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ABSTRACT

This paper models a Personal Pension with Risk sharing (PPR). We derive several relationships between the parameters of a PPR. For instance, we show how the Assumed Interest Rate affects the median growth rate of retirement income. Policyholders can adopt (at least) two approaches to a PPR – the investment approach and the consumption approach. In the investment approach, policyholders specify in each period how much to save or to withdraw, and how to allocate their retirement assets across different investment options. By contrast, in the consumption approach, policyholders specify the entire consumption stream in retirement exogenously. We explore these two approaches in full detail and show how they differ from each other. In accordance with (internal) habit formation, we allow for excess smoothness and excess sensitivity in retirement income.

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Private pension provision is in transition, shifting from defined benefit (DB) plans towards defined contribution (DC) plans (Investment Company Institute, 2015). Many employees find the trend towards DC plans undesirable (National Institute on Retirement Security, 2015). Indeed, a DC plan focusses primarily on wealth accumulation rather than providing stable lifelong income streams. Bovenberg and Nijman (2015) have introduced a new pension contract called a Personal Pension with Risk sharing (PPR) that can play an important role in the provision of old-age security. A PPR unbundles the three main functions of variable annuities: investment, (dis)saving and insurance.[1] In particular, it individualizes the investment and (dis)saving functions and organizes the insurance function (i.e., pooling of micro longevity risk) collectively.[2,3]

This paper models a PPR. We define a PPR in terms of seven parameters (e.g., investment policy, Assumed Interest Rate, volatility of retirement income). We show how the budget constraint implies several relationship between the parameters of a PPR. The budget constraint does, however, not uniquely identify the values of *all* parameters. As a consequence, policyholders must specify some parameter values exogenously. They can specify the parameter values according to (at least) two different approaches – the investment approach and the consumption approach. We explore these two approaches in full detail and show how they differ from each other.

In the investment approach, policyholders specify in each period how much to save or to withdraw, and how to allocate their retirement savings across different investment options. Insurers commonly adopt the investment approach. For instance, if a person wants to buy a variably annuity, insurers typically offer a selection of investment options and Assumed Interest Rates (AIRs). Pension payments (henceforth called annuity units) follow endogenously in the investment approach. In particular, the investment policy and the AIR together determine the median growth rate of current annuity units. Indeed, if a policyholder adopts a more aggressive investment policy (to earn a higher expected return on investment) and leaves the AIR unaffected, the median growth rate of current annuity units increases. Changes in not only the investment policy but also in the expected

[1] A PPR differs from a variable annuity in several important aspects. First, a PPR defines property rights in terms of a personal investment account rather than an income stream. Second, a PPR allows for more flexibility in customizing portfolio and drawdown policies to individual needs. Third, a PPR integrates the accumulation phase with the payout phase.

[2] Individualization of the investment function is possible without any welfare loss. Indeed, pooling of systematic risks does not generate any welfare gain. In fact, individualization of the investment function may even lead to an welfare improvement, because insurers can now tailor policies to individual needs.

[3] By pooling micro longevity risk and taking systematic risks, policyholders can achieve lifelong income streams at relatively low costs.

return on equities alter the median growth rate of current annuity units. Hence, if a person prefers a stable median income stream during the payout phase, he may not want to adopt the investment approach.

Standard variable annuities fully reflect a wealth shock into current annuity units; that is, current annuity units respond one-to-one to a change in wealth (see, e.g., Chai, Horneff, Maurer, and Mitchell, 2011; Maurer, Mitchell, Rogalla, and Kartashov, 2013b).[4] As a result, in a standard variable annuity, a wealth shock does not affect the AIR. By contrast, a PPR allows a wealth shock to be absorbed gradually to reduce the year-on-year volatility of retirement income. This so-called buffering of wealth shocks is optimal in the presence of (internal) habit formation (see Fuhrer, 2000). In the case of buffering, a wealth shock affects not only current annuity units but also the AIR. Furthermore, as the policyholder ages and the duration of the cash flows declines, the AIR becomes less effective in absorbing wealth shocks. Accordingly, in order to prevent extreme year-on-year volatility of retirement income at advanced ages, the policyholder may want to reduce the equity risk exposure as he grows older.

In the consumption approach, policyholders specify the entire consumption stream in retirement exogenously.[5] We characterize the consumption stream in terms of three parameters: the annuity units at the age of retirement, the median growth rate of annuity units and the volatility of annuity units. The contribution level, the investment policy and the discount rate follow endogenously from the (stochastic) liabilities. Policyholders thus adopt the principle of liability-driven investment (and *not* the principle of asset-driven liabilities). In fact, the investment policy consists of two endogenous components: a speculative component and an intertemporal hedge component. This decomposition is familiar from the literature on optimal consumption and portfolio choice under a stochastic investment opportunity set (see, e.g., Brennan and Xia, 2002; Wachter, 2002; Chacko and Viceira, 2005; Liu, 2007). The speculative portfolio allows policyholders to take advantage of the equity risk premium, while the hedge portfolio hedges changes in the investment opportunity set that affect the costs of annuity units (Merton, 1971).[6] The hedge portfolio thus allows policyholders to achieve a stable median income stream.

[4]Insurers have developed variable annuities for which payouts respond sluggishly to an unexpected wealth shock (see, e.g., Guillén, Jørgensen, and Nielsen, 2006; Maurer, Rogalla, and Siegelin, 2013a; Maurer, Mitchell, Rogalla, and Siegelin, 2014). However, these variable annuities are often based on complex profit-sharing rules and hence are difficult to value.

[5]Brown, Kling, Mullainathan, and Wrobel (2008, 2013) find that individuals value annuities more if framed in terms of consumption rather than investment.

[6]Changes in the investment opportunity set are due to shocks in e.g., the interest rate.

We allow the volatility of consumption to increase with the investment horizon (i.e., a wealth shock has a smaller impact on consumption in the near future than on consumption in the distant future). Buffering of wealth shocks in this way implies an excessively smooth and excessively sensitive consumption stream. Aggregate consumption data also exhibit these properties (see, e.g., Flavin, 1985; Deaton, 1987; Campbell and Deaton, 1989).[7] Excess smoothness and excess sensitivity in consumption is also consistent with internal habit formation (see Fuhrer, 2000).

I. Modelling a Personal Pension with Risk Sharing

This section models a PPR. Section I.A describes the financial market. Section I.B specifies the survival probabilities. These probabilities play an important role in a PPR. Indeed, a PPR pools micro longevity risk. Section I.C introduces the budget condition. This condition states that the policyholder's wealth should match the value of the pension liabilities in every state of nature. Section I.D examines the dynamics of the wealth process, while Section I.E investigates the dynamics of the value of the liabilities. Finally, Section I.F explores how the budget condition implies several relationships between the parameters of a PPR. Throughout, we consider only the payout phase.

A. *Financial Market*

This section introduces the financial market. Our specification of the financial market is closely related to Liu (2007). The only difference between Liu's specification and our specification is that he does not assume a complete financial market, whereas our specification does.[8] Denote by X_t an N-dimensional vector of state variables. This vector characterizes the asset prices in the financial market. The vector of state variables could include the short-term interest rate, the realized rate of inflation, or predictors of stock returns. We assume that X_t satisfies the following dynamic equation:

$$\mathrm{d}X_t = \mu^X \,\mathrm{d}t + \Sigma^X \,\mathrm{d}Z_t, \tag{1}$$

[7] Consumption is excessively smooth if consumption under-responds to wealth shocks; consumption is excessively sensitive if past wealth shocks have predictive power for future consumption growth.

[8] The financial market is complete only with respect to financial risk. Hence, micro longevity risk cannot be eliminated by trading in the financial market.

where the drift term μ^X and the diffusion matrix Σ^X are an N-by-1 vector function and an N-by-N matrix function of X_t, respectively, and Z_t is an N-dimensional standard Brownian motion. We denote the correlation structure of the stochastic processes by the N-by-N matrix ρ; here, the (i, j)th element of ρ represents the correlation coefficient between $\mathrm{d}Z_{it}$ and $\mathrm{d}Z_{jt}$.

We consider a financial market consisting of N risky assets and one (locally) risk-free asset. The vector of risky asset prices $P_t = (P_{1t}, \dots, P_{Nt})$ and the risk-free asset price P_{0t} satisfy, respectively, the following dynamic equations:[9]

$$\mathrm{d}P_t = \mu(X_t) P_t \, \mathrm{d}t + \Sigma(X_t) P_t \, \mathrm{d}Z_t, \tag{2}$$

$$\mathrm{d}P_{0t} = R_f(X_t) P_{0t} \, \mathrm{d}t. \tag{3}$$

The drift term $\mu(X_t)$, the diffusion matrix $\Sigma(X_t)$, and the risk-free interest rate $R_f(X_t)$ are functions of X_t. When there is no confusion, we write $\mu(X_t)$ as μ, $\Sigma(X_t)$ as Σ, and $R_f(X_t)$ as R_f.

B. Survival Probabilities

To protect policyholders against outliving their financial assets, a PPR distributes the accumulated retirement savings of someone who dies among the surviving policyholders of the same age group.[10] Hence, a PPR pools micro longevity risk. We assume that the risk-sharing pool is sufficiently large, so that the law of large numbers applies. Furthermore, we abstract away from macro longevity risk.

Denote by y the date of birth of a policyholder, by x_r the age at which policyholders retire, and by $x_{\max}$ the maximum age policyholders can reach. If the date of birth y falls between time $t - x_r$ and time $t - x_{\max}$ and the policyholder has survived to time t, then this policyholder receives a pension payment at time t. We denote the probability that a policyholder aged $x = t - y$ will survive to age $x + h$ by

$${}_h p_x = \exp\left\{ -\int_0^h \theta_{x+v} \mathrm{d}v \right\}. \tag{4}$$

[9]For notational convenience, we often write a column vector in the form $z = (z_1, \dots, z_N)$, where z_i represents the ith element of z.

[10]Alternatively, a PPR can distribute the accumulated retirement savings of someone who dies among the surviving policyholders of *all* age groups. In either case, micro longevity risk is eliminated only if the risk-sharing pool is sufficiently large.

Here, θ_{x+v} represents the force of mortality (or hazard rate) at age $x+v$.

C. Budget Condition

In a PPR, the value of the policyholder's wealth should match the value of the pension liabilities in every state of nature. Indeed, external risk absorbers are absent in a PPR. Let $W_{t,y}$ and $V_{t,y}$ denote, respectively, the value of the investment account and the value of the pension liabilities at time t of a policyholder born at time y. Mathematically, budget balance implies that for each $t \in [y + x_r, y + x_{\max}]$

$$W_{t,y} = V_{t,y}. \tag{5}$$

The budget condition (5) states that the personal 'balance sheet' funding ratio $W_{t,y}/V_{t,y}$ is equal to unity in every state of nature. By Itô's Lemma and the budget condition (5), $\mathrm{d}\log W_{t,y} = \mathrm{d}\log V_{t,y}$. We explore the dynamics of $\log W_{t,y}$ and $\log V_{t,y}$ in Sections I.D and I.E, respectively. Section I.F derives several relationships between the parameters of a PPR that follow from the budget condition (5).

D. Dynamics of the Wealth Process

The value of the investment account satisfies the following dynamic equation:

$$\mathrm{d}W_{t,y} = \left(\theta_{t-y} + R_f + \omega_{t,y}^{\top}\left[\mu - R_f\right]\right) W_{t,y}\,\mathrm{d}t + \omega_{t,y}^{\top}\Sigma W_{t,y}\,\mathrm{d}Z_t - B_{t,y}\,\mathrm{d}t. \tag{6}$$

Here, $\theta_{t-y} \geq 0$ represents the biometric rate of return at time t of a policyholder born at time y,[11] $\omega_{t,y}$ is an N-by-1 vector of portfolio weights (i.e., $\omega_{it,y}$ denotes the share of wealth invested in the ith risky asset at time t of a policyholder born at time y),[12] and $B_{t,y}$ denotes current annuity units at time t of a policyholder born at time y. The symbol '$\top$' represents the transpose sign.

Application of Itô's Lemma to $\log W_{t,y}$ yields

$$\mathrm{d}\log W_{t,y} = \left(\theta_{t-y} + \mu_{t,y}^{W}\right)\mathrm{d}t + \omega_{t,y}^{\top}\Sigma\,\mathrm{d}Z_t - \frac{B_{t,y}}{W_{t,y}}\,\mathrm{d}t, \tag{7}$$

[11] The accumulated retirement savings of someone who dies goes to the surviving policyholders of the same age group (and not to the heirs). Hence, surviving policyholders earn an additional return.

[12] The share of wealth invested in the risk-free asset is given by $1 - \sum_{i=1}^{N}\omega_{it,y}$.

where $\mu_{t,y}^W$ stands for the (geometric) expected financial return on wealth at time t of a policyholder born at time y:[13]

$$\mu_{t,y}^W = R_f + \omega_{t,y}^\top \left(\mu - R_f\right) - \frac{1}{2}\omega_{t,y}^\top \Sigma\rho\Sigma^\top \omega_{t,y}. \tag{8}$$

The last term on the right-hand side of (8) is called an Itô correction term.

E. *Dynamics of the Value of the Liabilities*

E.1. *Conversion Factor*

Denote by $C_{t,y}$ the conversion factor at time t of a policyholder born at time y. We define this factor as follows:

$$C_{t,y} = \frac{V_{t,y}}{B_{t,y}}. \tag{9}$$

The conversion factor $C_{t,y}$ is the factor at which policyholders can convert current annuity units $B_{t,y}$ into pension wealth $W_{t,y} = V_{t,y}$ (i.e., $B_{t,y}C_{t,y} = V_{t,y}$). It thus models how policyholders allocate pension wealth between current and future annuity units. We allow the conversion factor to depend on past speculative shocks and future expected financial rates of returns.[14]

By Itô's Lemma and equation (9), we find

$$\mathrm{d}\log V_{t,y} = \mathrm{d}\log C_{t,y} + \mathrm{d}\log B_{t,y}. \tag{10}$$

Hence, to derive the dynamics of $\log V_{t,y}$, we first need to derive the dynamics of $\log C_{t,y}$ and $\log B_{t,y}$. Sections I.E.2 and I.E.3 investigate the dynamics of $\log C_{t,y}$ and $\log B_{t,y}$, respectively.

[13] We note that the *geometric* expected financial return differs from the *arithmetic* expected financial return $R_f + \omega_{t,y}^\top \left(\mu - R_f\right)$.

[14] To dampen the impact of a speculative shock on current annuity units, we allow policyholders to adjust the conversion factor following a speculative shock.

E.2. Dynamics of the Conversion Factor

Denote by $V_{t,y,h}$ the amount of pension wealth that the policyholder needs to finance future annuity units $B_{t+h,y}$. The value of the total liabilities is thus given by

$$V_{t,y} = \int_0^{x_{\max}-(t-y)} V_{t,y,h}\,\mathrm{d}h. \tag{11}$$

We define $C_{t,y,h}$ as follows:

$$C_{t,y,h} = \frac{V_{t,y,h}}{B_{t,y}}. \tag{12}$$

Using (9), (11) and (12), we can write the conversion factor $C_{t,y}$ as follows:

$$C_{t,y} = \frac{1}{B_{t,y}} \int_0^{x_{\max}-(t-y)} V_{t,y,h}\,\mathrm{d}h = \int_0^{x_{\max}-(t-y)} C_{t,y,h}\,\mathrm{d}h. \tag{13}$$

Let $\delta_{t,y,v}$ be the so-called (forward) discount rate at time t for horizon $v \geq 0$ of a policyholder born at time y. We implicitly define the discount rate $\delta_{t,y,v}$ as follows:

$$C_{t,y,h} = {}_hp_{t-y} \exp\left\{ -\int_0^h \delta_{t,y,v}\,\mathrm{d}v \right\}. \tag{14}$$

The discount rate models the speed at which policyholders withdraw their pension wealth. We allow the discount rate to depend on past speculative shocks and future expected financial rates of return. Hence, we can write $\delta_{t,y,v}$ in terms of two components:

$$\delta_{t,y,v} = \delta^b_{t,y,v} + \delta^f_{t,y,v}. \tag{15}$$

Here, $\delta^b_{t,y,v}$ models how the discount rate depends on past speculative shocks. By increasing $\delta^b_{t,y,v}$ following a negative speculative shock, we allow policyholders to absorb (part of) a speculative shock into future (rather than current) annuity units. The term $\delta^f_{t,y,v}$ models how the discount rate depends on future expected financial returns. Depending on his intertemporal rate of substitution, a policyholder may want to adjust $\delta^f_{t,y,v}$ if future investment opportunities change. We refer to $\delta^b_{t,y,v}$ and $\delta^f_{t,y,v}$ as the backward-looking and forward-looking component of $\delta_{t,y,v}$, respectively.

Using (4), (14) and (15), we can write $C_{t,y,h}$ as follows:

$$C_{t,y,h} = F_{t,y,h} A_{t,y,h}, \tag{16}$$

where

$$F_{t,y,h} = \exp\left\{ -\int_0^h \delta^b_{t,y,v} \, \mathrm{d}v \right\}, \tag{17}$$

$$A_{t,y,h} = \exp\left\{ -\int_0^h \left(\theta_{t+v-y} + \delta^f_{t,y,v} \right) \mathrm{d}v \right\}. \tag{18}$$

We refer to $F_{t,y,h}$ and $A_{t,y,h}$ as the horizon-dependent funding ratio and the horizon-dependent annuity factor, respectively.[15]

E.2.1 Dynamics of the Horizon-Dependent Annuity Factor $A_{t,y,h}$

In order to derive the dynamics of the horizon-dependent annuity factor $A_{t,y,h}$, we assume that the forward-looking component of the discount rate $\delta^f_{t,y,v}$ depends on t and X_t (y and v are fixed). As a result, the average forward-looking component of the discount rate, i.e., $\bar{\delta}^f_{t,y,h} = \int_0^h \delta^f_{t,y,v} \, \mathrm{d}v/h$, also depends on t and X_t. By Itô's Lemma, the log horizon-dependent annuity factor $\log A_{t,y,h} = -\left(\bar{\delta}^f_{t,y,h} h + \int_0^h \theta_{t+v-y} \, \mathrm{d}v \right)$ satisfies now the following dynamic equation:

$$\begin{aligned} \mathrm{d} \log A_{t,y,h} = \Bigg(& \theta_{t-y} + \delta^f_{t,y,h} - \left(\mu^X\right)^\top D_{t,y,h} - \frac{\partial}{\partial t}\left(\bar{\delta}^f_{t,y,h} h\right) \\ & - \frac{1}{2} \mathrm{Tr}\left[\left(\Sigma^X\right)^\top H_X \left(\bar{\delta}^f_{t,y,h} h\right) \Sigma^X \right] \Bigg) \mathrm{d}t - \left(\Sigma^X\right)^\top D_{t,y,h} \, \mathrm{d}Z_t. \end{aligned} \tag{19}$$

Here, $D_{t,y,h} = \nabla_X \left(\bar{\delta}^f_{t,y,h} h\right)$ and $H_X \left(\bar{\delta}^f_{t,y,h} h\right)$ are the gradient and Hessian matrix of $\bar{\delta}^f_{t,y,h} h$ with respect to X_t, respectively, and Tr denotes the trace operator.

[15] We note that the budget condition is satisfied in every state of nature (i.e., the personal 'balance sheet' funding ratio $W_{t,y}/V_{t,y}$ always equals unity). By contrast, the 'cash flow' funding ratio $F_{t,y} = W_{t,y}/A_{t,y}$ can deviate from unity. Here, $A_{t,y} = \int_0^{x_{\max}-(t-y)} A_{t,y,h} \, \mathrm{d}h$.

E.2.2 Dynamics of the Horizon-Dependent Funding Ratio $F_{t,y,h}$

Equation (19) shows that the annuity factor $A_{t,y} = \int_0^{x_{\max}-(t-y)} A_{t,y,h}\,\mathrm{d}h$ is (typically) stochastic. The hedge portfolio aims at hedging stochastic variations in the annuity factor. If the hedge portfolio differs from the actual portfolio, the policyholder takes speculative risk. The budget condition is satisfied in every state of nature (see (5)). Hence, policyholders must absorb a speculative shock in either current annuity units $B_{t,y}$ or the conversion factor $C_{t,y}$ or a combination of both.

Denote by $\omega^S_{t,y}$ the N-dimensional vector of speculative portfolio weights at time t of a policyholder born at time y. The speculative shock at time t is thus given by $\omega^S_{t,y}\Sigma\,\mathrm{d}Z_t$. We assume that a policyholder absorbs a fraction $q_{t,y,h}$ of the current speculative shock into future annuity units $B_{t+h,y}$. That is, the exposure of $\log V_{t,y,h} = \log C_{t,y,h} + \log B_{t,y}$ to the current speculative shock equals $q_{t,y,h}$. We assume that the function $q_{t,y,h}$ (which we call the buffering function) increases with the horizon h, so that a current speculative shock has a larger impact on cash flows in the distant future than on cash flows in the near future. To absorb the entire speculative shock into current and future annuity units, we must have that

$$\int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} q_{t,y,h}\,\mathrm{d}h = 1, \tag{20}$$

where

$$\alpha_{t,y,h} = C_{t,y,h}/C_{t,y}. \tag{21}$$

The exposure of $\log C_{t,y,h} = \log V_{t,y,h} - \log V_{t,y,0}$ to the current speculative shock is equal to $q_{t,y,h} - q_{t,y,0}$. Hence, the horizon-dependent funding ratio (which models how the conversion factor depends on past speculative shocks) is given by

$$F_{t,y,h} = \exp\left\{ \int_{y+x_r}^{t} \left(q_{s,y,t+h-s} - q_{s,y,t-s}\right) \left(\omega^S_{s,y}\right)^\top \Sigma\,\mathrm{d}Z_s \right\}. \tag{22}$$

By comparing (17) with (22), we arrive at

$$\begin{aligned} \bar{\delta}^b_{t,y,h} &= \int_0^h \delta^b_{t,y,v}\,\mathrm{d}v/h = -\int_{y+x_r}^{t} \left(q_{s,y,t+h-s} - q_{s,y,t-s}\right) \left(\omega^S_{s,y}\right)^\top \Sigma\,\mathrm{d}Z_s/h \\ &= -\log F_{t,y,h}/h. \end{aligned} \tag{23}$$

Equation (23) shows how past speculative shocks affect the backward-looking component of the discount rate. In the case of no horizon differentiation in risk exposures (i.e., $q_{t,y,h}$ is independent of h), past speculative shocks do not affect the conversion factor $C_{t,y}$. Indeed, in the absence of horizon differentiation in risk exposures, policyholders fully absorb speculative shocks into current annuity units. The backward-looking component of the discount rate is thus the direct consequence of the gradual adjustment of current annuity units to speculative shocks. The horizon-dependent funding ratio satisfies the following dynamic equation (this follows from equation (22)):

$$\mathrm{d}\log F_{t,y,h} = \left(q_{t,y,h} - q_{t,y,0}\right)\left(\omega_{t,y}^{S}\right)^{\top}\Sigma\,\mathrm{d}Z_t - \int_{y+x_r}^{t} \mathrm{d}q_{s,y,t-s}\left(\omega_{s,y}^{S}\right)^{\top}\Sigma\,\mathrm{d}Z_s. \tag{24}$$

The first term at the right-hand side of (24) represents the effect of a current speculative shock on the funding ratio. The second term at the right-hand side of (24) denotes past speculative shocks that are gradually being absorbed into current annuity units.

E.2.3 Dynamics of the Horizon-Dependent Conversion Factor $C_{t,y,h}$

The log horizon-dependent conversion factor obeys the following dynamics (the first equality follows from (16) and Itô's Lemma, and the second equality follows from (19) and (24)):

$$\begin{aligned}
\mathrm{d}\log C_{t,y,h} &= \mathrm{d}\log F_{t,y,h} + \mathrm{d}\log A_{t,y,h} \\
&= \Bigg(\theta_{t-y} + \delta_{t,y,h}^{f} - D_{t,y,h}^{\top}\mu^{X} - \frac{\partial}{\partial t}\left(\bar{\delta}_{t,y,h}^{f}h\right) \\
&\quad -\frac{1}{2}\mathrm{Tr}\left[\left(\Sigma^{X}\right)^{\top} H_X\left(\bar{\delta}_{t,y,h}^{f}h\right)\Sigma^{X}\right]\Bigg)\,\mathrm{d}t \\
&\quad \left(\left[q_{t,y,h} - q_{t,y,0}\right]\omega_{t,y}^{S\top}\Sigma - D_{t,y,h}^{\top}\Sigma^{X}\right)\mathrm{d}Z_t - \int_{y+x_r}^{t}\mathrm{d}q_{s,y,t-s}\omega_{s,y}^{S\top}\Sigma\,\mathrm{d}Z_s.
\end{aligned} \tag{25}$$

E.2.4 Dynamics of the Conversion Factor $C_{t,y}$

The dynamic equation of the log conversion factor is given by (this follows from Itô's Lemma and equations (13), (20) and (25))

$$\mathrm{d}\log C_{t,y} = \left(\theta_{t-y} + \mu_{t,y}^{C}\right)\mathrm{d}t + \left(\left[1 - q_{t,y,0}\right]\left(\omega_{t,y}^{S}\right)^{\top}\Sigma - \widehat{D}_{t,y}^{\top}\Sigma^{X}\right)\mathrm{d}Z_t - \frac{1}{C_{t,y}}\,\mathrm{d}t. \tag{26}$$

Here $\widehat{D}_{t,y} = \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} D_{t,y,h} \,\mathrm{d}h$ denotes the sensitivity vector of the conversion factor with respect to the underlying state variables and $\mu_{t,y}^C \,\mathrm{d}t$ is the expected financial rate of return on the conversion factor:

$$\begin{aligned}\mu_{t,y}^C \,\mathrm{d}t = &\int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} \mathbb{E}_t \left[\mathrm{d}\log C_{t,y,h}\right] \mathrm{d}h + \frac{1}{2} \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} \,\mathrm{d}\log C_{t,y,h} \,\mathrm{d}\log C_{t,y,h} \\ &- \frac{1}{2} \int_0^{x_{\max}-(t-y)} \int_0^{x_{\max}-(t-y)} \alpha_{t,y,i}\alpha_{t,y,j} \,\mathrm{d}\log C_{t,y,i} \,\mathrm{d}\log C_{t,y,j} - \theta_{t-y} \,\mathrm{d}t.\end{aligned} \tag{27}$$

The quantity $\mathrm{d}\log C_{t,y,i} \,\mathrm{d}\log C_{t,y,j}$ denotes the quadratic covariation between $\log C_{t,y,i}$ and $\log C_{t,y,j}$. More specifically,

$$\begin{aligned}\mathrm{d}\log C_{t,y,i} \,\mathrm{d}\log C_{t,y,j} = &\left(\left[q_{t,y,i} - q_{t,y,0}\right] \left(\omega_{t,y}^S\right)^\top \Sigma - D_{y,i}^\top \Sigma^X \right) \times \rho \\ &\times \left(\left[q_{t,y,j} - q_{t,y,0}\right] \left(\omega_{t,y}^S\right)^\top \Sigma - D_{y,j}^\top \Sigma^X \right)^\top \mathrm{d}t.\end{aligned} \tag{28}$$

E.3. Dynamics of Current Annuity Units

We specify the annuity units at time $t+h$ of a policyholder born at time y as follows:

$$B_{t+h,y} = B_{y+x_r,y} \exp\left\{ \int_{y+x_r}^{t+h} \gamma_{s,y}^f \,\mathrm{d}s + \int_{y+x_r}^{t+h} \Sigma_{s,y,t+h-s}^{B\top} \,\mathrm{d}Z_s \right\}. \tag{29}$$

Here, $\gamma_{t,y}^f$ denotes the (unconditional) median growth rate of annuity units at time t of a policyholder born at time y and the vector $\Sigma_{t,y,h}^B$ models the exposure of future log annuity units $\log B_{t+h,y}$ to a current Brownian shock $\mathrm{d}Z_t$. We require that $\Sigma_{t,y,h}^B$ increases with the horizon h, so that a current Brownian shock has a larger impact on annuity units in the distant future than on annuity units in the near future. It follows from (29) that

$$B_{t+h,y} = B_{t,y} F_{t,y,h} \exp\left\{ \int_t^{t+h} \gamma_{s,y}^f \,\mathrm{d}s + \int_t^{t+h} \Sigma_{s,y,t+h-s}^{B\top} \,\mathrm{d}Z_s \right\}, \tag{30}$$

where

$$F_{t,y,h} = \exp\left\{ \int\limits_{y+x_r}^{t} \left(\Sigma^{B\top}_{s,y,t+h-s} - \Sigma^{B\top}_{s,y,t-s} \right) \mathrm{d}Z_s \right\}. \tag{31}$$

Comparison of (31) with (22) yields

$$\Sigma^{B\top}_{t,y,h} = q_{t,y,h} \left(\omega^{S}_{t,y} \right)^{\top} \Sigma. \tag{32}$$

Equation (32) relates the vector of speculative portfolio weights to the volatility of annuity units.

Log annuity units $\log B_{t,y}$ evolve according to (this follows from (29)):

$$\mathrm{d}\log B_{t,y} = \gamma^{f}_{t,y}\,\mathrm{d}t + \Sigma^{B\top}_{t,y,0}\,\mathrm{d}Z_t + \int\limits_{y+x_r}^{t} \mathrm{d}\Sigma^{B\top}_{s,y,t-s}\,\mathrm{d}Z_s = \gamma_{t,y}\,\mathrm{d}t + \Sigma^{B\top}_{t,y,0}\,\mathrm{d}Z_t. \tag{33}$$

Here, $\gamma_{t,y}$ denotes the *actual* median growth rate of annuity units. We allow the actual median growth rate of annuity units to depend on past Brownian shocks and current expected financial rates of return. Hence, we can decompose $\gamma_{t,y}$ as follows:

$$\gamma_{t,y} = \gamma^{b}_{t,y} + \gamma^{f}_{t,y}, \tag{34}$$

where

$$\gamma^{b}_{t,y} = \int\limits_{y+x_r}^{t} \mathrm{d}\Sigma^{B\top}_{s,y,t-s}\,\mathrm{d}Z_s. \tag{35}$$

The parameters $\gamma^{b}_{t,y}$ and $\gamma^{f}_{t,y}$ are referred to as the backward-looking and forward-looking component of $\gamma_{t,y}$, respectively. Equation (35) represents past Brownian shocks that are absorbed into the median growth rate of annuity units.

F. Relationships between the Parameters of a PPR

The previous sections have modelled a PPR in terms of seven parameters. The first column of Table I lists these parameters. This section derives several relationship between the parameters of a PPR that follow from the budget condition (5). This condition is equivalent to the following two conditions:

$$W_{y+x_r,y} = V_{y+x_r,y} = B_{y+x_r,y} C_{y+x_r,y}. \tag{36}$$

$$\mathrm{d}\log W_{t,y} = \mathrm{d}\log V_{t,y} = \mathrm{d}\log B_{t,y} + \mathrm{d}\log C_{t,y}. \tag{37}$$

Substitution of (7), (26) and (33) into (37) yields

$$0 = \left(\mu^W_{t,y} - \mu^C_{t,y} - \gamma_{t,y}\right)\mathrm{d}t + \left(\omega^\top_{t,y}\Sigma - \left(\omega^S_{t,y}\right)^\top \Sigma + \widehat{D}^\top_{t,y}\Sigma^X\right)\mathrm{d}Z_t. \tag{38}$$

Using (36) and (38), we find the following system of equations:

$$W_{y+x_r,y} = B_{y+x_r,y} C_{y+x_r,y}, \tag{39}$$

$$\gamma_{t,y} = \mu^W_{t,y} - \mu^C_{t,y}, \tag{40}$$

$$\omega^\top_{t,y} = \left(\omega^S_{t,y}\right)^\top - \widehat{D}^\top_{t,y}\Sigma^X\Sigma^{-1}. \tag{41}$$

Policyholders can use this system of equations to determine the values of the parameters. The second equation (see (40)) shows that the median growth rate of annuity units equals the difference between the expected financial return on log pension wealth and the expected financial return on the log conversion factor. The third equation (see (41)) shows that the vector of portfolio weights $\omega_{t,y}$ is equal to the sum of the vector of speculative portfolio weights $\omega^S_{t,y}$ and the vector of hedge portfolio weights $\omega^H_{t,y} = -\left(\Sigma^X\Sigma^{-1}\right)^\top \widehat{D}_{t,y}$. We also have a restriction for every horizon h (see also (32)):

$$\Sigma^B_{t,y,h} = q_{t,y,h}\left(\omega^S_{t,y}\right)^\top \Sigma. \tag{42}$$

Equations (39) – (41) do not uniquely identify the parameters of a PPR. Policyholders must thus specify the values of some parameters exogenously. The parameters of a PPR can be specified according to at least two approaches – the investment approach and the consumption approach. In the investment approach, policyholders specify the discount rate, the investment policy and the buffering function exogenously; see also the second column of Table I. In the consumption approach, policyholders specify the initial annuity units, the median growth rate of annuity units and the volatility of annuity units exogenously; see also the third column of Table I. The next sections explore the investment approach and the consumption approach in more detail.

Table I
Investment Approach and Consumption Approach

The second (third) column of this table summarizes the exogenous and endogenous parameters of the investment (consumption) approach.

Parameter	Investment Approach	Consumption Approach
Initial Account Value $W_{y+x_r,y}$	Exogenous	Endogenous
Portfolio Strategy $\omega_{t,y}$	Exogenous	Endogenous
Discount Rate $\delta^f_{t,y,h}$	Exogenous	Endogenous
Buffering Function $q_{t,y,h}$	Exogenous	Endogenous
Initial Annuity Units $B_{y+x_r,y}$	Endogenous	Exogenous
Volatility Vector $\Sigma^B_{t,y,h}$	Endogenous	Exogenous
Growth Rate $\gamma^f_{t,y}$	Endogenous	Exogenous

II. Investment Approach

This section explores the investment approach. In this approach, policyholders specify the initial account value, the forward-looking component of the discount rate, the vector of portfolio weights and the buffering function $q_{t,y,h}$; see also Table I. The annuity units at the age of retirement, the forward-looking component of the median growth of annuity units and the volatility function $\Sigma^B_{t,y,h}$ follow endogenously. Section II.A specifies the vector of state variables and its dynamics. Section II.B considers the investment approach with the restriction that $q_{t,y,h}$ is independent of h (i.e., past speculative shocks do not affect the discount rate). We relax this restriction in Section II.C.

A. State Variables

For the sake of simplicity, we characterize asset prices by the following three state variables: the rate of inflation π_t, the (short-term) real interest rate r_t, and the nominal stock price S_t. Hence, $X_t = (\pi_t, r_t, S_t)$. Following Brennan and Xia (2002), the rate of inflation and the real interest rate are driven by mean-reverting processes of the Ornstein-Uhlenbeck type. We describe the stock price by a geometric Brownian motion. The drift term μ^X and the diffusion coefficient Σ^X are thus specified as follows:

$$\mu^X = \begin{pmatrix} \eta\left(\bar{\pi} - \pi_t\right) \\ \kappa\left(\bar{r} - r_t\right) \\ S_t R_f + S_t \lambda_S \sigma_S \end{pmatrix}, \quad \Sigma^X = \begin{pmatrix} \sigma_\pi & 0 & 0 \\ 0 & \sigma_r & 0 \\ 0 & 0 & S_t \sigma_S \end{pmatrix}. \tag{43}$$

Here, $\eta > 0$ and $\kappa > 0$ are mean reversion coefficients, $\bar{\pi}$ and $\bar{r}$ denote long-term means, λ_S is the constant equity risk premium per unit of risk, and $\sigma_\pi > 0$, $\sigma_r > 0$ and $\sigma_S > 0$ correspond to diffusion coefficients.

Policyholders invest their wealth in three risky assets: two nominal zero-coupon bonds (with different times of maturity $h_1 = T_1 - t$ and $h_2 = T_2 - t$) and a risky stock. We find the following expressions for the expected excess return $\mu - R_f$ and the the diffusion matrix Σ (see Appendix):

$$\mu - R_f = \begin{pmatrix} -\lambda_\pi \sigma_\pi K_{h_1} - \lambda_r \sigma_r L_{h_1} \\ -\lambda_\pi \sigma_\pi K_{h_2} - \lambda_r \sigma_r L_{h_2} \\ \lambda_S \sigma_S \end{pmatrix}, \quad \Sigma = \begin{pmatrix} -\sigma_\pi K_{h_1} & -\sigma_r L_{h_1} & 0 \\ -\sigma_\pi K_{h_2} & -\sigma_r L_{h_2} & 0 \\ 0 & 0 & \sigma_S \end{pmatrix}. \tag{44}$$

Here, $\lambda = (\lambda_\pi, \lambda_r, \lambda_S)$ is the vector of market prices of risk, $K_h = \frac{1}{\eta}\left(1 - e^{-\eta h}\right)$ and $L_h = \frac{1}{\kappa}\left(1 - e^{-\kappa h}\right)$.

B. *Direct Absorption of Speculative Shocks*

Figure 1 illustrates the investment approach with the restriction that the buffering function $q_{t,y,h}$ is independent of h and the discount rate $\delta_{t,y,h}$ is constant (i.e., $\delta_{t,y,h} = \delta$). Standard variable annuity products typically satisfy these two assumptions (see, e.g., Chai et al., 2011; Maurer et al., 2013b). A constant discount rate implies that the vector of hedge portfolio weights is equal to zero:

$$\omega^H_{t,y} = 0. \tag{45}$$

The annuity units at the age of retirement, the forward-looking component of the median growth rate of annuity units and the function $\Sigma^B_{t,y,h}$ follow from equations (39), (40) and

(41), respectively. We find[16]

$$B_{y+x_r,y} = \frac{W_{y+x_r,y}}{C_{y+x_r,y}}, \tag{46}$$

$$\gamma^f_{t,y} = r_t + \pi_t + \omega_{t,y}^\top \left(\mu - R_f\right) - \frac{1}{2}\omega_{t,y}^\top \Sigma\rho\Sigma^\top \omega_{t,y} - \delta, \tag{47}$$

$$\Sigma^B_{t,y,h} = \left(\omega^S_{t,y}\right)^\top \Sigma. \tag{48}$$

As shown by equation (47), the expected return on pension wealth and the discount rate together determine the median growth rate of annuity units. As a result, the median growth rate of annuity units is not constant, but rather depends on the real interest rate and the rate of inflation. If policyholders aim to achieve a stable median pension stream during the payout phase, the dependence of the median growth rate on the state variables is undesirable. To obtain a constant median growth rate, policyholders should determine the discount rate endogenously. Section III derives the discount rate under the assumption that policyholders specify the median growth rate of annuity units exogenously. Furthermore, equation (48) shows that every cash flow has the same exposure to a current speculative shock. Indeed, the annuity factor remains unchanged following a speculative shock.

C. Gradual Absorption of Speculative Shocks

This section allows $q_{t,y,h}$ to depend on the investment horizon h. As a result, the backward-looking component of the median growth rate of annuity units no longer equals zero (see equation (23)). The forward-looking component of the median growth rate of annuity units is still given by equation (47). However, equation (48) is no longer valid. The exposure of future annuity units to a current Brownian shock is now given by

$$\Sigma^B_{t,y,h} = q_{t,y,h}\left(\omega^S_{t,y}\right)^\top \Sigma. \tag{49}$$

It follows from equation (20) that $q_{t,y,h}$ converges to one as age approaches death. Hence, under the condition that the vector of speculative portfolio weights $\omega^S_{t,y}$ is constant over time, the volatility of future annuity units $\Sigma^B_{t,y,h}$ increases as policyholders grow older. To maintain a stable consumption stream over the life cycle, policyholders may thus want

[16] We note that the backward-looking component of the median growth rate of annuity units is equal to zero.

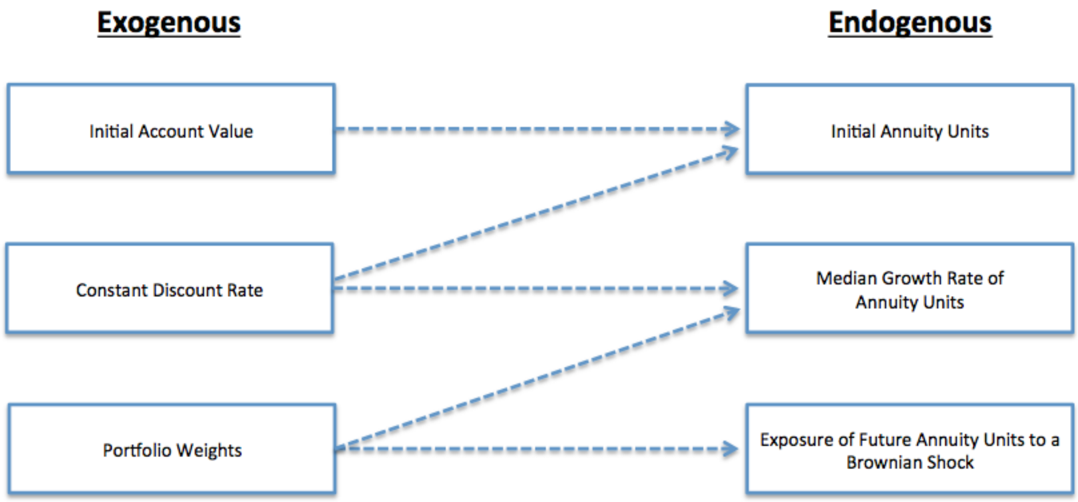

Figure 1. Illustration of the investment approach: A special case. The figure illustrates the investment approach with the restriction that the discount rate is constant and the function $q_{t,y,h}$ is independent of h. The left-hand side of the figure shows the exogenous design parameters of the pension contract. These exogenous parameters determine the parameters of the pension contract on the right-hand side of the figure.

to adopt a life cycle investment strategy, so that $\Sigma^B_{t,y,h}$ remains constant over the course of their lives.

III. Consumption Approach

This section explores the consumption approach. In this approach, policyholders specify the entire consumption stream in retirement exogenously. Section III.A examines the Defined Benefit (DB) approach in which consumption is constant in either nominal or real terms. Section III.B extends the DB approach to stochastic pension payments.

A. *Defined Benefit Approach*

In the DB approach, policyholders specify how much to consume at the age of retirement and the rate at which consumption grows over time. Pension payments either grow with the inflation rate (guaranteed real pension payments) or do not grow at all (guaranteed nominal pension payments). In this section, we derive the discount rate and the vector of portfolio weights endogenously from the liabilities of the pension contract. This contrasts with Section II in which policyholders specify the discount rate and the vector of portfolio weights exogenously. Figure 2 illustrates the DB approach.

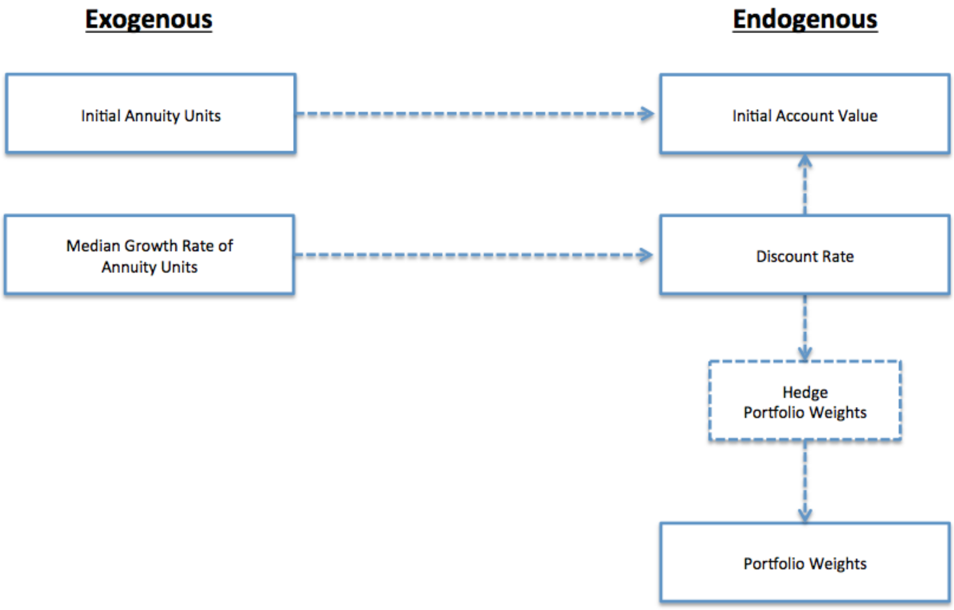

Figure 2. Illustration of the DB approach. The figure illustrates the DB approach. The left-hand side of the figure shows the exogenous parameters of the pension contract. These exogenous parameters determine the parameters on the right-hand side of the figure.

The DB approach specifies the growth rate of annuity units as follows:

$$\gamma_{t,y} = \gamma^f_{t,y} = g \cdot \pi_t, \tag{50}$$

where $g \in \{0, 1\}$. If g equals zero (unity), consumption is constant in nominal (real) terms. The coefficient g determines the discount rate and the vector of hedge portfolio weights.[17] The following specification of the discount rate yields budget balance (see Appendix):

$$\delta_{t,y,h} = \delta^f_{t,y,h} = d_{0h} + d_{1h} \cdot \pi_t + d_{2h} \cdot r_t, \tag{51}$$

where the coefficients d_{0h}, d_{1h} and d_{2h} follow endogenously from the liabilities of the contract. The vector of portfolio weights $\omega^H_{t,y}$ depends on $\widehat{D}_{t,y} = \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} D_{t,y,h}\, \mathrm{d}h$ as follows (see also (41)):

$$\omega^H_{t,y} = -\left(\Sigma^X \Sigma^{-1}\right)^\top \widehat{D}_{t,y}. \tag{52}$$

Here, $D_{t,y,h} = \nabla_X \int_0^h \delta_{t,v}\, \mathrm{d}v = \left(\int_0^h d_{1v}\, \mathrm{d}v, \int_0^h d_{2v}\, \mathrm{d}v, 0\right) = (D_{1h}, D_{2h}, D_{3h}) = D_h$ denotes

[17]We note that the vector of speculative portfolio weights is zero by definition.

the sensitivity vector of the discount rate $\bar{\delta}_{t,y,h}h$ with respect to the state variables. The coefficients d_{0h}, d_{1h} and d_{2h} solve the following system of equations (see Appendix):

$$d_{0h} = -D_h^\top \Sigma^X \Sigma^{-1} \left(\mu - R_f\right) + \eta D_{1h}\bar{\pi} + \kappa D_{2h}\bar{r} - \frac{1}{2} D_h^\top \Sigma^X \rho D_h \Sigma^{X\top}, \tag{53}$$

$$d_{1h} = 1 - g - \eta D_{1h}, \tag{54}$$

$$d_{2h} = 1 - \kappa D_{2h}. \tag{55}$$

This system of equations shows that the coefficient g determines the discount rate (51).

A.1. Guaranteed Nominal Pension Payments

Policyholders receive guaranteed nominal annuity units if $g = 0$. Solving (54) - (55), we arrive at $d_{1h} = e^{-\eta h}$ and $d_{2h} = e^{-\kappa h}$.[18] Substituting the expressions of d_{0h}, d_{1h} and d_{2h} into equation (51) and using $e^{-\kappa h} r_t + e^{-\eta h}\pi_t + \left(1 - e^{-\kappa h}\right)\bar{r} + \left(1 - e^{-\eta h}\right)\bar{\pi} = \mathbb{E}_t\left[r_{t+h} + \pi_{t+h}\right]$, we find

$$\begin{aligned} \delta_{t,y,h} = {} & \mathbb{E}_t\left[r_{t+h} + \pi_{t+h}\right] - D_{1h}\lambda_r\sigma_r - D_{2h}\lambda_\pi\sigma_\pi \\ & - \frac{1}{2}\left(D_{1h}\sigma_r\right)^2 - \frac{1}{2}\left(D_{2h}\sigma_\pi\right)^2 - \rho_{12}D_{1h}D_{2h}\sigma_r\sigma_\pi. \end{aligned} \tag{56}$$

The discount rate (56) equals the nominal forward interest rate $r_{t,h}$ (see equation (A11) in the Appendix). Indeed, payments are risk-free. As a result, funds should use the risk-free term structure to discount future liabilities. The inflation sensitivity and the real interest rate sensitivity of the liabilities are given by $\widehat{D}_{1t,y} = \frac{1}{\eta}\int_0^{x_{\max}-(t-y)} \alpha_{t,y,h}\left(1 - e^{-\eta h}\right) \mathrm{d}h$ and $\widehat{D}_{2t,y} = \frac{1}{\kappa}\int_0^{x_{\max}-(t-y)} \alpha_{t,y,h}\left(1 - e^{-\kappa h}\right) \mathrm{d}h$, respectively. Pension funds can replicate the pension contract by investing in a portfolio of nominal bonds with inflation sensitivity $\widehat{D}_{1t,y}$ and the real interest rate sensitivity $\widehat{D}_{2t,y}$.

A.2. Guaranteed Real Pension Payments

Policyholders receive guaranteed real annuity units if $g = 1$. Solving (54) - (55), we arrive at $d_{1h} = 0$ and $d_{2h} = e^{-\kappa h}$. The discount rate is equal to the real forward interest

[18] Substituting $g = 0$ into (54) and using $D_{1h} = \int_0^h d_{1v}\,\mathrm{d}v$, we arrive at $d_{1h} = 1 - \eta\int_0^h d_{1v}\,\mathrm{d}v$. Substitution of $d_{1v} = e^{-\eta v}$ into this equation yields $e^{-\eta h} = 1 - \eta\int_0^h e^{-\eta v}\,\mathrm{d}v = 1 - \eta\frac{1}{\eta}\left(1 - e^{-\eta h}\right) = e^{-\eta h}$.

rate:

$$\delta_{t,y,h} = \mathbb{E}_t\left[r_{t+h}\right] - D_{2h}\lambda_r\sigma_r - \frac{1}{2}\left(D_{2h}\sigma_r\right)^2. \tag{57}$$

The value of the liabilities is insensitive to changes in the inflation rate. Indeed, pension payments are guaranteed in real terms. The real interest rate sensitivity of the liabilities is given by $\widehat{D}_{2t,y} = \frac{1}{\kappa}\int_0^{x_{\max}-(t-y)} \alpha_{t,y,h}\left(1-e^{-\kappa h}\right) \mathrm{d}h$. To guarantee the pension payments to the policyholders, pension funds should invest in an investment portfolio that is insensitive to changes in the inflation rate and has the same real interest rate sensitivity as the liabilities.

B. Defined Ambition Approach

The Defined Ambition (DA) approach generalizes the DB approach to stochastic pension payments. In the DA approach, policyholders specify how much to consume at the age of retirement, the median rate at which consumption grows over time and the degree of uncertainty in consumption. As in the DB approach, the discount rate and the vector of portfolio weights follow endogenously from the liabilities of the pension contract. The DA approach generalizes the growth rate of annuity units (50) as follows:

$$\gamma_{t,y} = \gamma_{t,y}^f = g_0 + g_1 \cdot \pi_t + g_2 \cdot r_t, \tag{58}$$

where the coefficients g_0, g_1 and g_2 are given exogenously. We note that if $g_1 \in \{0,1\}$ and $g_0 = g_2 = 0$, then (58) coincides with (50). In the Appendix, we show that the following specification of the discount rate yields budget balance:

$$\delta_{t,y,h} = \delta_{t,y,h}^f = d_{0h} + d_{1h} \cdot \pi_t + d_{2h} \cdot r_t. \tag{59}$$

Here, the coefficients d_{0h}, d_{1h} and d_{2h} are determined by the liabilities of the pension contract. The vector of hedge portfolio weights $\omega_{t,y}^H$ is given by

$$\omega_{t,y}^H = -\left(\Sigma^X \Sigma^{-1}\right)^\top \widehat{D}_{t,y}, \tag{60}$$

where $\widehat{D}_{t,y} = \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} D_{t,y,h}\, \mathrm{d}h$ with $D_{t,y,h} = \left(\int_0^h d_{1v}\,\mathrm{d}v, \int_0^h d_{2v}\,\mathrm{d}v, 0\right)$. The vector $D_{t,y,h}$ measures how sensitive the discount rate $\bar{\delta}_{t,y,h}h$ is with respect to the state variables. Section III.B.1 assumes that the exposure of future annuity units to a current Brownian shock does not depend on the horizon h; that is, $\Sigma_{t,y,h}^B = \Sigma^B$. Figure 3 illustrates this

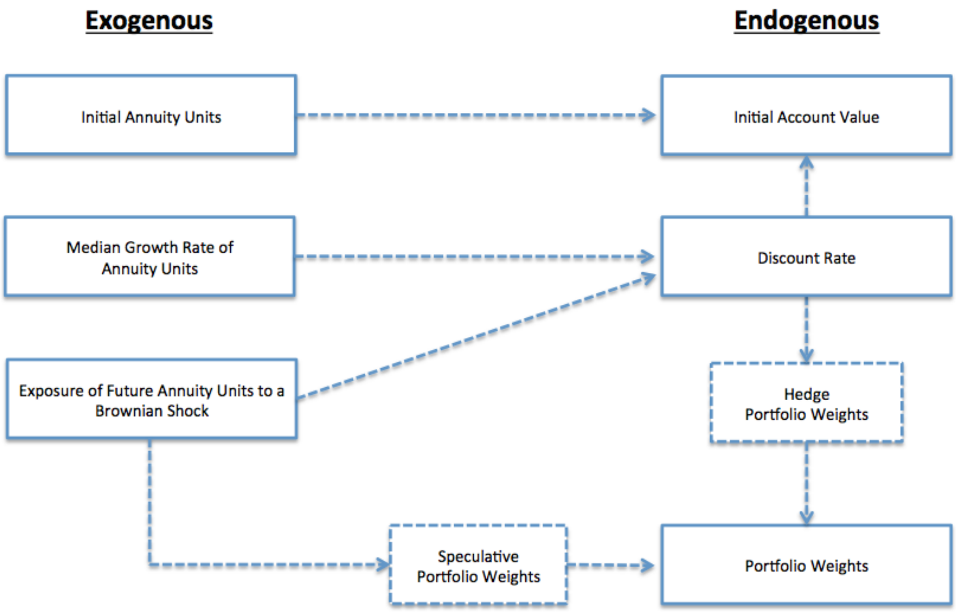

Figure 3. Illustration of the DA approach: Direct adjustment of annuity units. The figure illustrates the DA approach with direct adjustment of annuity units. The left-hand side of the figure shows the exogenous parameters of the pension contract. These exogenous parameters determine the parameters on the right-hand side of the figure.

case. Section III.B.2 considers the DA approach with gradual adjustment of annuity units to Brownian shocks. This case is illustrated by Figure 4.

B.1. Direct Adjustment of Annuity Units to Speculative Shocks

The volatility vector Σ^B determines the vector of speculative portfolio weights as follows:

$$\omega^S_{t,y} = \omega^S = \left(\Sigma^B \Sigma^{-1}\right)^\top . \tag{61}$$

The vector of total portfolio weights is thus given by

$$\omega_{t,y} = \omega^S + \omega^H_{t,y} = \left(\Sigma^B \Sigma^{-1}\right)^\top - \left(\widehat{D}^\top_{t,y} \Sigma^X \Sigma^{-1}\right)^\top . \tag{62}$$

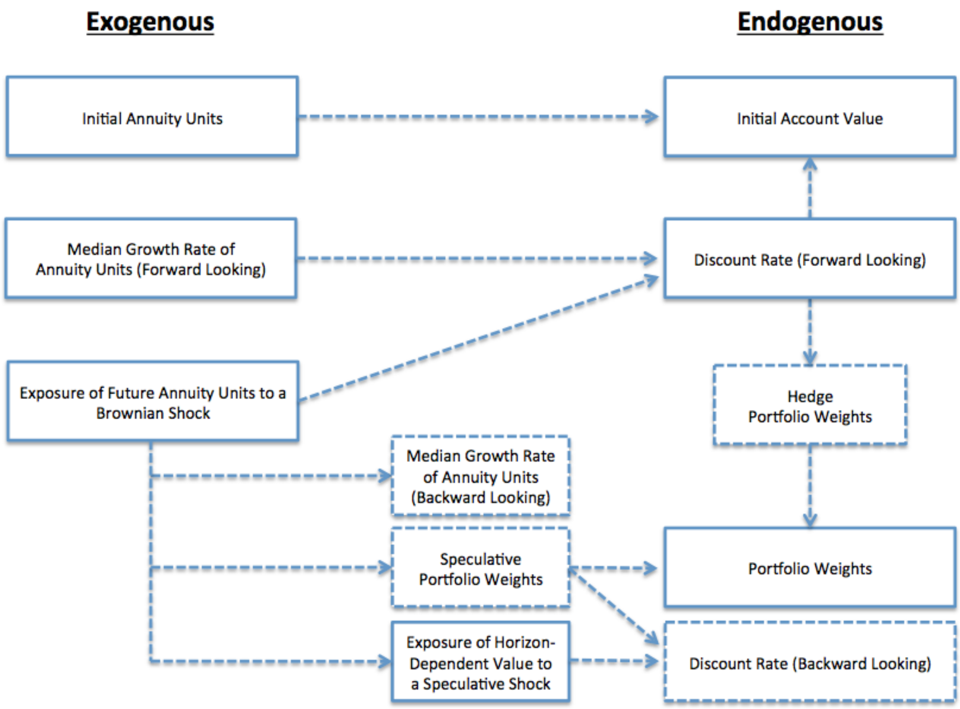

Figure 4. Illustration of the DA approach: Gradual adjustment of annuity units. The figure illustrates the DA approach with gradual adjustment of annuity units. The left-hand side of the figure shows the exogenous parameters of the pension contract. These exogenous parameters determine the parameters on the right-hand side of the figure.

The coefficients d_{0h}, d_{1h} and d_{2h} solve the following system of equations:

$$\begin{aligned} d_{0h} = & -g_0 + \kappa D_{2h}\bar{r} + \eta D_{1h}\bar{\pi} + \left(\Sigma^B - D_h^\top \Sigma^X\right)\Sigma^{-1}\left(\mu - R_f\right) \\ & - \frac{1}{2} D_h^\top \Sigma^X \rho D_h \Sigma^{X\top} - \frac{1}{2}\Sigma^B \rho \left(\Sigma^B\right)^\top + \Sigma^B \rho \left(D_h^\top \Sigma^X\right)^\top, \end{aligned} \tag{63}$$

$$d_{1h} = 1 - g_1 - \eta D_{1h}, \tag{64}$$

$$d_{2h} = 1 - g_2 - \kappa D_{2h}. \tag{65}$$

Solving (64) – (65), we arrive at $d_{1h} = (1 - g_1)\, e^{-\eta h}$ and $d_{2h} = (1 - g_2)\, e^{-\kappa h}$. The forward-looking component of the discount rate is thus given by

$$\begin{aligned} \delta_h^f = & \mathbb{E}_t\left[(1 - g_2)\, r_{t+h} + (1 - g_1)\, \pi_{t+h}\right] + \left(\Sigma^B - D_h^\top \Sigma^X\right)\Sigma^{-1}\left(\mu - R_f\right) \\ & - \frac{1}{2} D_h^\top \Sigma^X \rho D_h \Sigma^{X\top} - \frac{1}{2}\Sigma^B \rho \left(\Sigma^B\right)^\top + \Sigma^B \rho \left(D_h^\top \Sigma^X\right)^\top - g_0. \end{aligned} \tag{66}$$

The inflation sensitivity $\widehat{D}_{1t,y}$ and the real interest rate sensitivity $\widehat{D}_{2t,y}$ of the pension liabilities are given by $\widehat{D}_{1t,y} = (1-g_1)\frac{1}{\eta}\int_0^{x_{\max}-(t-y)} \alpha_{t,y,h}\left(1-e^{-\eta h}\right) \mathrm{d}h$ and $\widehat{D}_{2t,y} = (1-g_2)\frac{1}{\kappa}\int_0^{x_{\max}-(t-y)} \alpha_{t,y,h}\left(1-e^{-\kappa h}\right) \mathrm{d}h$, respectively. The pension should choose a hedge portfolio with the same sensitivities.

B.2. Gradual Adjustment of Annuity Units to Speculative Shocks

The vector $\widehat{\Sigma}_{t,y}^B = \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h}\Sigma_h^B \,\mathrm{d}h$ determines the vector of speculative weights:

$$\omega_{t,y}^S = \left(\widehat{\Sigma}_{t,y}^B \Sigma^{-1}\right)^\top . \tag{67}$$

It follows from (67) that the investor implements a life cycle strategy if Σ_h^B increases with the horizon h (because $\widehat{\Sigma}_{t,y}^B$ decreases as the investor ages). The vector of total portfolio weights is given by

$$\omega_{t,y} = \omega_{t,y}^S + \omega_{t,y}^H = \left(\widehat{\Sigma}_{t,y}^B \Sigma^{-1}\right)^\top - \left(\widehat{D}_{t,y}^\top \Sigma^X \Sigma^{-1}\right)^\top . \tag{68}$$

The coefficients d_{1h} and d_{2h} are the same as in the previous section. The coefficient d_{0h} is given by

$$\begin{aligned} d_{0h} = & -g_0 + \kappa D_{2h}\bar{r} + \eta D_{1h}\bar{\pi} + \left(\Sigma_h^B - D_h^\top \Sigma^X\right)\Sigma^{-1}\left(\mu - R_f\right) \\ & - \frac{1}{2} D_h^\top \Sigma^X \rho D_h \Sigma^{X\top} - \frac{1}{2}\Sigma_h^B \rho \left(\Sigma_h^B\right)^\top + \Sigma_h^B \rho \left(D_h^\top \Sigma^X\right)^\top . \end{aligned} \tag{69}$$

IV. Concluding Remarks

This paper has explored how to model a PPR. We have derived a system of restrictions on the parameters of a PPR. These restrictions do not uniquely identify the parameters. As a result, policyholders must specify the values of some parameters exogenously. The parameter values can be specified according to (at least) two approaches - the investment approach and the consumption approach. In the investment approach, policyholders specify the speed of decumulation and the investment policy exogenously. We have showed how these exogenous parameters determine the median growth rate and volatility of retirement income. The consumption approach specifies the entire income stream exogenously. We have showed which discount rate and investment policy correspond to a particular consumption profile.

Appendix A. Bond Price Dynamics

We start by deriving the analytical solution to the stochastic differential equation (SDE) for the Ornstein-Uhlenbeck process. After applying Itô's Lemma to the function $f(t, \pi_t) = e^{\eta t}(\pi_t - \bar{\pi})$, we find

$$\begin{aligned} \mathrm{d}f(t, \pi_t) &= \eta e^{\eta t}(\pi_t - \bar{\pi})\,\mathrm{d}t + e^{\eta t}\mathrm{d}\pi_t \\ &= \eta e^{\eta t}(\pi_t - \bar{\pi})\,\mathrm{d}t - e^{\eta t}\eta(\pi_t - \bar{\pi})\,\mathrm{d}t + e^{\eta t}\sigma_\pi \mathrm{d}Z_{1t} = \sigma_\pi e^{\eta t}\mathrm{d}Z_{1t}. \end{aligned} \tag{A1}$$

The solution of (A1) is given by

$$f(t, \pi_{t+v}) = f(t, \pi_t) + \sigma_\pi \int_t^{t+v} e^{\eta u}\mathrm{d}Z_{1u}. \tag{A2}$$

The inflation rate at time $t + v > t$ is given by (the first and third equality follow from the definition of $f(t, \pi_t)$, and the second equality follows from (A2))

$$\begin{aligned} \pi_{t+v} &= \bar{\pi} + e^{-\eta(t+v)} f(t, \pi_{t+v}) = \bar{\pi} + e^{-\eta(t+v)} f(t, \pi_t) + \sigma_\pi \int_t^{t+v} e^{-\eta(t+v-u)}\mathrm{d}Z_{1u} \\ &= \bar{\pi} + e^{-\eta v}(\pi_t - \bar{\pi}) + \sigma_\pi \int_0^v e^{-\eta(v-u)}\mathrm{d}Z_{1(t+u)} \\ &= \pi_t + \left(1 - e^{-\eta v}\right)(\bar{\pi} - \pi_t) + \sigma_\pi \int_0^v e^{-\eta(v-u)}\mathrm{d}Z_{1(t+u)}. \end{aligned} \tag{A3}$$

In a similar fashion, we find

$$r_{t+v} = r_t + \left(1 - e^{-\kappa v}\right)(\bar{r} - r_t) + \sigma_r \int_0^v e^{-\kappa(v-u)}\mathrm{d}Z_{2(t+u)}.$$

The (conditional) expectation of the inflation rate $\mathbb{E}_t[\pi_{t+v}]$ and the (conditional) expectation of the real interest rate $\mathbb{E}_t[r_{t+v}]$ are given by

$$\mathbb{E}_t[\pi_{t+v}] = \pi_t + \eta(\bar{\pi} - \pi_t) K_v, \tag{A4}$$

$$\mathbb{E}_t[r_{t+v}] = r_t + \kappa(\bar{r} - r_t) L_v. \tag{A5}$$

The average inflation rate $\check{\pi}_{t,h} = \frac{1}{h}\int_0^h \pi_{t+v}\mathrm{d}v$ and the average real interest rate $\check{r}_{t,h} = \frac{1}{h}\int_0^h r_{t+v}\mathrm{d}v$ play a key role in determining the yield to maturity. We find (the first equality follows from substituting (A3) to eliminate π_{t+v})

$$\begin{aligned}
\check{\pi}_{t,h} &= \frac{1}{h}\int_0^h \pi_{t+v}\mathrm{d}v \\
&= \frac{1}{h}\int_0^h \left(\pi_t + (\bar{\pi}-\pi_t)\left(1-e^{-\eta v}\right)\right)\mathrm{d}v + \frac{\sigma_\pi}{h}\int_0^h\int_0^v e^{-\eta(v-u)}\mathrm{d}Z_{1(t+u)}\mathrm{d}v \\
&= \frac{1}{h}\int_0^h \left(\pi_t + (\bar{\pi}-\pi_t)\left(1-e^{-\eta v}\right)\right)\mathrm{d}v + \frac{\sigma_\pi}{h}\int_0^h\int_v^h e^{-\eta(h-u)}\mathrm{d}u\mathrm{d}Z_{1(t+v)} \qquad \text{(A6)} \\
&= \frac{1}{h}\int_0^h \left(\pi_t + (\bar{\pi}-\pi_t)\,\eta K_v\right)\mathrm{d}v + \frac{\sigma_\pi}{\eta h}\int_0^h \left(1-e^{-\eta(h-v)}\right)\mathrm{d}Z_{1(t+v)} \\
&= \frac{1}{h}\int_0^h \mathbb{E}_t\left[\pi_{t+v}\right]\mathrm{d}v + \frac{\sigma_\pi}{h}\int_0^h K_{h-v}\mathrm{d}Z_{1(t+v)}.
\end{aligned}$$

In a similar fashion, we find that the average real interest rate $\check{r}_{t,h}$ is given by

$$\check{r}_{t,h} = \frac{1}{h}\int_0^h r_{t+v}\mathrm{d}v = \frac{1}{h}\int_0^h \mathbb{E}_t\left[r_{t+v}\right]\mathrm{d}v + \frac{\sigma_r}{h}\int_0^h L_{h-v}\mathrm{d}Z_{2(t+v)}. \qquad \text{(A7)}$$

The pricing kernel is given by (see, e.g., Brennan and Xia, 2002):

$$\xi_t = \exp\left\{-\int_0^t \left(\pi_s + r_s + \frac{1}{2}\phi^\top\rho\phi\right)\mathrm{d}s + \phi Z_t\right\}. \qquad \text{(A8)}$$

Here, ϕ is a vector of factor loadings. We can determine the price of a bond is as follows:

$$\begin{aligned}
P_{t,h} &= \mathbb{E}_t\left[\frac{\xi_{t+h}}{\xi_t}\right], \\
&= \mathbb{E}_t\left[\exp\left\{-\int_0^h \left(\pi_{t+v} + r_{t+v} + \frac{1}{2}\phi^\top\rho\phi\right)\mathrm{d}v + \phi\left(Z_{t+h}-Z_t\right)\right\}\right].
\end{aligned} \qquad \text{(A9)}$$

Substituting (A6) and (A7) into the pricing formula (A9) to eliminate $\int_0^h \pi_{t+v}\mathrm{d}v$ and $\int_0^h r_{t+v}\mathrm{d}v$, we arrive at

$$\begin{aligned}
P_{t,h} &= \exp\left\{-\int_0^h \left(\mathbb{E}_t\left[\pi_{t+v}+r_{t+v}\right]+\frac{1}{2}\phi^\top\rho\phi\right)\mathrm{d}v\right\}\mathbb{E}_t\left[\exp\left\{\int_0^h \phi_3\mathrm{d}Z_{3(t+v)}\right\}\right.\\
&\left.\exp\left\{\int_0^h\left(\phi_1-\sigma_\pi K_{h-v}\right)\mathrm{d}Z_{1(t+v)}+\int_0^h\left(\phi_2-\sigma_r L_{h-v}\right)\mathrm{d}Z_{2(t+v)}\right\}\right]\\
&= \exp\left\{-\int_0^h \left(\mathbb{E}_t\left[\pi_{t+v}+r_{t+v}\right]-\lambda_\pi\sigma_\pi K_v-\lambda_r\sigma_r L_v\right.\right.\\
&\left.\left.-\frac{1}{2}\left(\sigma_\pi K_v\right)^2-\frac{1}{2}\left(\sigma_r L_v\right)^2-\rho_{12}\sigma_\pi\sigma_r K_v L_v\right)\mathrm{d}v\right\}\\
&= \exp\left\{-\int_0^h r_{t,v}\mathrm{d}v\right\}.
\end{aligned} \tag{A10}$$

The instantaneous forward interest rate $r_{t,v}$ is defined as follows:

$$\begin{aligned}
r_{t,v} &= \mathbb{E}_t\left[\pi_{t+v}+r_{t+v}\right]-\lambda_\pi\sigma_\pi K_v-\lambda_r\sigma_r L_v\\
&\quad-\frac{1}{2}\left(\sigma_\pi K_v\right)^2-\frac{1}{2}\left(\sigma_r L_v\right)^2-\rho_{12}\sigma_\pi\sigma_r K_v L_v.
\end{aligned} \tag{A11}$$

The log bond price is given by (this follows from (A4), (A5), (A10) and (A11))

$$\begin{aligned}
\log P_{t,h} = -\int_0^h &\left(\pi_t+\eta\left(\bar{\pi}-\pi_t\right)K_v+r_t+\kappa\left(\bar{r}-r_t\right)L_v-\lambda_\pi\sigma_\pi K_v\right.\\
&\left.-\lambda_r\sigma_r L_v-\frac{1}{2}\left(\sigma_\pi K_v\right)^2-\frac{1}{2}\left(\sigma_r L_v\right)^2-\rho_{12}\sigma_\pi\sigma_r K_v L_v\right)\mathrm{d}v.
\end{aligned} \tag{A12}$$

Solving the integral (A12), we arrive at[19]

$$\begin{aligned}
\log P_{t,h} &= -\pi_t h-\left(\bar{\pi}-\pi_t\right)\left(h-K_h\right)-r_t h-\left(\bar{r}-r_t\right)\left(h-L_h\right)\\
&\quad+\frac{\lambda_\pi\sigma_\pi}{\eta}\left(h-K_h\right)+\frac{\lambda_r\sigma_r}{\kappa}\left(h-L_h\right)+\frac{1}{2}\left(\frac{\sigma_\pi}{\eta}\right)^2\left(h-2K_h+\frac{1}{2}K_{2h}\right)\\
&\quad+\frac{1}{2}\left(\frac{\sigma_r}{\kappa}\right)^2\left(h-2L_h+\frac{1}{2}L_{2h}\right)+\frac{\rho_{12}\sigma_\pi\sigma_r}{\kappa\eta}\left(h-K_h-L_h+\frac{1-e^{-(\eta+\kappa)h}}{\eta+\kappa}\right)\\
&= -\pi_t K_h-r_t L_h-M_h,
\end{aligned}$$

[19]The first equality follows from $K_v^2=\left(1-2e^{-\eta v}+e^{-2\eta v}\right)/\eta^2$ and the second equality follows from $K_h^2=\left(2K_h-K_{2h}\right)/\kappa$.

where the horizon-dependent constant M_h is defined as follows:

$$\begin{aligned} M_h = & \left(\bar{\pi} - \frac{\lambda_\pi \sigma_\pi}{\eta} - \frac{1}{2}\left[\frac{\sigma_\pi}{\eta}\right]^2\right)(h - K_h) + \frac{1}{4\eta}(\sigma_\pi K_h)^2 \\ & + \left(\bar{r} - \frac{\lambda_r \sigma_r}{\kappa} - \frac{1}{2}\left[\frac{\sigma_r}{\kappa}\right]^2\right)(h - L_h) + \frac{1}{4\kappa}(\sigma_r L_h)^2 \\ & + \frac{\rho_{12}\sigma_\pi \sigma_r}{\eta\kappa}\left(h - K_h - L_h + \frac{1 - e^{-(\kappa+\eta)h}}{\eta + \kappa}\right). \end{aligned} \tag{A13}$$

To calculate how the value of the bond with a fixed maturity $t + h$ develops as time proceeds (i.e., $t + h$ is fixed but t changes), we apply Itô's Lemma to

$$P_{t,h} = \exp\left\{-\pi_t K_h - r_t L_h - M_h\right\}.$$

We find

$$\begin{aligned} \frac{\mathrm{d}P_{t,h}}{P_{t,h}} = & \Big(r_{t,h} - \eta(\bar{\pi} - \pi_t)K_h - \kappa(\bar{r} - r_t)L_h + \frac{1}{2}(\sigma_\pi K_h)^2 + \frac{1}{2}(\sigma_r L_h)^2 \\ & + \rho_{12}\sigma_\pi\sigma_r K_h L_h\Big)\mathrm{d}t - \sigma_\pi K_h \mathrm{d}Z_{1t} - \sigma_r L_h \mathrm{d}Z_{2t} \\ = & \left(r_t + \pi_t - \lambda_\pi \sigma_\pi K_h - \lambda_r \sigma_r L_h\right)\mathrm{d}t - \sigma_\pi K_h \mathrm{d}Z_{1t} - \sigma_r L_h \mathrm{d}Z_{2t}. \end{aligned}$$

Appendix B. Derivation of d_{0h}, d_{1h} and d_{2h}

Budget balance implies that (see also (40))

$$\gamma_{t,y} = \mu_{t,y}^W - \mu_{t,y}^C. \tag{B1}$$

Substituting (58) into (B1), we arrive at

$$g_0 + g_1 \cdot \pi_t + g_2 \cdot r_t = \mu_{t,y}^W - \mu_{t,y}^C. \tag{B2}$$

The vector of portfolio weights is given by

$$\omega_{t,y} = \omega_{t,y}^H + \omega_{t,y}^S = -\left(\Sigma^X \Sigma^{-1}\right)^\top \widehat{D}_{t,y} + \left(\Sigma^{-1}\right)^\top \widehat{\Sigma}_{t,y}^B. \tag{B3}$$

Substituting the vector of portfolio weights into (B3) yields

$$
\begin{aligned}
\mu_{t,y}^{W} = {} & \pi_t + r_t + \left(\widehat{\Sigma}_{t,y}^{B\top} - \widehat{D}_{t,y}^{\top}\Sigma^{X}\right)\Sigma^{-1}\left(\mu - R_f\right) \\
& - \frac{1}{2}\widehat{\Sigma}_{t,y}^{B\top}\rho\widehat{\Sigma}_{t,y}^{B} - \frac{1}{2}\widehat{D}_{t,y}^{\top}\Sigma^{X}\rho\Sigma^{X\top}\widehat{D}_{t,y} + \widehat{D}_{t,y}^{\top}\Sigma^{X}\rho\widehat{\Sigma}_{t,y}^{B}.
\end{aligned}
\tag{B4}
$$

Substituting (59) and the expression for μ^X (see (43)) into (27), we arrive at

$$
\begin{aligned}
\mu_{t,y}^{C} = {} & \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} d_{0h}\,\mathrm{d}h + \pi_t \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} d_{1h}\,\mathrm{d}h + r_t \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} d_{2h}\,\mathrm{d}h \\
& - \widehat{D}_{1t,y}^{\top}\eta\left(\bar{\pi} - \pi_t\right) - \widehat{D}_{2t,y}^{\top}\kappa\left(\bar{r} - r_t\right) + \frac{1}{2}\int_0^{x_{\max}-(t-y)} \alpha_{t,y,h}\,\mathrm{d}\log C_{t,y,h}\,\mathrm{d}\log C_{t,y,h} \\
& - \frac{1}{2}\int_0^{x_{\max}-(t-y)}\int_0^{x_{\max}-(t-y)} \alpha_{t,y,i}\alpha_{t,y,j}\,\mathrm{d}\log C_{t,y,i}\,\mathrm{d}\log C_{t,y,j}.
\end{aligned}
\tag{B5}
$$

It follows from substituting (B4) and (B5) into (B2) that d_{1h} and d_{2h} must satisfy the following two conditions:

$$
d_{1h} = 1 - g_1 + \eta D_{1h}, \tag{B6}
$$

$$
d_{2h} = 1 - g_2 + \kappa D_{2h}. \tag{B7}
$$

The coefficient d_{0h} must satisfy

$$
\begin{aligned}
\int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} d_{0h}\,\mathrm{d}h = {} & -g_0 + \left(\widehat{\Sigma}_{t,y}^{B\top} - \widehat{D}_{t,y}^{\top}\Sigma^{X}\right)\Sigma^{-1}\left(\mu - R_f\right) - \frac{1}{2}\widehat{\Sigma}_{t,y}^{B\top}\rho\widehat{\Sigma}_{t,y}^{B} \\
& - \frac{1}{2}\widehat{D}_{t,y}^{\top}\Sigma^{X}\rho\Sigma^{X\top}\widehat{D}_{t,y} + \widehat{D}_{t,y}^{\top}\Sigma^{X}\rho\widehat{\Sigma}_{t,y}^{B} + \widehat{D}_{1t,y}^{\top}\eta\bar{\pi} + \widehat{D}_{2t,y}^{\top}\kappa\bar{r} \\
& - \frac{1}{2}\int_0^{x_{\max}-(t-y)} \alpha_{t,y,h}\,\mathrm{d}\log C_{t,y,h}\,\mathrm{d}\log C_{t,y,h} \\
& + \frac{1}{2}\int_0^{x_{\max}-(t-y)}\int_0^{x_{\max}-(t-y)} \alpha_{t,y,i}\alpha_{t,y,j}\,\mathrm{d}\log C_{t,y,i}\,\mathrm{d}\log C_{t,y,j}.
\end{aligned}
\tag{B8}
$$

Straightforward computations yield (69).

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