ are the prices of the zero-coupon bonds with 9-year and 10-year maturity, respectively. The prices are determined by the corresponding bond yields $P^{(\tau)}_{b,t} = \exp\left(-\tau \cdot y^{(\tau)}_t\right)$ for τ-year maturity.[9]

4 Affine Pricing Models

As discussed, bond yields are fundamental and responsible for the impact of mis-specification uncertainty on the funding ratio. They can be modeled in many ways.

[9]Alternatively, given the total return index $I_{s,t}$, and the bond yields $y^{(\tau)}_t$, one may obtain the stock gross return by $r_{s,t} = \log \frac{I_{s,t}}{I_{s,t-1}}$, and the bond's holding return by $r_{b,t} = \log \frac{P^{(9)}_{b,t}}{P^{(10)}_{b,t-1}} = 10y^{(10)}_{t-1} - 9y^{(9)}_t$.

A significant one is by the class of affine term structure models (ATSMs). Models in this class establish an affine relation between bond yields and factors. These factors could include bond yields with different maturities, macroeconomic variables, latent variables, unspanned variables, etc. Moreover, the ATSMs could be imposed with an arbitrage condition, or estimated by various approaches (e.g., maximum likelihood approach, Kalman filter approach, minimum chi-squared estimation [Hamilton and Wu, 2012], etc.), providing different estimates of the parameters and distributions of bond yields. To study the impact of misspecification uncertainty on bond yields, we include several types of ATSMs: the canonical three-factor model [Litterman and Scheinkman, 1991], the dynamic Nelson-Siegel model [Diebold and Li, 2006, Diebold et al., 2006], and the arbitrage-free model estimated by the 3-step OLS approach [Adrian et al., 2013]. In this section, we first describe the estimation of the model. The second subsection describes the performance evaluation to select the pseudo true DGP with historical data. In the end, we discuss the simulation approaches used to forecast for different types of models.

4.1 Description of Models

This part describes the setting and the estimation of the types of models under consideration.

4.1.1 The Canonical Three-Factor Model

The canonical three-factor model is proposed by Litterman and Scheinkman [1991]. It explains the bond yields using the first three principal components as the factors. Many sequential studies find it fitting the empirical data rather well; see Driessen et al. [2000], Dai and Singleton [2003], Piazzesi [2005, 2010], et al. It is popular also for its high tractability in practice. The model is postulated as

$$y_t^{(\tau)} = A_\tau + B_\tau' F_t + \varepsilon_t,$$

where $y_t^{(\tau)}$ is the yield with τ-period maturity, $F_t \in \mathbb{R}^{3\times 1}$ is the vector compiling the three factors, and ε_t is the estimation error with zero mean assumption.

The key point of this model is forming the factors in F_t, the first three principal components, from the sample. The sample data typically includes k bond yields with different maturities, including the one(s) to be priced. It is found in the literature that the first three principal components already account for at least 90% of the variation in the bond yields. Thus, it is suggested to use the first three principal components as the factors F_t, referred to as level, slope, and curvature, respectively. B_τ is a vector of the loadings for the three factors. The parameter A_τ is the intercept.

4.1.2 The Dynamic Nelson Siegel Model

The dynamic Nelson Siegel (DNS) model [Diebold and Li, 2006, Diebold et al., 2006] is an affine model without imposing an arbitrage-free restriction. This model is built on the parsimonious approximation of bond yields by Nelson and Siegel [1987]. This approximation is a linear formula associated with some latent parameters, weighted by the quantities related to the maturity τ. The DNS regards the dynamic form of the estimated latent parameters as the factors representing the level L_t, the slope S_t, and the curvature C_t, and regards the constructed τ-specific parameters constructed as the loadings B_τ. In particular, the bond yield is given by

$$y_t^{(\tau)} = B'_\tau F_t + \varepsilon_t, \tag{10}$$

where

$$B_\tau = \left(1, \quad \frac{1-e^{-\lambda\tau}}{\lambda\tau}, \quad \frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right)', \qquad F_t = \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix}.$$

Moreover, Diebold et al. [2006] assume that the factors follow a VAR(1) process. For convenience, the VAR(1) process is formulated by mean-centered factors, with the factor means μ_L, μ_S and μ_C, respectively, given by

$$(F_t - \mu) = \Psi\,(F_{t-1} - \mu) + \eta_t, \tag{11}$$

where $\mu = (\mu_L, \mu_S, \mu_C)'$, and the parameter Ψ is estimated by the dynamic factors. The disturbances ε_t and η_t are white noises, orthogonal to each other and to the initial states,[10]

$$\begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix} \sim WN\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_\varepsilon & 0 \\ 0 & \Sigma_\eta \end{pmatrix}\right], \qquad \mathbb{E}(F_0\eta'_t) = 0, \qquad \mathbb{E}(F_0\varepsilon'_t) = 0.$$

In particular, the Σ_ε matrix is assumed to be diagonal, implying that zero correlation between the bond yields of different maturities. The Σ_η matrix is assumed non-diagonal, allowing correlation between the factors. λ is a fixed parameter chosen such that the loading for C_t is maximized in the approach by Nelson and Siegel [1987].

The factors in this model may include yields only, as well as with extra macro-economic variables. In case of yield-only factors, the latent factor F_t is estimated by multiple linear regressions for a number of dependent variables $y_t^{(\tau)}$, with the designed B_τ in (10). When k macro-economic factors are included, the new factor vector $\widetilde{F}_t \in \mathbb{R}^{(3+k)\times 1}$ is complied with the estimated F_t and the macro-economic

[10]In case that ε_t and η_t are one-dimensioned, we replace Σ_ε^2 and Σ_η^2 by σ_ε^2 and σ_η^2.

variables. The new loadings matrix $\widetilde{B}_\tau$ is an augmented B_τ by appropriate rows of zeros. The estimator of the parameter Ψ for $\widetilde{F}_t$ is constructed the same way as for F_t in (11). In other words, the macro-economic factors do not affect the bond yield pricing process (10), but only the VAR(1) process (11). Such factors are called unspanned factors; see Joslin et al. [2014], Duffee [2011], Wright [2011], etc.

This model can be estimated by straightforwardly applying OLS to (10) and (11). In this approach λ is typically set to 0.0609, as in Diebold and Li [2006]. Alternatively, one may employ a one-step Kalman filter approach.When long-term liabilities greatly increase in a low interest rate, and the probability of very low yields should be modeled accurately, the Kalman filter approach is preferred.

4.1.3 The Arbitrage-Free ATSM

An arbitrage-free ATSM imposes the arbitrage-free restriction, besides modelling the evolution of the factors. It can be estimated by many approaches. Here we apply the 3-step OLS approach developed by Adrian et al. [2013].

The factors are assumed to follow a VAR(1) process,

$$F_t = \mu + \Psi F_{t-1} + \eta_t, \tag{12}$$

where μ and Ψ are the parameters characterizing the autoregressive factor vector F_t. The arbitrage-free condition rules out arbitrage opportunities, and guarantees a unique price in a complete market. This price can be used to replicate other contingent claims in the market, a property favored in the financial theory. The replication is implemented through the pricing kernel, which discounts a bond's future price such that the expected discounted price equals today's bond price. In this model, the pricing kernel is assumed to be exponentially affine to the price of risk, and the price of risk is affine to the factors. In this way, the bond yields can be expressed by the recursively determined A_τ and B_τ,

$$y_t^{(\tau)} = A_\tau + B_\tau' F_t + \varepsilon_t, \tag{13}$$

where $\varepsilon_t \sim \mathcal{N}(0, \Sigma_\varepsilon)$. All parameters, including the ones in the VAR(1) process, the pricing kernel and the price of risk, are estimated by the 3-step OLS approach. We refer to Adrian et al. [2013] for more details to the set-up and estimation.

The number of factors is not restricted. One may include any yield factors conjectured to affect the bond being priced. One can also include unspanned macroeconomic factors that affect only the VAR(1) process (12) but not the affine pricing process (13), by adjusting the model structure and estimation as appropriate. In this paper, we take into account a number of such models containing a variety of factors.

4.2 Evaluation of the Models

We evaluate a large set of plausible models that includes the three types as described, in order to select the best-performing one to be the pseudo true DGP. Using historical data, a model's performance is judged according to a loss function that compares the historical realizations and estimates.

A typical loss function is built on the likelihood function. In this paper, for a sample Z collecting T-length time series of a bond yield with τ-maturity $\left(y_1^{(\tau)}, \cdots, y_T^{(\tau)}\right)$, the conditional quasi-log-likelihood function is

$$\log l(\phi) = \frac{T}{2}\log\sigma_\varepsilon^2 - \frac{1}{2}\sum_{t=1}^{T}\frac{\left(y_t^{(\tau)} - A_\tau - B_\tau' F_t\right)^2}{\sigma_\varepsilon^2},$$

where $\phi = \{A_\tau, B_\tau, \sigma_\varepsilon^2\}$. The loss function is constructed by

$$Q(Z,\phi) = -2\log l(\phi) = T\log\sigma_\varepsilon^2 - \sum_{t=1}^{T}\frac{\left(y_t^{(\tau)} - A_\tau - B_\tau' F_t\right)^2}{\sigma_\varepsilon^2}.$$

A smaller $Q(Z,\phi)$ indicates better performance, and closer to the true DGP. We rank the values of the loss function given by all the models considered. The model with the least loss function value is regarded as the best-performing one, and used as the psuedo true DGP.

4.3 Simulation of the Models

Based on a model, the prediction of bond yields is implemented by simulation. Given the types of model under consideration, we apply two simulation methods, depending on the imposed conditions.

The first simulation method is historical simulation. It is applied to simulating the canonical three-factor model. This model imposes no assumption on the evolution of the factors, and no arbitrage-free condition. In other words, we know little about the distribution of the factors. Therefore, one possible simulation method is the historical simulation, i.e., drawing samples with replacement from the historical data.

More precisely, the historical factors are assumed to satisfy

$$F_t = \mu + \eta_t,$$

where μ is the expectation of the historical data F_t, estimated by $\widehat{\mu} = \frac{1}{T}\sum_{t=1}^{T} F_t$, and $\eta_t = F_t - \widehat{\mu}$ is the error following the empirical distribution. In practice, simulating F_t is simply drawing with replacement from the historical record. Suppose the simulated factors are $\widetilde{F}_t$. Then the simulated bond yields $\widetilde{y}_t^{(\tau)}$ can be calculated with the estimated characterizing parameters $\widehat{A}_\tau$ and $\widehat{B}_\tau$, i.e., $\widetilde{y}_t^{(\tau)} = \widehat{A}_\tau + \widehat{B}_\tau' \widetilde{F}_t$. Since the simulate is based on a random draw, no initial values are assumed.

For other types of models, the factors' behavior is assumed. Therefore, we can forecast by the Monte-Carlo approach. We use the Kalman filter in the simulation. Using the estimated characterizing parameters pertaining to the affine bond yield and VAR(1) factors processes, this approach allows us to simulate bond yields and factors together in one step. The initial value to simulate bond yield is the end-period value in the sample. Simulating N paths, the forecast is simply the mean of the simulated paths.

5 Empirical Analysis

Considering that we need the information of bond yields with maturities up to 30 years for the liabilities' analysis, and particularly the maturities of 9 years and 10 years for the assets' analysis, we choose to model the bond yields with maturities of 1-year, 5-year, 9-year, 10-year, 15-year, 19-year 20-year, 25-year, and 30-year. The bond yields with other maturities are recovered by interpolating the yield curve formed by the modelled bond yields.

This section first describes the basis data information, followed by determining the pseudo true DGPs and the prediction intervals of bond yields when misspecification uncertainty is incorporated, and comparing with the prediction intervals without such uncertainty. The next part analyzes the impact of misspecification uncertainty transmitted to the assets, the liabilities, and the funding ratio. In the last part, we conduct a sensitivity analysis, using an alternative nominal model and adjusting the duration of the bond investment.

5.1 Data Description

Since we model bond yields, the data needed are the data for factors, including bond yields and macroeconomic variables. Stock returns are needed as well, to predict the investment returns. We collect the recent 10 years' data, i.e., from January 2007 until December 2016. The data are of monthly frequency, with 120 observations in total.

We assume that the yield factors are the European AAA bond yields, whose information is collected in terms of the Nielson-Siegel-Svensson (NSS) parameters from the European Central Bank (ECB). The NSS parameters allow us to generate the annualized AAA bond yield for any maturity. Macroeconomic factors include the Dutch CPI, OECD year over year GDP growth rate, and the ECB fixed rate tenders used for the main refinancing operations (MRO). These variables represent inflation, economic growth and monetary policy, which are macro-economic indexes. We assume that the consumption in retirement occurs in the domestic market only, and is thus affected by the Dutch CPI. We consider OECD year over year GDP growth rates instead of the Dutch GDP, because the bond yields to be modelled are affected by more than just the Dutch economy. The raw data of the fixed rate tenders are documented from Oct 2008; missing observations are replaced by zero in the ECB data. The stock return is obtained by the STOXX 600 total return index.

Table 1 reports the descriptive statistics for part of the data ready to apply. It does not report all the NSS-estimated bond yields to be used as factors, but only those with the same maturities for the bonds to be modelled. We use 44 bonds with different maturities including the reported ones, to construct the principle components for the factors in the canonical three-factor model. These bonds are with maturities of 1, 3, 6, 9, 10, 15, 18, 21, 30, 40, 50, 80, 90, 110 months, and 1 year to 30 years.

Figure 3a plots the reported NSS-estimated yields, and Figure 3b shows the historical data of the stock return and the macroeconomic variables. The NSS-estimated bond yields generally decrease. The 1-year yield fell below zero since 2013, reaching almost -1% in the recent months. The 5-year bond yield also fell below zero since 2015. The plot for the stock return shows fluctuations around a mean value. We assume the stock return normally distributed.

5.2 The Prediction Intervals of the Bonds, with Misspecification Uncertainty

To incorporate misspecification uncertainty into the prediction interval of a specific maturity bond yield for the next 10 years ahead, we firstly determine the pseudo true DGP for this bond using the past 10 years' information, and assume the pseudo true DGP remains unchanged for the future 10 years. We determine the amount of model uncertainty and uncertainty parameters in the second part. The uncertainty parameters allow us to incorporate misspecification uncertainty into the prediction interval for this bond. The same implementation is carried out for the other specific maturities bonds.

5.2.1 The Pseudo True DGPs and the Benchmark Nominal Model

The pseudo true DGP for a specific maturity bond is selected from a large set of empirically relevant models, namely the ones described in Section 4.1. The factors of the first three principle components in the canonical three-factor model are constructed by the 44 bond yields mentioned earlier. For the DNS model using the yield-only factors, the factors are the reported bond yields. For the DNS model using the macro-yield factors, they are the reported bond yields and macroeconomic variables.

For arbitrage-free ATSMs, some factors are only yields by choice, and some includes macroeconomic variables as well. The recent 10-year data are used to estimate the parameters for each empirically relevant model in the selection set. The value of the loss function for each model is obtained by the estimates of these parameters.

Table 2 collects 45 empirically relevant models, and reports the rankings of their performances by the loss function values in descending order. The pseudo true DGP for each maturity is ranked in the last place. The pseudo true DGPs for all the maturities are assumed unchanged in the 10 years forecast. All the pseudo true DGPs are trending upward in prediction, indicating mean-reverting processes. Figure 4 illustrates such a process, by including the historical data and the forecast using the DNS models estimated by Kalman filter.

We choose the canonical three-factor model as the benchmark nominal model to model bond yields. The first main reason is that it is popular and easy to implement. It fits empirical data fairly well, and typically explains more than 90% of the variation in the bond yields. The second reason is that the considerable model uncertainty implied by its simple structure could help in studying the impact of model uncertainty. This model imposes no assumption on the evolution of the factors, and no arbitrage-free condition. As a consequence, it might allow for considerable misspecification uncertainty, and cause problematic forecast.

The second reason can be verified by recalling the method to forecast by this model. The employed historical simulation method draws an observation with replacement from the historical data for each period and each simulation path. The forecast of each period is simply the mean of the drawn observations for each period in all paths. As long as the number of paths is large enough, the forecast is constant over time. Hence, there could be a big discrepancy from the mean-reverting pseudo true DGP to this nominal model in forecasting. Consequently, we expect to find large amount of model uncertainty κ_t^*, large values of θ_t and wide prediction intervals with misspecification uncertainty, providing a clearer result to study the impact of model uncertainty.

In other words, the first reason represents practitioners' preference, while the second reason fits our study purpose.

5.2.2 The Prediction Intervals of Bond Yields with Misspecification Uncertainty

We apply monthly data, and predict monthly the bond yields, their prediction intervals without and with misspecification uncertainty respectively, for the future 10 years.

As mentioned, the bond yield for each specific maturity τ is modelled and forecasted individually. The forecast of the bond yield $y_t^{(\tau)}$ is based on $N = 50,000$ simulation paths of 120 months ahead starting from Jan 2017, using the data in Dec 2016 as the initial values. With the variance of the simulations for each period t, the prediction interval for the nominal model $PI_t\left(y_t^{(\tau)}\right)$ can be recovered by (7) .

The prediction interval of $y_t^{(\tau)}$ with misspecification concern, $MUPI_t\left(y_t^{(\tau)}\right)$, combines the misspecification interval and the prediction intervals of the bound model models. It is constructed by the uncertainty parameter θ_t, the forecast μ_t and the forecast variance σ_t^2. The uncertainty parameter θ_t implies the impact of the model uncertainty amount κ_t^* at time t, between the nominal model and the pseudo true DGP for τ-maturity bond. As κ_t^* is obtained by (1) with the forecasts and the variances of the nominal model and the pseudo true DGP, θ_t is recovered as well by (6), given the condition $\kappa_t^* = \kappa_{\theta_t}$.

Table 3 reports the estimates of forecasted yield for each year, $\widehat{\mu}_t$, and the standard deviation $\widehat{\sigma}_t$. The former is generally increasing over maturity, while the latter is decreasing. But both of them remain almost stable over time. Table 4 reports the estimates $\widehat{\kappa}_t^*$ and $\widehat{\theta}_t$ for the end of each year. Because the pseudo true DGP for each maturity remains the same for the whole forecast horizon, we can compare the size of model uncertainty over time. For instance, for the 1-year maturity bond, the amount of model uncertainty $\widehat{\kappa}_t^*$, decreases dramatically until 2022, and then increases slightly until the end of the horizon. Since $\widehat{\sigma}_t$ is fairly stable over time, correspondingly $\widehat{\theta}_t$ shows the same pattern as $\widehat{\kappa}_t^*$. For the 30-year maturity bond, $\widehat{\kappa}_t^*$ is too small to observe model uncertainty. Thus, the resulting prediction interval with misspecification uncertainty is supposed to be almost identical to that without misspecification uncertainty. $\widehat{\kappa}_t^*$ and $\widehat{\theta}_t$ are remarkably large in the near future and for shorter maturities, but much smaller in the further future and for longer maturities.

The $PI_t\left(y_t^{(\tau)}\right)$ and $MUPI_t\left(y_t^{(\tau)}\right)$, are obtained by (7) and (9) respectively. Since $\widehat{\theta}_t$ is generally decreasing over time and over maturity, the misspecification interval width determined by $\widehat{\theta}_t\widehat{\sigma}_t^2$ shrinks over time and over maturity. So is the $MUPI_t\left(y_t^{(\tau)}\right)$.

The resulting $PI_t\left(y_t^{(\tau)}\right)$ and $MUPI_t\left(y_t^{(\tau)}\right)$ are plotted in Figure 5. Figure 5a and Figure 5b show the structure of bond yields from the perspective of maturity and time respectively. The dashed lines indicate the $MUPI_t\left(y_t^{(\tau)}\right)$, while the dotted lines indicate the $PI_t\left(y_t^{(\tau)}\right)$. The gap between them is the impact of misspecification uncertainty. We see that the misspecification impact in the near future is stronger, causing the prediction interval with misspecification uncertainty wider than that of the nominal model. As $\widehat{\kappa}^*$ becomes more insignificant over time, $MUPI_t\left(y_t^{(\tau)}\right)$ becomes closer to $PI_t\left(y_t^{(\tau)}\right)$; see the 2026 plot in Figure 5b. These observations are in line with the implications from combining the results in Table 3 and Table 4.

As discussed, the forecast of the nominal model is obtained by historical simulation. Thus, it is a constant, being the mean of historical data. The forecast of the pseudo true DGP, capturing the mean-reverting trend, is time-varying with a low level of initial yield. Hence there will be a big discrepancy at the beginning. As the yield given the pseudo true DGP picks up and recovers the historical mean level in the further future, the discrepancy shrinks, and therefore there is less and less misspecification uncertainty in the longer run.

5.3 Funding Ratio Analysis

This section firstly focuses on analyzing the funding ratio using the benchmark nominal model, namely the canonical three-factor model. Then, we implement a sensitivity analysis considering two aspects. One uses a different nominal model, the DNS model with yield-only factors. Another one changes the duration in the bond investment to be about 5 years and 10 years.

5.3.1 The Impact of Misspecification Uncertainty

The simulations of the nominal model and the bound models are used to simulate the assets, the liabilities, and the funding ratio. Their 95% prediction intervals are decided by the 2.5% and 97.5% quantiles.

In simulations, we set a lower bound constraint for the bond yields to be -2%. Whenever the modelled bond yield is lower than -2%, the rate applied to in assets and liabilities is -2%. This constraint mainly works on the pseudo true DGP(s), particularly at the beginning of the forecast horizon on the lower bounds and for shorter maturity bonds. This is because of the low initial values in the simulations. It is not likely to constrain later as the mean-reverting force drives the yields upwards.

Table 5 reports the outcomes of the forecast, the 95% prediction interval of the nominal model, and the prediction intervals incorporating misspecification uncertainty. Figure 7a plots these outcomes.

Firstly, the forecasted value of assets decreases constantly. The 95%-quantile prediction interval for the nominal model is wider at the beginning, but becomes narrower at the end of 2017. This means that the variance of the assets under the nominal model becomes smaller. Secondly, $MUPI_t(A_t)$ behaves similarly, but there is another reason resulting in the wider interval in the beginning. The misspecification uncertainty is stronger at the beginning, seen in the 9-year maturity and 10-year maturity in Figure 5a. Therefore, the return rates based on the bound models are more distant from those based on the nominal model, and result in a wider misspecification interval in the beginning. Thirdly, the gap between $PI_t(A_t)$ and $MUPI_t(A_t)$ is almost the same in the further future. This means that the assets, given by the nominal model and the bound models, move at almost the same rate. This rate is mainly affected by the bond return $r_{b,t}$, associated with the difference between the 9-year and 10-year bond yields. There is a big difference between these two yields due to the significant misspecification uncertainty at the beginning; see the graphs of 9-year and 10-year bond yields in Figure 5a. Therefore, the rate is not the same. As the misspecification uncertainty becomes insignificant, their difference becomes stable, and thus $r_{b,t}$ given by the nominal model and the bound models, becomes the same. At last, the prediction intervals are positively skewed.

The liabilities at time t involve the bonds with $(30 - t)$ years to maturity. The misspecification intervals of the bond yields become narrower over time. Together with the discounting effect, the impact of misspecification uncertainty on the liabilities becomes almost unobservable. Both $PI_t(L_t)$ and $MUPI_t(L_t)$ gradually become narrower, due to a smaller variance of the liabilities under the nominal model. Like the asset side, the interval is positively skewed.

The forecast of the funding ratio decreases slowly, and ends up at 0.95 in 2026. The funding ratio might not be sufficient to pay the future liabilities. However, the funding ratio is positively skewed, implying a potential upward trend. Additionally, the $PI_t(FR_t)$ in 2026 tells that the funding ratio may be as low as 0.66. Taking into account misspecification uncertainty, the lower bound of $MUPI_t(FR_t)$ drops to 0.51. The gap is as large as 0.15. On the upper bound side, the difference is even larger. The upper bound of $PI_t(FR_t)$ is 1.36, while the one with misspecification uncertainty might become even 1.88. The difference is as substantial as 0.55, implying that the impact of misspecification uncertainty is non-negligible when using the canonical three-factor model as the nominal model.

5.3.2 Sensitivity Analysis

This part implements the sensitivity analysis. The analysis in terms of the canonical three-factor model shows a substantial impact of misspecification uncertainty on the funding ratio, when the investment strategy is simply buying and selling the 10-year and 9-year bonds, respectively. We would like to conduct a sensitivity analysis to see what would change if using a better structured model and different durations.

5.3.2.1 An Alternative Nominal Model

The DNS model imposes an extra assumption on the evolution of the factors. This helps in performing better in-sample and out-of-sample. The forecast in Figure 3a is made by this DNS model, showing mean-reverting trends, like the pseudo true DGPs. Consequently, we do not expect the same large amount of model uncertainty as in using the benchmark nominal model; see earlier discussion. When the nominal model is replaced by a better structured model, the both prediction intervals, without and with misspecification uncertainty, are expected to shrink.

Figure 6 plots the forecasts of the bond yields, their prediction intervals of the alternative nominal model and the bound models, using the DNS model with yield-only factors. We see more impact of misspecification uncertainty is decreasing over maturity, but increasing over time, unlike the case using the benchmark nominal model which is decreasing over time only.

Table 6 reports the forecasts of the assets, the liabilities, and the funding ratio, as well as their prediction intervals with and without misspecification uncertainty. Figure 7b shows the corresponding graphs. We find that for the assets and liabilities, the prediction intervals with and without model uncertainty are very close. The difference is more obvious in the funding ratio. The difference grows larger over time, but not as dramatically as in using the benchmark model.

The forecasted funding ratio ends with 1.07 at the end of 2026. Compared to those under the benchmark nominal model, both the prediction intervals with and without misspecification uncertainty are narrower, indicating more precise estimation. Like the benchmark model, both the prediction intervals are positively skewed, implying a force of moving upwards. But the median is almost the same as the mean.

However, the impact of the misspecification uncertainty is still significant for the further future. The interval difference is 0.05 for the upper bounds, and 0.04 for the lower bounds. These are not easily neglected amounts, in particular, in case that the pension fund has not completely recovered from the financial crisis.

5.3.2.2 Investment with Other Durations

We also analyze the impact when changing the duration in the bond investment. We investigate two cases, shortening the duration, and lengthening it. In both cases, the liabilities are not affected. The duration change concerns only the asset side, and hence affects the funding ratio.

In the first case, we allocate 30% of the assets to 1-year bonds, and keep 25% to invest in 10-year bonds. The duration becomes approximately 5 years. The 1-year bonds mature the next period, increasing the liquidity of the pension fund. We apply this investment strategy to both the benchmark and alternative nominal models. The results are reported in Table 7 and Table 9, and plotted in Figure 8 and Figure 10. We see that the intervals become wider for both nominal models. This is mainly because more misspecification uncertainty is embedded in the 1-year bonds; see the graphs in fFigure 5 and 6. The differences are not particularly remarkable, although the changes for the benchmark nominal model is relatively larger than those for the alternative nominal model. Other features remain almost the same as in the original strategy.

The second case considers investing 27% of the assets in 20-year bonds. The allocation to stocks remains the same, and the rest are invested in 10-year bonds. The next period, the bonds are sold and replaced by new bonds with the designed maturities. The duration of the bond investment now becomes about 15 years. The results are reported in Table 8 and Table 10, and plotted in Figure 9 and Figure 11, respectively. They show that the misspecification interval becomes narrower.

This sensitivity analysis shows that longer duration of bond investment provides narrower prediction intervals, due to smaller variance for longer term bonds and insignificant misspecification uncertainty.

6 Summary and Conclusions

This paper studies the impact of misspecification uncertainty on predicting the funding ratio of a Dutch defined benefit pension fund 10 years ahead from 2017 to 2026. The ambition of the pension fund management to offset the inflation and thus to invest in stocks breaks the balance of the funding ratio. Bond yields become crucial to affect the funding ratio through both the asset and liability sides. They could be determined by a number of models, but some simple though clearly misspecified ones are typically preferred in practice. This situation motivates the need to take into account model uncertainty. In this research, we ignore the model uncertainty caused by stock return and survival probabilities. Moreover, when modelling bond yields, we only focus on misspecification uncertainty, but ignore parameter uncertainty.

We assume that the investor prefers the canonical three-factor model to predict bond yields; this model does not model the evolution of the factors, or impose the arbitrage free condition, but is simple and preferred. We also apply a lower bound constraint on the bond yields to stay in line with the real financial market. The empirical analysis shows that the impact of misspecification uncertainty on the bond yields is stronger for short term bonds and the near future. The transmitted impact of misspecification uncertainty on assets, liabilities, and funding ratio is significant. The funding ratio over time is below 1. The prediction interval with misspecification uncertainty can be up to 0.67 wider than the one for nominal model.

The sensitivity analysis is carried out to investigate two aspects. One is to use a better structured model as an alternative nominal model, and the other is to alter the duration of the bond investment. The first analysis dramatically reduces the impact of misspecification uncertainty, and results in a funding ratio above 1 in the forecast. The misspecification interval is reduce to 0.09. The second analysis shows that the influence of the bond investment duration is limited. This sensitivity analysis implies that the fundamental reason for a large misspecification interval of the funding ratio is the poor prediction under a nominal bond yield model.

The bound models in this paper are constructed based on the first moment of a bond yield, in case of normality. They can be constructed based on the second moment as well. In case of non-normality, higher moments are also needed to characterize the bound models. These aspects can be the topics of future research.

Last but not least, this paper studies the impact of misspecification uncertainty essentially on bonds. The impact on the funding ratio is in fact considered as transmitted from the bond investments in the economy. We do not study the impact on the funding ratio in terms of its distribution, and thus the results are not a straightforward impact of model uncertainty on funding ratio. Moreover, for simplicity, model uncertainty from other economic elements, such as stock returns, demographic change, mortality rates, etc., are not considered. The purpose of this paper is to show the significance of taking into account misspecification uncertainty, and to propose the incorporation of a misspecification interval. These mentioned discrepancies could be improved with extensions, depending on research interests.

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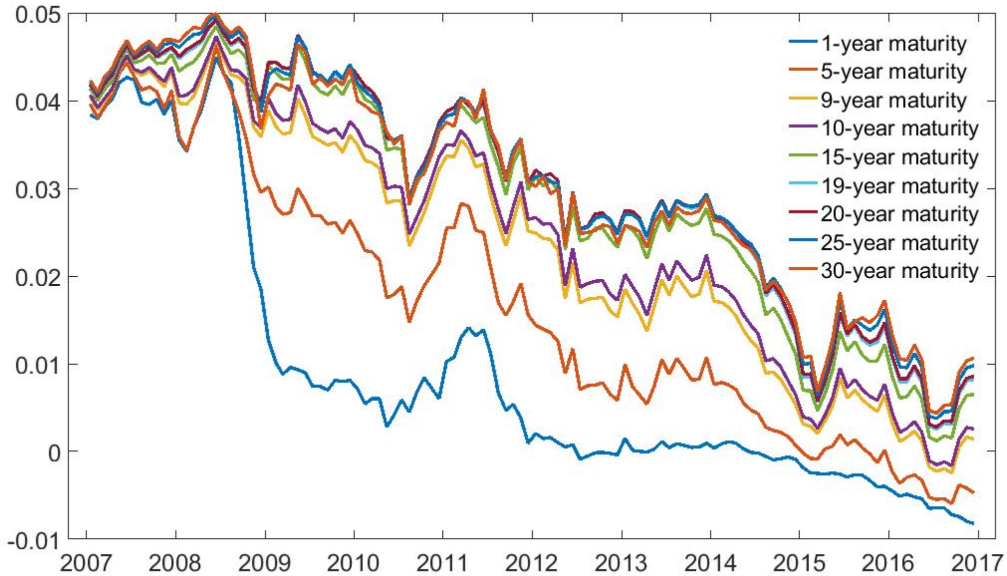
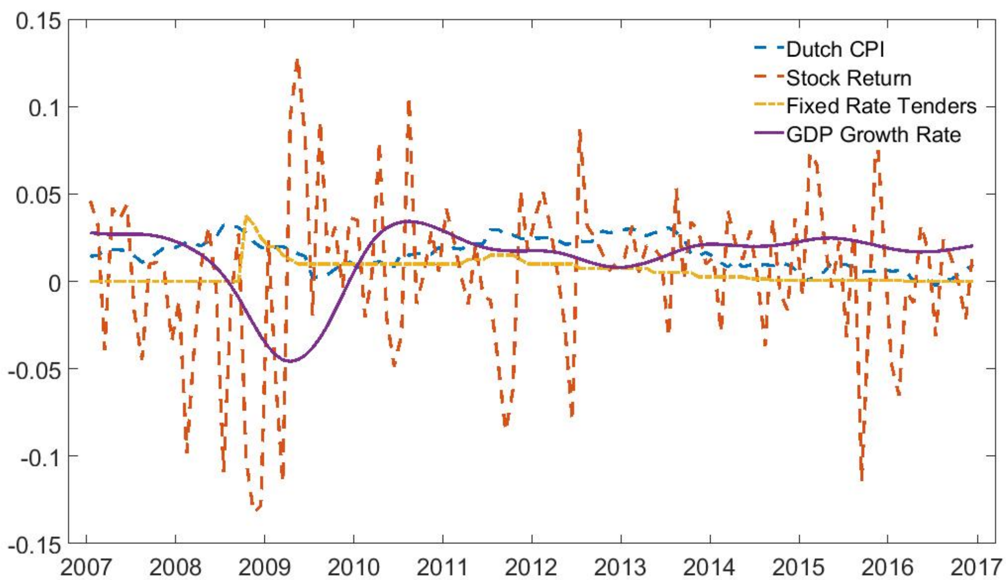
Appendix A Tables and Figures

Table 1: *Data Description. The first panel reports the NSS-estimated yields with τ-year maturity, while the second panel reports the other variables. "STOXX" is short for the stock return, "NLCPI" the dutch CPI, "FRT" the fixed rate tenders, and "GDP Growth" the OECD economic growth rate. The data spans from Jan 2007 to Dec 2016 in monthly frequency.*

variables	mean	std	min	max
		NSS Yields with τ-year maturity		
1-year	0.0091	0.0154	-0.0082	0.0449
5-year	0.0167	0.0150	-0.0059	0.0463
9-year	0.0235	0.0144	-0.0024	0.0470
10-year	0.0247	0.0143	-0.0016	0.0473
15-year	0.0287	0.0138	0.0012	0.0485
19-year	0.0301	0.0134	0.0025	0.0491
20-year	0.0303	0.0134	0.0028	0.0492
25-year	0.0306	0.0131	0.0037	0.0496
30-year	0.0303	0.0128	0.0044	0.0499
		Other Variables		
STOXX	0.0039	0.0476	-0.1324	0.1285
NLCPI	0.0158	0.0089	-0.0024	0.0323
FRT	0.0059	0.0067	0.0000	0.0375
GDP Growth	1.3423	1.8529	-4.5800	3.4300



(a) *NSS-estimated bond yields*



(b) *Historical data of stock returns and macro variables*

Figure 3: *Descriptive plots of the bond yields, stock returns and other variables. The horizon is from the beginning of 2007 to the beginning of 2017.*

Table 2: *The model rankings according to the values of the loss function.*

Index	Model	Macro	τ-year Maturity								
			1	5	9	10	15	19	20	25	30
1	***y3F***	0	42	42	42	42	42	42	42	42	42
2	yYOSSM	0	43	44	44	44	43	39	43	43	43
3	***yYO2S***	0	45	45	43	43	44	43	44	44	44
4	yYMCRSSM	1	44	43	45	45	45	38	45	45	40
5	yYMCR2S	1	25	41	41	41	41	45	39	39	36
6	1/12	1	41	39	39	39	39	44	38	41	41
7	5	1	24	34	35	33	35	40	41	38	45
8	10	1	9	37	33	35	40	35	40	32	38
9	15	1	29	35	34	34	37	37	35	35	35
10	25	1	10	40	37	19	38	32	33	36	32
11	[1/12,5]	1	30	33	40	40	34	33	32	33	34
12	[25,29]	1	32	36	19	38	33	19	34	34	33
13	[10,12.5]	1	8	26	38	37	21	34	37	22	39
14	[1,5,7.5]	1	28	38	23	23	23	25	19	40	37
15	[7.5,12.5,16.7]	1	12	6	25	25	32	24	25	37	22
16	[7.5,20,29]	1	40	12	13	13	19	21	24	21	10
17	[1/12,5,7.5,1/120]	1	27	32	31	24	22	22	21	30	30
18	[5,10,15,20]	1	7	9	36	22	25	26	22	19	25
19	[10,20,25,35]	1	33	29	22	31	15	41	9	25	23
20	[5,7.5,10,20,35]	1	39	30	24	6	24	9	29	10	12
21	[1/12,10,15,20,25]	1	36	10	6	26	18	6	6	29	29
22	[7.5,12.5,15,20,25,35]	1	13	8	26	17	6	30	10	9	9
23	[1/12,5,15,16.7,25,35]	1	19	28	7	21	26	29	26	12	19
24	[1/12,5,7,100,12.5,16.7,20.8]	1	16	31	30	18	36	23	30	6	31
25	[1/12,5,10,15,20,25,35]	1	35	4	27	7	31	36	23	26	28

To be continued

Table 2 continued

Index	Model	Macro	τ-year Maturity								
			1	5	9	10	15	19	20	25	30
26	1/12	0	15	13	12	27	7	10	31	28	6
27	5	0	26	19	29	36	28	15	28	31	26
28	10	0	31	2	9	30	13	7	7	8	8
29	15	0	38	11	32	11	27	8	8	11	11
30	25	0	37	16	11	12	11	11	11	7	7
31	[1/12,5]	0	18	27	10	29	8	27	27	27	14
32	[25,29]	0	11	7	16	15	12	31	15	23	27
33	[10,12.5]	0	22	3	8	9	10	28	14	14	13
34	[1,5,7.5]	0	6	5	28	32	14	14	36	17	17
35	[7.5,12.5,16.7]	0	21	21	17	10	9	12	12	24	2
36	[7.5,20,29]	0	20	22	15	8	3	3	3	13	4
37	[1/12,5,7.5,1/120]	0	34	15	21	28	5	5	5	2	20
38	[5,10,15,20]	0	14	23	3	16	29	17	17	4	3
39	[10,20,25,35]	0	17	25	5	3	30	13	13	15	5
40	[5,7.5,10,20,35]	0	23	17	1	5	2	16	16	3	15
41	[1/12,10,15,20,25]	0	1	18	20	20	4	1	4	5	1
42	[7.5,12.5,15,20,25,35]	0	3	1	18	1	1	20	1	1	18
43	[1/12,5,15,16.7,25,35]	0	5	24	14	14	17	18	20	18	16
44	[1/12,5,7,100,12.5,16.7,20.8]	0	4	20	4	2	16	4	18	16	24
45	[1/12,5,10,15,20,25,35]	0	**2**	**14**	**2**	**4**	**20**	**2**	**2**	**20**	**21**

Notes: "*y3F*" is short for the canonical three-factor model, "*yYOSSM*" the DNS model estimated by Kalman filter approach with yield factors only, "*yYO2S*" the DNS model estimated by 2-step approach with yield factors only, "*yYMCRSSM*" the DNS model estimated by Kalman filter approach with yield factors and macroeconomic factors, "*yYO2S*" the DNS model estimated by 2-step approach with yield factors and macroeconomic factors. Other models are estimated by 3-step OLS, using only the yield factors with the maturities of the number of *years* indicated in the vectors. The first column provides the index to the model. The *MACRO* column indicates whether macroeconomic factors are included; 1 for yes while 0 for no. The remaining columns show the performance rankings for modelling each maturity bond, in descending order according to the values of the loss function.

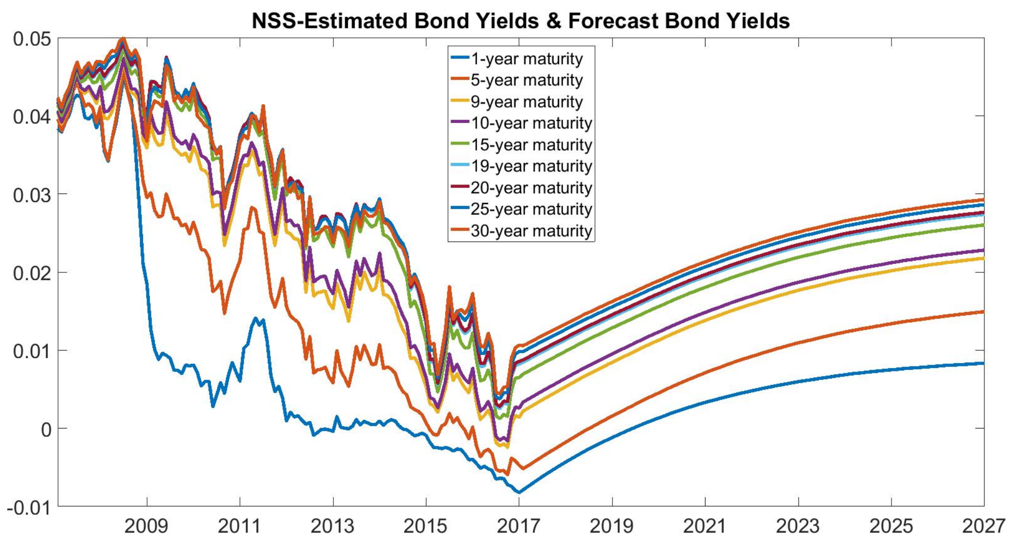


Figure 4: *The NSS-estimated bond yields, and bond yield forecasts by DNS with yield-only factors.*

t	Year	$\widehat{\mu}_t$ of τ-year maturity $(\times 10^{-2})$								
		1	5	9	10	15	19	20	25	30
0	2016	0.91	1.67	2.35	2.48	2.88	3.01	3.03	3.06	3.04
1	2017	0.91	1.67	2.36	2.48	2.88	3.02	3.03	3.06	3.04
2	2018	0.90	1.66	2.34	2.47	2.87	3.00	3.02	3.05	3.03
3	2019	0.90	1.66	2.34	2.47	2.87	3.00	3.02	3.05	3.03
4	2020	0.89	1.65	2.33	2.45	2.86	2.99	3.01	3.04	3.02
5	2021	0.90	1.67	2.35	2.48	2.88	3.01	3.03	3.06	3.03
6	2022	0.89	1.66	2.34	2.46	2.87	3.00	3.02	3.05	3.02
7	2023	0.92	1.68	2.36	2.48	2.88	3.02	3.03	3.06	3.04
8	2024	0.90	1.66	2.34	2.47	2.87	3.00	3.02	3.05	3.03
9	2025	0.91	1.67	2.35	2.47	2.87	3.01	3.02	3.05	3.03
10	2026	0.90	1.67	2.35	2.47	2.87	3.01	3.03	3.06	3.03

t	Year	$\widehat{\sigma}_t$ of τ-year maturity $(\times 10^{-2})$								
		1	5	9	10	15	19	20	25	30
0	2016	1.54	1.50	1.44	1.43	1.37	1.34	1.33	1.30	1.28
1	2017	1.53	1.49	1.43	1.42	1.37	1.33	1.33	1.30	1.27
2	2018	1.53	1.50	1.44	1.42	1.37	1.34	1.33	1.30	1.28
3	2019	1.53	1.49	1.43	1.42	1.37	1.33	1.33	1.30	1.27
4	2020	1.53	1.49	1.44	1.43	1.37	1.34	1.33	1.30	1.28
5	2021	1.53	1.49	1.43	1.42	1.37	1.33	1.33	1.30	1.27
6	2022	1.52	1.49	1.44	1.42	1.37	1.34	1.33	1.30	1.28
7	2023	1.54	1.50	1.44	1.42	1.37	1.34	1.33	1.30	1.28
8	2024	1.53	1.50	1.44	1.43	1.38	1.34	1.34	1.30	1.28
9	2025	1.54	1.50	1.44	1.43	1.37	1.34	1.33	1.30	1.28
10	2026	1.53	1.49	1.44	1.42	1.37	1.34	1.33	1.30	1.28

Table 3: *This table collects the estimates of forecast $\widehat{\mu}_t$ and standard deviations $\widehat{\sigma}_t$ for τ-maturity bonds, using the benchmark nominal model. The forecasts are obtained by simulation.*

t	Year	$\widehat{\kappa}_t^*$ of τ-year maturity $(\times 10^{-2})$								
		1	5	9	10	15	19	20	25	30
0	2016	166.07	154.07	148.71	147.71	138.68	135.51	132.80	126.31	118.22
1	2017	48.58	44.35	41.34	38.37	30.39	34.18	27.90	29.12	35.70
2	2018	21.29	19.07	21.95	14.39	11.63	16.32	10.79	11.41	15.18
3	2019	8.32	9.96	13.54	5.39	4.96	9.38	4.47	4.84	6.37
4	2020	2.36	6.72	8.65	2.36	2.39	5.98	2.18	2.30	3.17
5	2021	0.21	4.93	4.82	1.21	1.04	3.27	1.00	1.07	1.52
6	2022	0.15	3.90	2.23	0.77	0.55	1.51	0.52	0.63	0.79
7	2023	0.68	3.41	0.71	0.44	0.26	0.38	0.24	0.34	0.36
8	2024	1.65	2.87	0.10	0.36	0.21	0.02	0.14	0.22	0.18
9	2025	2.29	2.66	0.02	0.24	0.12	0.12	0.06	0.11	0.06
10	2026	2.96	2.49	0.22	0.20	0.04	0.45	0.03	0.07	0.01

t	Year	$\widehat{\theta}_t$ of τ-year maturity								
		1	5	9	10	15	19	20	25	30
0	2016	118.66	117.29	119.86	120.55	121.32	122.81	122.23	122.02	120.10
1	2017	64.44	63.13	63.41	61.67	57.03	61.96	56.28	58.87	66.32
2	2018	42.56	41.31	46.11	37.69	35.20	42.73	34.93	36.77	43.15
3	2019	26.72	29.96	36.31	23.12	23.05	32.46	22.53	24.00	28.02
4	2020	14.21	24.54	28.91	15.24	15.92	25.79	15.66	16.49	19.67
5	2021	4.27	21.06	21.66	10.93	10.53	19.18	10.68	11.29	13.69
6	2022	3.65	18.72	14.69	8.74	7.66	12.99	7.63	8.59	9.82
7	2023	7.58	17.42	8.27	6.61	5.25	6.55	5.23	6.31	6.65
8	2024	11.85	16.00	3.16	5.98	4.69	1.55	3.95	5.07	4.66
9	2025	13.93	15.38	1.49	4.84	3.52	3.64	2.58	3.60	2.62
10	2026	15.90	14.94	4.66	4.40	2.14	7.11	1.73	2.91	1.28

Table 4: *This table collects the estimated amount of model uncertainty $\widehat{\kappa}_t^*$, and the corresponding uncertainty parameter $\widehat{\theta}_t$, for τ-year maturity bonds, under the benchmark nominal model.*

t	Year	Assets				
		$\mu(A_t)$	$Q_{2.5\%}(A_t)$	$Q_{97.5\%}(A_t)$	$Q^l_{2.5\%}(A^l_t)$	$Q^u_{97.5\%}(A^u_t)$
0	2016	30.63	24.83	40.39	20.02	52.24
1	2017	29.27	24.00	36.14	20.56	44.96
2	2018	28.07	22.82	35.35	19.79	43.81
3	2019	26.87	21.52	34.29	18.56	42.76
4	2020	25.64	20.20	33.19	17.14	42.03
5	2021	24.39	18.82	32.10	15.64	41.47
6	2022	23.14	17.53	30.91	14.22	40.61
7	2023	21.88	16.17	29.88	12.82	39.83
8	2024	20.62	14.84	28.67	11.46	38.59
9	2025	19.37	13.50	27.61	10.07	37.68
10	2026	18.08	12.13	26.51	8.62	36.90

t	Year	Liabilities				
		$\mu(L_t)$	$Q_{2.5\%}(L_t)$	$Q_{97.5\%}(L_t)$	$Q^l_{2.5\%}(L^l_t)$	$Q^u_{97.5\%}(L^u_t)$
0	2016	30.63	24.83	40.39	20.02	52.24
1	2017	29.52	23.93	38.54	21.50	44.07
2	2018	28.40	23.15	36.87	21.60	40.15
3	2019	27.30	22.41	35.10	21.36	37.18
4	2020	26.16	21.62	33.34	20.89	34.73
5	2021	24.98	20.69	31.59	20.19	32.50
6	2022	23.80	19.89	29.84	19.52	30.47
7	2023	22.62	18.99	28.11	18.74	28.51
8	2024	21.43	18.07	26.37	17.89	26.63
9	2025	20.24	17.12	24.67	16.96	24.91
10	2026	19.01	16.18	22.98	16.02	23.22

t	Year	Funding Ratio				
		$\mu(FR_t)$	$Q_{2.5\%}(FR_t)$	$Q_{97.5\%}(FR_t)$	$Q^l_{2.5\%}(FR^l_t)$	$Q^u_{97.5\%}(FR^u_t)$
0	2016	1.00	1.00	1.00	1.00	1.00
1	2017	1.00	0.83	1.23	0.82	1.38
2	2018	1.00	0.82	1.23	0.77	1.42
3	2019	0.99	0.81	1.23	0.74	1.47
4	2020	0.99	0.80	1.24	0.70	1.53
5	2021	0.98	0.78	1.25	0.67	1.58
6	2022	0.98	0.77	1.27	0.64	1.64
7	2023	0.97	0.75	1.29	0.60	1.70
8	2024	0.97	0.72	1.31	0.56	1.75
9	2025	0.96	0.70	1.33	0.52	1.81
10	2026	0.95	0.66	1.36	0.48	1.88

Table 5: *This table reports the 95% prediction intervals without and with misspecification uncertainty, for of the assets, the liabilities and the funding ratios. They are built on the bond yields estimated by the* ***benchmark nominal model****. Results are plotted in Figure 7a.*

t	Year	Assets				
		$\mu(A_t)$	$Q_{2.5\%}(A_t)$	$Q_{97.5\%}(A_t)$	$Q_{2.5\%}(A_t^l)$	$Q_{97.5\%}(A_t^u)$
0	2016	38.18	36.43	40.03	36.37	40.09
1	2017	36.01	32.97	39.23	32.92	39.29
2	2018	33.89	30.27	37.77	30.23	37.82
3	2019	31.90	27.99	36.15	27.98	36.15
4	2020	30.03	25.93	34.51	25.76	34.73
5	2021	28.27	24.02	32.93	23.66	33.41
6	2022	26.60	22.25	31.41	21.78	32.06
7	2023	25.02	20.59	30.00	20.08	30.70
8	2024	23.50	18.96	28.60	18.51	29.22
9	2025	22.04	17.40	27.22	17.04	27.70
10	2026	20.63	15.90	25.94	15.69	26.19

t	Year	Liabilities				
		$\mu(L_t)$	$Q_{2.5\%}(L_t)$	$Q_{97.5\%}(L_t)$	$Q_{2.5\%}(L_t^l)$	$Q_{97.5\%}(L_t^u)$
0	2016	38.18	36.43	40.03	36.37	40.09
1	2017	35.54	31.07	40.72	30.93	40.94
2	2018	32.99	27.95	39.07	27.70	39.46
3	2019	30.69	25.68	36.85	25.34	37.41
4	2020	28.62	23.87	34.49	23.48	35.15
5	2021	26.76	22.32	32.29	21.90	32.99
6	2022	25.05	20.94	30.09	20.49	30.85
7	2023	23.48	19.69	28.13	19.24	28.91
8	2024	22.00	18.57	26.19	18.09	27.00
9	2025	20.59	17.49	24.34	17.04	25.10
10	2026	19.25	16.41	22.58	15.98	23.31

t	Year	Funding Ratio				
		$\mu(FR_t)$	$Q_{2.5\%}(FR_t)$	$Q_{97.5\%}(FR_t)$	$Q_{2.5\%}(FR_t^l)$	$Q_{97.5\%}(FR_t^u)$
0	2016	1.00	1.00	1.00	1.00	1.00
1	2017	1.02	0.93	1.11	0.93	1.11
2	2018	1.03	0.92	1.15	0.91	1.16
3	2019	1.04	0.92	1.18	0.90	1.19
4	2020	1.05	0.91	1.20	0.89	1.23
5	2021	1.06	0.91	1.22	0.88	1.27
6	2022	1.07	0.90	1.24	0.86	1.30
7	2023	1.07	0.89	1.26	0.85	1.33
8	2024	1.07	0.88	1.28	0.84	1.35
9	2025	1.07	0.87	1.30	0.82	1.36
10	2026	1.07	0.85	1.33	0.81	1.38

Table 6: *This table reports the 95% prediction intervals without and with misspecification uncertainty, for of the assets, the liabilities and the funding ratios. They are built on the **alternative nominal model**. Results are plotted in Figure 7b.*

t	Year	Assets				
		$\mu(A_t)$	$Q_{2.5\%}(A_t)$	$Q_{97.5\%}(A_t)$	$Q_{2.5\%}(A_t^l)$	$Q_{97.5\%}(A_t^u)$
0	2016	30.63	24.83	40.39	20.02	52.24
1	2017	29.03	23.79	36.64	19.72	47.06
2	2018	27.55	22.23	35.55	18.45	45.61
3	2019	26.08	20.59	34.36	16.88	44.42
4	2020	24.60	18.90	33.07	15.19	43.33
5	2021	23.10	17.26	31.84	13.48	42.32
6	2022	21.61	15.62	30.58	11.80	41.32
7	2023	20.11	14.05	29.27	10.16	40.16
8	2024	18.62	12.39	27.98	8.50	39.00
9	2025	17.13	10.77	26.70	6.85	37.85
10	2026	15.64	9.17	25.44	5.20	36.76

t	Year	Liabilities				
		$\mu(L_t)$	$Q_{2.5\%}(L_t)$	$Q_{97.5\%}(L_t)$	$Q_{2.5\%}(L_t^l)$	$Q_{97.5\%}(L_t^u)$
0	2016	30.63	24.83	40.39	20.02	52.24
1	2017	29.52	23.93	38.54	21.50	44.07
2	2018	28.40	23.15	36.87	21.60	40.15
3	2019	27.30	22.41	35.10	21.36	37.18
4	2020	26.16	21.62	33.34	20.89	34.73
5	2021	24.98	20.69	31.59	20.19	32.50
6	2022	23.80	19.89	29.84	19.52	30.47
7	2023	22.62	18.99	28.11	18.74	28.51
8	2024	21.43	18.07	26.37	17.89	26.63
9	2025	20.24	17.12	24.67	16.96	24.91
10	2026	19.01	16.18	22.98	16.02	23.22

t	Year	Funding Ratio				
		$\mu(FR_t)$	$Q_{2.5\%}(FR_t)$	$Q_{97.5\%}(FR_t)$	$Q_{2.5\%}(FR_t^l)$	$Q_{97.5\%}(FR_t^u)$
0	2016	1.00	1.00	1.00	1.00	1.00
1	2017	1.00	0.74	1.35	0.70	1.56
2	2018	0.98	0.72	1.35	0.65	1.61
3	2019	0.97	0.70	1.34	0.61	1.65
4	2020	0.95	0.68	1.34	0.57	1.69
5	2021	0.94	0.66	1.34	0.53	1.73
6	2022	0.92	0.63	1.34	0.49	1.78
7	2023	0.90	0.60	1.34	0.44	1.82
8	2024	0.88	0.57	1.35	0.40	1.86
9	2025	0.85	0.53	1.35	0.34	1.90
10	2026	0.83	0.48	1.37	0.28	1.96

Table 7: *This table reports the 95% prediction intervals without and with misspecification uncertainty, for of the assets, the liabilities and the funding ratios. They are built on the* ***benchmark nominal model*** *with* **30**% *investment in* ***1-year bond****. Results are plotted in Figure 8.*

t	Year	Assets				
		$\mu(A_t)$	$Q_{2.5\%}(A_t)$	$Q_{97.5\%}(A_t)$	$Q_{2.5\%}(A_t^l)$	$Q_{97.5\%}(A_t^u)$
0	2016	30.63	24.83	40.39	20.02	52.24
1	2017	29.52	23.93	37.04	21.04	43.81
2	2018	28.51	22.87	36.65	20.47	43.73
3	2019	27.51	21.76	35.81	19.42	42.86
4	2020	26.48	20.64	34.83	18.18	42.24
5	2021	25.41	19.41	33.79	16.87	41.63
6	2022	24.34	18.27	32.91	15.60	41.19
7	2023	23.27	17.13	31.83	14.43	40.21
8	2024	22.19	15.92	30.83	13.21	39.23
9	2025	21.12	14.77	29.88	12.01	38.56
10	2026	20.00	13.51	28.93	10.63	38.24

t	Year	Liabilities				
		$\mu(L_t)$	$Q_{2.5\%}(L_t)$	$Q_{97.5\%}(L_t)$	$Q_{2.5\%}(L_t^l)$	$Q_{97.5\%}(L_t^u)$
0	2016	30.63	24.83	40.39	20.02	52.24
1	2017	29.52	23.93	38.54	21.50	44.07
2	2018	28.40	23.15	36.87	21.60	40.15
3	2019	27.30	22.41	35.10	21.36	37.18
4	2020	26.16	21.62	33.34	20.89	34.73
5	2021	24.98	20.69	31.59	20.19	32.50
6	2022	23.80	19.89	29.84	19.52	30.47
7	2023	22.62	18.99	28.11	18.74	28.51
8	2024	21.43	18.07	26.37	17.89	26.63
9	2025	20.24	17.12	24.67	16.96	24.91
10	2026	19.01	16.18	22.98	16.02	23.22

t	Year	Funding Ratio				
		$\mu(FR_t)$	$Q_{2.5\%}(FR_t)$	$Q_{97.5\%}(FR_t)$	$Q_{2.5\%}(FR_t^l)$	$Q_{97.5\%}(FR_t^u)$
0	2016	1.00	1.00	1.00	1.00	1.00
1	2017	1.00	0.93	1.14	0.93	1.26
2	2018	1.01	0.90	1.15	0.88	1.29
3	2019	1.01	0.89	1.17	0.84	1.35
4	2020	1.01	0.87	1.19	0.80	1.41
5	2021	1.02	0.86	1.22	0.77	1.48
6	2022	1.02	0.84	1.25	0.74	1.55
7	2023	1.03	0.83	1.28	0.71	1.62
8	2024	1.04	0.81	1.32	0.68	1.68
9	2025	1.04	0.79	1.36	0.65	1.75
10	2026	1.05	0.77	1.41	0.61	1.85

Table 8: *This table reports the 95% prediction intervals without and with misspecification uncertainty, for of the assets, the liabilities and the funding ratios. They are built on the* ***benchmark nominal model*** *with* **27**% *investment in* ***20-year bond****. Results are plotted in Figure 9.*

t	Year	Assets				
		$\mu(A_t)$	$Q_{2.5\%}(A_t)$	$Q_{97.5\%}(A_t)$	$Q_{2.5\%}(A_t^l)$	$Q_{97.5\%}(A_t^u)$
0	2016	38.18	36.45	40.02	36.41	40.07
1	2017	36.05	33.61	38.58	33.56	38.64
2	2018	34.01	31.16	37.00	31.14	37.03
3	2019	32.07	28.92	35.40	28.83	35.50
4	2020	30.21	26.82	33.81	26.58	34.11
5	2021	28.44	24.86	32.29	24.47	32.79
6	2022	26.73	22.99	30.81	22.47	31.47
7	2023	25.08	21.17	29.35	20.58	30.13
8	2024	23.47	19.41	27.94	18.77	28.79
9	2025	21.90	17.71	26.56	17.06	27.45
10	2026	20.36	16.04	25.20	15.40	26.08

t	Year	Liabilities				
		$\mu(L_t)$	$Q_{2.5\%}(L_t)$	$Q_{97.5\%}(L_t)$	$Q_{2.5\%}(L_t^l)$	$Q_{97.5\%}(L_t^u)$
0	2016	38.18	36.45	40.02	36.41	40.07
1	2017	35.50	31.04	40.67	30.86	40.95
2	2018	32.97	27.94	39.05	27.65	39.53
3	2019	30.68	25.64	36.81	25.25	37.46
4	2020	28.63	23.89	34.49	23.44	35.23
5	2021	26.77	22.29	32.26	21.84	33.03
6	2022	25.07	20.94	30.14	20.48	30.92
7	2023	23.50	19.70	28.13	19.24	28.91
8	2024	22.02	18.56	26.19	18.10	26.97
9	2025	20.61	17.46	24.35	17.01	25.12
10	2026	19.26	16.46	22.57	16.01	23.31

t	Year	Funding Ratio				
		$\mu(FR_t)$	$Q_{2.5\%}(FR_t)$	$Q_{97.5\%}(FR_t)$	$Q_{2.5\%}(FR_t^l)$	$Q_{97.5\%}(FR_t^u)$
0	2016	1.00	1.00	1.00	1.00	1.00
1	2017	1.02	0.90	1.14	0.90	1.14
2	2018	1.04	0.89	1.20	0.88	1.21
3	2019	1.05	0.88	1.24	0.86	1.26
4	2020	1.06	0.88	1.26	0.85	1.30
5	2021	1.07	0.87	1.29	0.84	1.34
6	2022	1.07	0.87	1.31	0.83	1.37
7	2023	1.07	0.86	1.32	0.81	1.39
8	2024	1.07	0.84	1.34	0.79	1.41
9	2025	1.07	0.83	1.35	0.77	1.43
10	2026	1.06	0.81	1.36	0.75	1.45

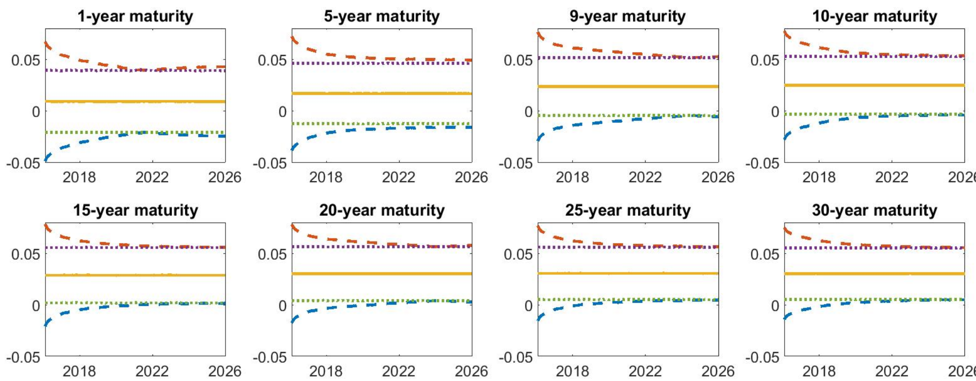
Table 9: *This table reports the 95% prediction intervals without and with misspecification uncertainty, for of the assets, the liabilities and the funding ratios. They are built on the* ***alternative nominal model*** *with* **30**% *investment in* ***1-year bond****. Results are plotted in Figure 10.*

t	Year	Assets				
		$\mu(A_t)$	$Q_{2.5\%}(A_t)$	$Q_{97.5\%}(A_t)$	$Q_{2.5\%}(A_t^l)$	$Q_{97.5\%}(A_t^u)$
0	2016	38.18	36.45	40.02	36.41	40.07
1	2017	35.79	32.00	39.96	31.92	40.06
2	2018	33.47	28.91	38.64	28.86	38.72
3	2019	31.30	26.33	36.87	26.04	37.31
4	2020	29.29	24.20	35.08	23.62	35.94
5	2021	27.40	22.25	33.28	21.44	34.52
6	2022	25.64	20.49	31.60	19.54	33.08
7	2023	23.98	18.80	29.96	17.82	31.50
8	2024	22.40	17.24	28.38	16.29	29.83
9	2025	20.88	15.74	26.86	14.92	28.09
10	2026	19.42	14.27	25.43	13.62	26.32

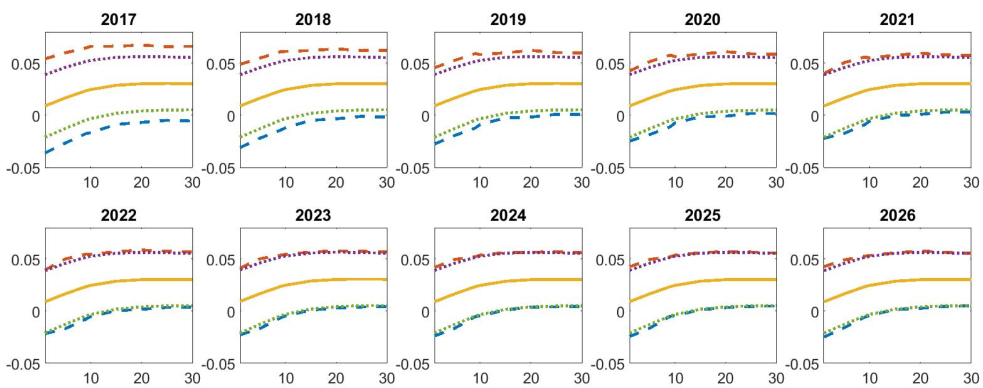
t	Year	Liabilities				
		$\mu(L_t)$	$Q_{2.5\%}(L_t)$	$Q_{97.5\%}(L_t)$	$Q_{2.5\%}(L_t^l)$	$Q_{97.5\%}(L_t^u)$
0	2016	38.18	36.45	40.02	36.41	40.07
1	2017	35.50	31.04	40.67	30.86	40.95
2	2018	32.97	27.94	39.05	27.65	39.53
3	2019	30.68	25.64	36.81	25.25	37.46
4	2020	28.63	23.89	34.49	23.44	35.23
5	2021	26.77	22.29	32.26	21.84	33.03
6	2022	25.07	20.94	30.14	20.48	30.92
7	2023	23.50	19.70	28.13	19.24	28.91
8	2024	22.02	18.56	26.19	18.10	26.97
9	2025	20.61	17.46	24.35	17.01	25.12
10	2026	19.26	16.46	22.57	16.01	23.31

t	Year	Funding Ratio				
		$\mu(FR_t)$	$Q_{2.5\%}(FR_t)$	$Q_{97.5\%}(FR_t)$	$Q_{2.5\%}(FR_t^l)$	$Q_{97.5\%}(FR_t^u)$
0	2016	1.00	1.00	1.00	1.00	1.00
1	2017	1.01	0.95	1.07	0.95	1.08
2	2018	1.02	0.94	1.10	0.93	1.12
3	2019	1.02	0.93	1.12	0.90	1.15
4	2020	1.02	0.92	1.14	0.88	1.19
5	2021	1.02	0.90	1.15	0.85	1.22
6	2022	1.02	0.89	1.17	0.83	1.25
7	2023	1.02	0.87	1.19	0.80	1.28
8	2024	1.02	0.85	1.20	0.78	1.30
9	2025	1.01	0.82	1.22	0.76	1.31
10	2026	1.01	0.80	1.24	0.74	1.32

Table 10: *This table reports the 95% prediction intervals without and with misspecification uncertainty, for of the assets, the liabilities and the funding ratios. They are built on the **alternative nominal model** with* **27**% *investment in **20-year bond**. Results are plotted in Figure 11.*

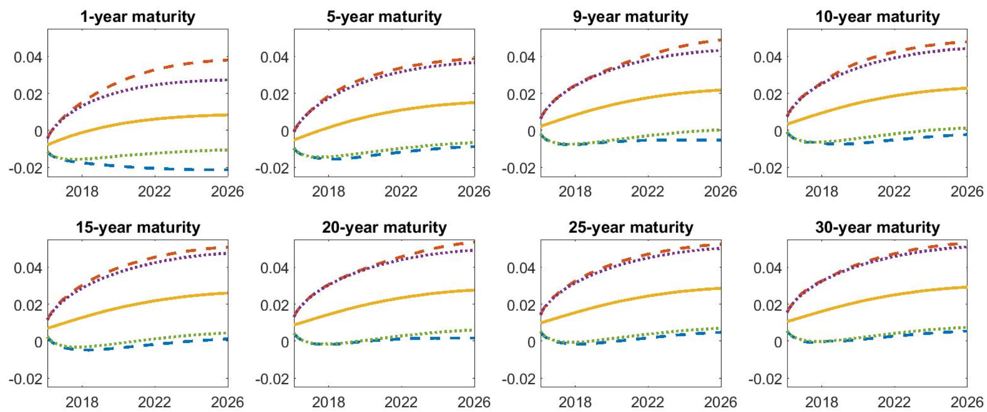


(a) *Bond yields over time*

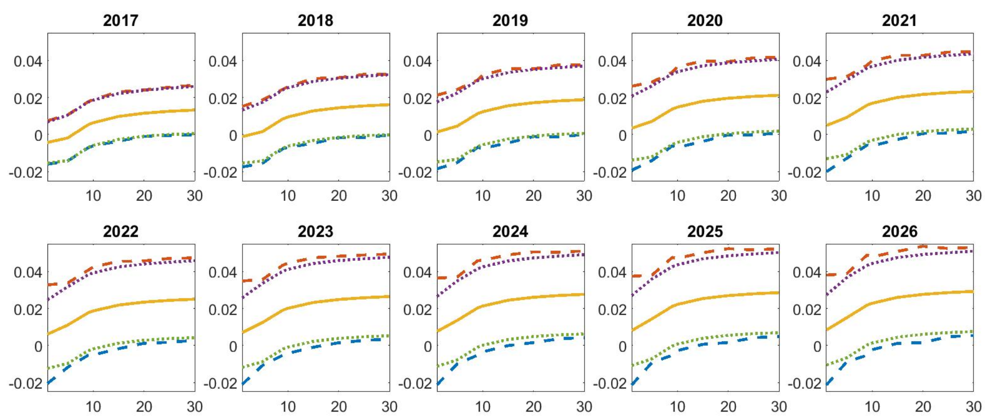


(b) *Term structure in the end of the Year*

Figure 5: *The prediction intervals with and without model misspecification uncertainty, for the term structure of the bond yields under the* ***benchmark nominal model****. The dotted lines indicate the* 95% *prediction intervals for the nominal model, while the dashed lines indicate the prediction intervals with misspecification uncertainty.*

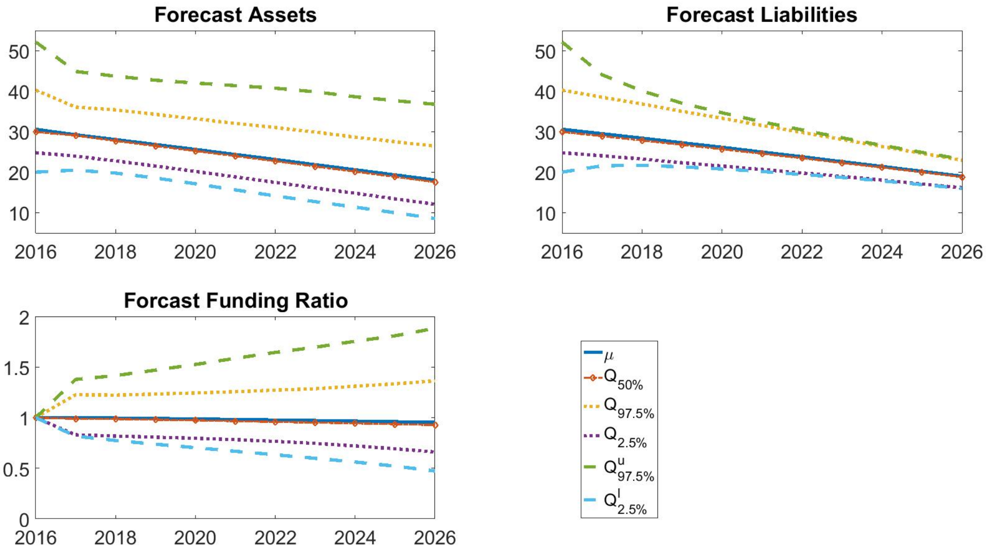


(a) *Bond yields over Years*

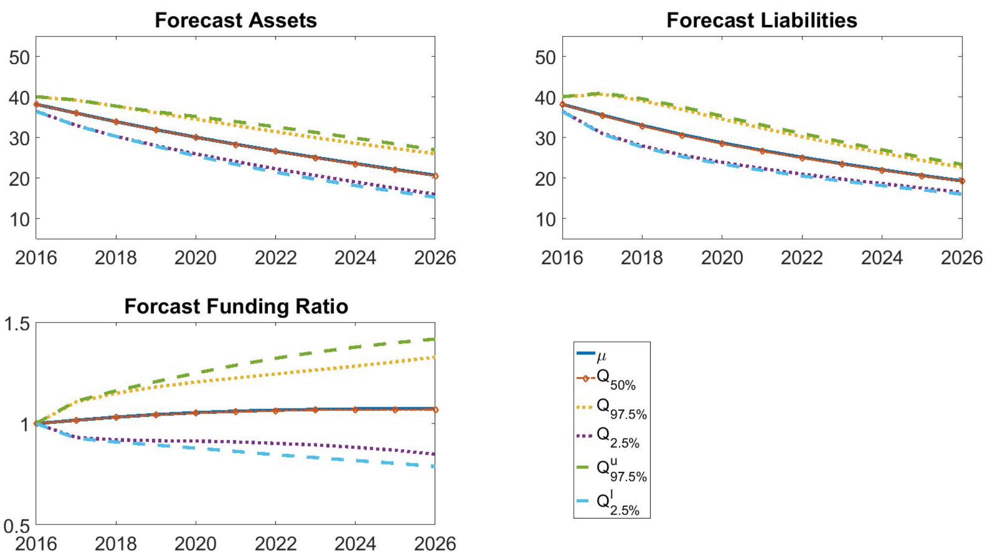


(b) *Term structure in the end of the Year*

Figure 6: *The The prediction intervals with and without model misspecification uncertainty, for the term structure of the bond yields under the **alternative nominal model**. The dotted lines indicate the* 95% *prediction intervals for the nominal model, while the dashed lines indicate the prediction intervals with misspecification uncertainty.*



(a) *Forecasts of the assets, the liabilities and the funding ratio, using the* ***benchmark nominal model****.*



(b) *Forecasts of the assets, the liabilities and the funding ratio, using the* ***alternative nominal model****.*

Figure 7: *The results of the forecasts are reported in Table 5 and 6, respectively. The solid line and the dashed-diamond line are for the expectations and the medians derived from the nominal model. The dotted lines indicate the* 95% *prediction intervals for the nominal model, while the dashed lines indicate the prediction intervals with misspecification uncertainty.*

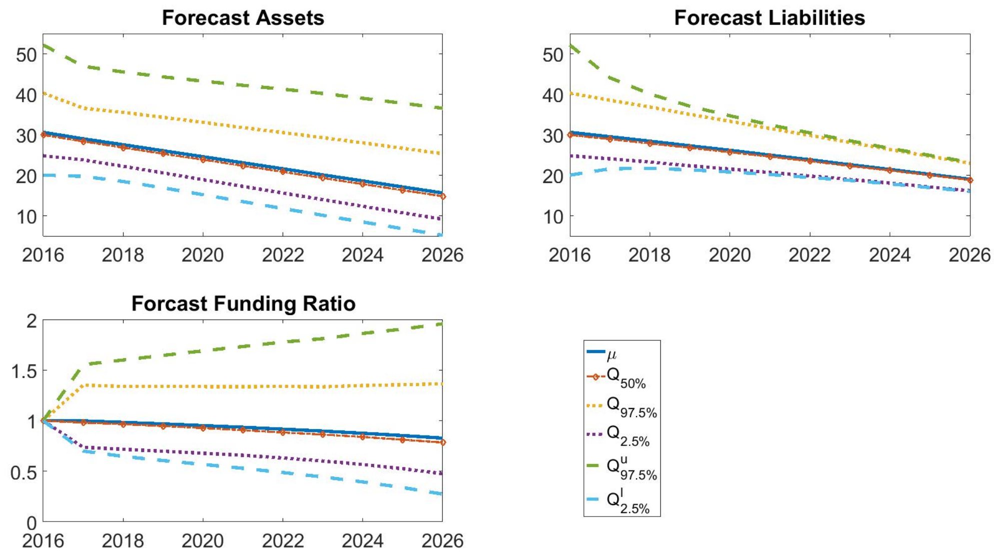


Figure 8: *Forecasts of the assets, the liabilities and the funding ratio, using the **benchmark nominal model**, with* **30**% ***1-year bond*** *in the investment.*

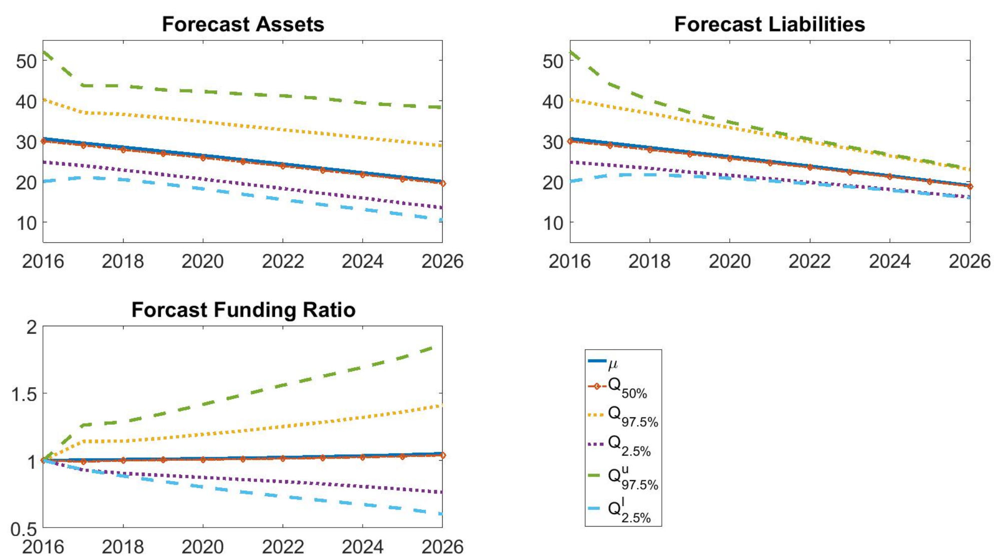


Figure 9: *Forecasts of the assets, the liabilities and the funding ratio, using the **benchmark nominal model**, with* **27**% ***20-year bond*** *in the investment.*

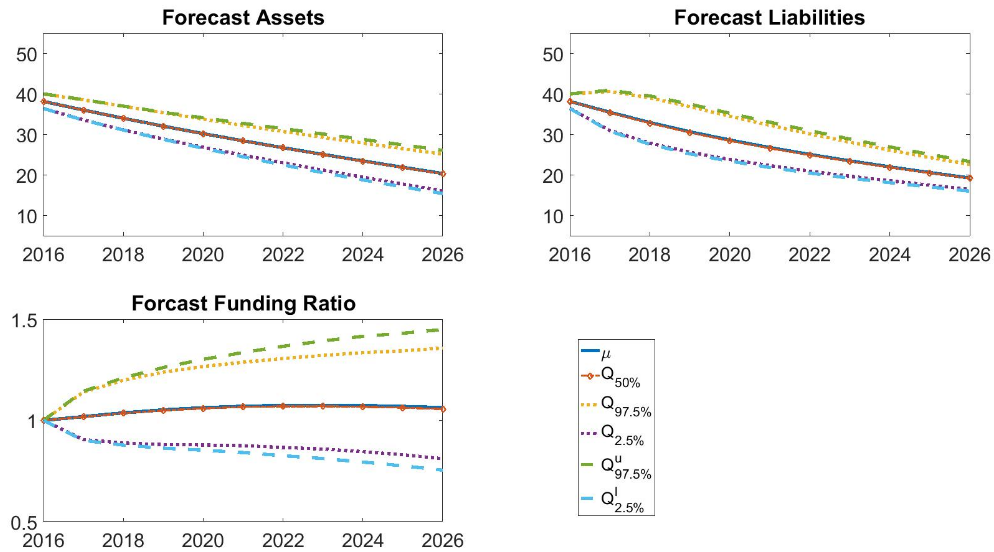


Figure 10: *Forecasts of the assets, the liabilities and the funding ratio, using the* ***alternative nominal model****, with* **30**% ***1-year bond*** *in the investment.*

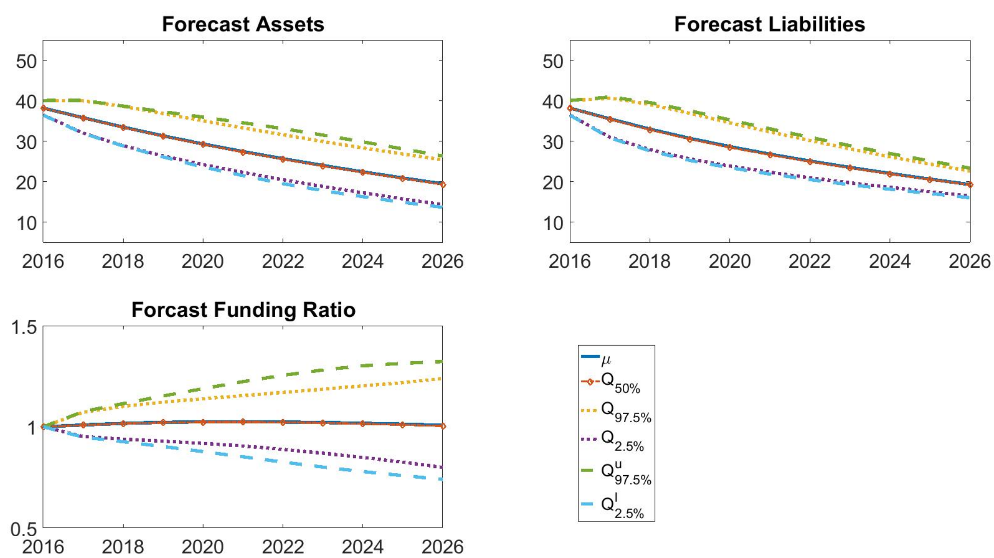


Figure 11: *Forecasts of the assets, the liabilities and the funding ratio, using the* ***alternative nominal model****, with* **27**% ***20-year bond*** *in the investment.*