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Robust Pricing of Fixed Income Securities

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Abstract

We analyze a dynamic investment problem with interest rate risk and ambiguity. After deriving the optimal terminal wealth and investment policy, we expand our model into a robust general equilibrium model and calibrate it to U.S. data. We confirm the bond premium puzzle, i.e., we need an unreasonably high relative risk-aversion parameter to explain excess returns on long-term bonds. Our model with robust investors reduces this risk-aversion parameter substantially: a relative risk aversion of less than four suffices to match market data. Additionally we provide a novel formulation of robust dynamic investment problems together with an alternative solution technique: the robust version of the martingale method.

JEL classification: C61, G11, G12.
Keywords: dynamic asset allocation, robustness, uncertainty, ambiguity, bond premium puzzle.

1 Introduction

Similarly to the equity premium puzzle, there exists a bond premium puzzle. The risk premium on long-term bonds is higher than predicted by mainstream models using reasonable parameter values. This phenomenon was first described in Backus, Gregory and Zin (1989), who show that in bond markets "*... the representative agent model with additively separable preferences fails to account for the sign or the magnitude of risk premiums*". Although the bond premium puzzle has received much less attention in the literature than the equity premium puzzle, it is by no means of less importance. As Rudebusch and Swanson (2008) remark, the value of outstanding long-term bonds in the U.S. is much larger than the value of outstanding equity. The present literature about equilibrium bond pricing estimates an unreasonably high relative risk aversion parameter. For instance Piazzesi and Schneider (2007) estimate the relative risk-aversion parameter to be 57; van Binsbergen, Fernández-Villaverde, Koijen and Rubio-Ramrez (2012) estimate a value of relative risk-aversion in their model around 80; while Rudebusch and Swanson (2012) estimate it to be 110. Although several potential explanations can be found in the literature, none of the provided solutions are generally accepted – just as in the case of the equity premium puzzle.

In this paper, we approach the bond premium puzzle from a new angle. A key parameter in any investment allocation model is the risk premium earned on investing in bonds or, equivalently, the prices of risk in a factor model. However, given a limited history of data, the investor faces substantial uncertainty about the magnitude of these risk premiums. We build on the literature on robust decision making and asset pricing and formulate a dynamic investment problem under robustness in a market for bonds and stocks. Our model features stochastic interest rates driven by a two-factor Gaussian affine term structure model. The investor chooses optimal portfolios of bonds and stocks taking into account the uncertainty about the bond and stock risk premiums. We solve the representative investor's optimization problem and give an explicit solution to the optimal terminal wealth, the least-favorable physical probability measure, and the optimal investment policy.

We calibrate the risk aversion and robustness parameters by equating the optimal portfolio weights implied by the model to weights observed in actual aggregate portfolio holdings. This is different from the existing literature, which typically calibrates first-order conditions of a consumption-based asset pricing model to the observed expected returns. Given the very low volatility of consumption, and the low correlation of stock and bond returns with consumption growth, that approach requires high levels of risk aversion to fit the observed risk premiums. Our approach only uses the optimal asset demands, which do not directly involve the volatility of or correlations with consumption; only the volatilities of the returns and intertemporal hedging demands for the assets are required. We then use the concept of detection error probabilities to disentangle the risk- and uncertainty-aversion parameters.

We estimate the parameters of our model using 42 years of U.S. market data by Maximum Likelihood. We find that matching the optimal equity demand to market data gives reasonable values for the risk-aversion parameter. If we assume that the financial market consists of only a stock market index and a money market account, the calibrated risk-aversion parameter in the non-robust version of our terminal wealth utility model is reasonable, 1.92 (see Table 5). For the bond demand the situation is very different: reasonable values of the risk-aversion parameter imply far too low bond risk premiums. The reason for this are the high bond risk premiums relative to the bond price volatilities. Only with robustness we can find reasonable values for the risk-aversion parameter to match observed demands. When calibrating the non-robust version of our model to U.S. market data, we need a value between 7.9 and 69.1 (depending on the investment horizon of the representative investor) for the risk-aversion parameter to explain the market data. After accounting for robustness, these values decrease to 2.9 and 25.5, respectively (see Table 6). Thus, our model can to a large extent resolve the bond premium puzzle.

Apart from our results on the bond premium puzzle, another contribution of our paper is of a more technical nature. We develop a novel method to solve the robust portfolio problem, the robust martingale method. We show in Theorem 1 that the robust dynamic investment problem can be interpreted as a non-robust dynamic investment problem with so-called least-favorable risk premiums, which are time-dependent but deterministic. This means that, based on our Theorem 1, we can formulate the objective function of the investor at time zero *non-recursively*. The importance of this contribution is stressed by Maenhout (2004), who writes: "*Using the value function V itself to scale θ may make it difficult to formulate a time-zero problem, as V is only known once the problem is solved.*" In Theorem 1 we prove that the time-zero robust dynamic investment problem can be formulated ex ante, before solving for the value function itself, and we also provide this alternative (but equivalent) formulation of the investment problem in closed form. Since without this non-recursive formulation the problem can only be solved recursively, the literature so far had to rely exclusively on the Hamilton-Jacobi-Bellman differential equation to solve robust dynamic investment problems. Our alternative formulation of the problem makes it possible to apply an alternative technique to solve robust dynamic investment problems, namely a robust version of the martingale method. This method is likely also applicable in other settings.

Our paper relates to the literature on the bond premium puzzle. Backus, Gregory and Zin (1989) use a consumption-based endowment economy to study the behavior of risk premiums. They conclude that in order for their model to match the risk premium observed in market data, the coefficient of relative risk aversion must be at least around 8-10. This value for the relative risk-aversion parameter is considered too high by the majority of the literature to reconcile with both economic intuition and economic experiments. Further early discussion on the bond premium puzzle can be found in Donaldson, Johnsen and Mehra (1990) and Den Haan (1995). They demonstrate that the bond premium puzzle is not a peculiarity of the consumption-based endowment economy, but it is also present in real business-cycle models. This remains true even if one allows for variable labor and capital or for nominal rigidities. Rudebusch and Swanson (2008) examine whether the bond premium puzzle is still present if they use a more sophisticated macroeconomic model instead of either the consumption-based endowment economy or the real business-cycle model. They use several DSGE setups and find that the bond premium puzzle remains even if they extend their model to incorporate large and persistent habits and real wage bargaining rigidities. However, Wachter (2008) provides a resolution to the puzzle by incorporating habit-formation into an endowment economy. Piazzesi and Schneider (2006) use Epstein-Zin preferences, but to match market data, they still need a relative risk aversion of 59.

Our paper obviously relates to the literature on robust dynamic asset allocation. Investors are uncertain about the parameters of the distributions that describe returns. In robust decision making, the investor makes decisions that "*not only work well when the underlying model for the state variables holds exactly, but also perform reasonably well if there is some form of model misspecification*" (Maenhout (2004)). We use the minimax approach to robust decision making. A comparison of the minimax approach with other approaches, such as the recursive smooth ambiguity preferences approach, can be found in Peijnenburg (2010). The minimax approach assumes that the investor considers a set of possible investment paths regarding the parameters she is uncertain about. She chooses the worst case scenario, and then she makes her investment decision using this worst case scenario to maximize her value function. To determine the set of possible parameters we use the penalty approach. This means that we do not set an explicit constraint on the parameters about which the investor is uncertain, but we introduce a *penalty term* for these parameters. Deviations of the parameters from a so-called base model are penalized by this function. Then the investor solves her unconstrained optimization problem using this new goal function.

The penalty approach was introduced into the literature first by Anderson, Hansen and Sargent (2003), and it was applied by Maenhout (2004) and Maenhout (2006) to analyze equilibrium equity prices. Maenhout (2004) finds that in the case of a constant investment opportunity set robustness increases the equilibrium equity premium and it decreases the risk-free rate. Concretely, a robust Duffie-Epstein-Zin representative investor with reasonable risk-aversion and uncertainty-aversion parameters generate a 4% to 6% equity premium. Furthermore, Maenhout (2006) finds that, if the investment opportunity set is stochastic, robustness increases the importance of intertemporal hedging compared to the non-robust case. We confirm this result in our setting (see Corollary 1). While our paper sheds light on the importance of parameter uncertainty for asset prices, several papers analyzed the effects of parameter uncertainty on asset allocation. Branger, Larsen, and Munk (2013) solve a stock-cash allocation problem with a constant risk-free rate and uncertainty aversion. The model of Flor and Larsen (2014) features a stock-bond-cash allocation problem with stochastic interest rates and ambiguity, while Munk and Rubtsov (2014) also account for inflation ambiguity. Feldhütter, Larsen, Munk, and Trolle (2012) investigate the importance of parameter uncertainty for bond investors empirically.

The paper is organized as follows. Section 2 introduces our model, i.e., the financial market and the robust dynamic optimization problem. Section 2 also provides the solution to the robust investment problem, using the martingale method. In Section 3 we calibrate our model to our data. In Section 4 we solve for the equilibrium prices and in Section 5 we disentangle the risk aversion from the uncertainty aversion using detection error probabilities. Section 6 concludes.

2 Robust Investment Problem

We consider agents that have access to an arbitrage-free complete financial market consisting of a money market account, constant maturity bond funds, and a stock market index. The short rate r_t is assumed to be affine in an N-dimensional factor $\boldsymbol{F}_t$, i.e.,

$$r_t = A_0 + \boldsymbol{\iota}'\boldsymbol{F}_t, \tag{1}$$

where $\boldsymbol{\iota}$ denotes a column vector of ones. The factors $\boldsymbol{F}_t$ follow an N-dimensional Ornstein-Uhlenbeck process, i.e.,

$$\mathrm{d}\boldsymbol{F}_t = -\boldsymbol{\kappa}(\boldsymbol{F}_t - \boldsymbol{\mu}_F)\mathrm{d}t + \boldsymbol{\sigma}_F \mathrm{d}\boldsymbol{W}^{\mathbb{Q}}_{F,t}. \tag{2}$$

Here $\boldsymbol{\mu}_F$ is an N-dimensional column vector of long-term averages, $\boldsymbol{\kappa}$ is an $N \times N$ diagonal mean-reversion matrix, $\boldsymbol{\sigma}_F$ is an $N \times N$ lower triangular matrix with strictly positive elements in its diagonal, and $\boldsymbol{W}^{\mathbb{Q}}_{F,t}$ is an N-dimensional column vector of independent standard Wiener processes under the risk-neutral measure $\mathbb{Q}$. The value of the available stock market index is denoted by S_t and satisfies

$$\mathrm{d}S_t = S_t r_t \mathrm{d}t + S_t \left(\boldsymbol{\sigma}'_{FS}\mathrm{d}\boldsymbol{W}^{\mathbb{Q}}_{F,t} + \sigma_{N+1}\mathrm{d}W^{\mathbb{Q}}_{N+1,t}\right), \tag{3}$$

where σ_{N+1} is strictly positive, $\boldsymbol{\sigma}_{FS}$ is an N-dimensional column vector governing the covariance between stock and bond returns, and $W^{\mathbb{Q}}_{N+1,t}$ is a standard Wiener process (still under the risk-neutral measure $\mathbb{Q}$) that is independent of $\boldsymbol{W}^{\mathbb{Q}}_{F,t}$. As our financial market is arbitrage-free and complete, such a risk-neutral measure $\mathbb{Q}$ indeed exists and is unique.

Although we will study the effect of ambiguity on investment decisions and equilibrium prices below, it is important to note that, due to the market completeness, the risk-neutral measure $\mathbb{Q}$ is unique and agents cannot be ambiguous about it. Indeed, the risk-neutral measure $\mathbb{Q}$ is uniquely determined by market prices and, thus, if all investors accept that there is no arbitrage opportunity on the market and they observe the same market prices, then they all have to agree on the risk-neutral measure $\mathbb{Q}$ as well. Investors will be ambiguous in our model about the physical probability measure or, equivalently, about the prices of risk of the Wiener processes $\boldsymbol{W}^{\mathbb{Q}}_{F,t}$ and $W^{\mathbb{Q}}_{N+1,t}$. We denote $\boldsymbol{W}^{\mathbb{Q}}_{F,t}$ and $W^{\mathbb{Q}}_{N+1,t}$ jointly as

$$\boldsymbol{W}^{\mathbb{Q}}_{t} = \begin{bmatrix} \boldsymbol{W}^{\mathbb{Q}}_{F,t} \\ W^{\mathbb{Q}}_{N+1,t} \end{bmatrix}. \tag{4}$$

Now consider an investor with investment horizon T. She derives utility from terminal wealth. This investor is ambiguous about the physical probability measure. She has a physical probability measure $\mathbb{B}$ in mind which she considers the most probable, but she is uncertain about whether this is the true physical probability measure or not. This measure $\mathbb{B}$ is called the *base measure.* As the investor is not certain that the measure $\mathbb{B}$ is the true physical probability measure, she considers other possible physical probability measures as well. These measures are called *alternative (physical) measures* and denoted by $\mathbb{U}$. We formalize the relationship between $\mathbb{Q}$, $\mathbb{B}$, and $\mathbb{U}$ as

$$\begin{aligned} \mathrm{d}\boldsymbol{W}_t^{\mathbb{B}} &= \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} - \boldsymbol{\lambda}\mathrm{d}t, \quad (5) \\ \mathrm{d}\boldsymbol{W}_t^{\mathbb{U}} &= \mathrm{d}\boldsymbol{W}_t^{\mathbb{B}} - \boldsymbol{u}(t)\mathrm{d}t, \quad (6) \end{aligned}$$

where $\boldsymbol{W}_t^{\mathbb{B}}$ and $\boldsymbol{W}_t^{\mathbb{U}}$ are $(N+1)$-dimensional standard Wiener processes under the measures $\mathbb{B}$ and $\mathbb{U}$, respectively. Thus, $\boldsymbol{\lambda}$ can be identified as the $(N+1)$-dimensional vector of prices of risk of the base measure $\mathbb{B}$, while $\boldsymbol{u}(t)$ denotes the $(N+1)$-dimensional vector of prices of risk of $\mathbb{U}$[1]. It is important to emphasize that the investor assumes $\boldsymbol{u}(t)$ to be a deterministic function of time, i.e., $\boldsymbol{u}(t)$ is assumed to be non-stochastic.[2]

We can now formalize the investor's optimization problem, given a CRRA utility function with risk aversion $\gamma > 1$[3], time-preference parameter $\delta > 0$, and a stochastic and non-negative parameter Υ_t, which expresses the investor's attitude towards uncertainty, and which we will describe in more details later in this section.

[1] In order for $\mathbb{U}$ to be well defined, we assume throughout $\int_0^T \|\boldsymbol{u}(s)\|^2 \mathrm{d}s < \infty$.

[2] A natural question here is why we allow $\boldsymbol{u}$ to be a deterministic function of time, but assume $\boldsymbol{\lambda}$ to be constant. Allowing $\boldsymbol{\lambda}$ to be a deterministic function of time would not change our conclusions, but it would result in more complicated expressions due to time-integrals involving $\boldsymbol{\lambda}(t)$. Moreover, we could not calibrate our model to market data without assuming some functional form for $\boldsymbol{\lambda}(t)$. Thus since for our purposes a constant $\boldsymbol{\lambda}$ suffices and it allows straightforward model calibration, we throughout take $\boldsymbol{\lambda}$ to be constant.

[3] In the case of $\gamma = 1$ the investor has log-utility. All of our results can be shown to hold in this case as well.

Problem 1. *Given initial wealth x, find an optimal pair $(X_T, \mathbb{U})$ for the robust utility maximization problem*

$$V_0(x) = \inf_{\mathbb{U}} \sup_{X_T} \mathrm{E}^{\mathbb{U}} \left\{ \exp(-\delta T) \frac{X_T^{1-\gamma}}{1-\gamma} \right.$$
$$\left. + \int_0^T \Upsilon_s \exp(-\delta s) \frac{\partial \mathrm{E}^{\mathbb{U}} \left[\log \left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{B}} \right)_s \right]}{\partial s} \mathrm{d}s \right\}, \qquad (7)$$

subject to the budget constraint

$$\mathrm{E}^{\mathbb{Q}} \left[\exp \left(- \int_0^T r_s \mathrm{d}s \right) X_T \right] = x. \qquad (8)$$

The investor's optimization problem as it is formulated here follows the so-called martingale method. Given that our financial market is complete, the martingale method maximization is over terminal wealth X_T only. It is not necessary, mathematically, to consider optimization of the portfolio strategy as the optimal strategy will simply be that one that achieves the optimal terminal wealth X_T. For (mathematical) details we refer to Karatzas and Shreve (1998).

The outer inf in Problem 1 adds robustness to the investment problem as the investor considers the worst case scenario, i.e., she chooses the measure $\mathbb{U}$ which minimizes the value function (evaluated at the optimal terminal wealth). The investor considers all alternative probability measures $\mathbb{U}$ which are equivalent[4] to the base measure $\mathbb{B}$.

[4]Two probability measures are said to be equivalent if and only if each is absolutely continuous with respect to the other. That is, the investor may be uncertain about the exact probability of events, but she is certain about which events happen for sure (i.e., with probability 1) or with probability zero. This is a common, sometimes implicit, assumption in this literature.

The first part of the expression in brackets in (7) expresses that the investor cares about her discounted power utility from terminal wealth X_T. The second term represents a penalty: if the investor calculates her value function using a measure $\mathbb{U}$ which is very different from $\mathbb{B}$, then the penalty term will be high. We will be more explicit about what we mean by two probability measures being *very different* from each other in the next paragraph. The fact that the investor considers a worst-case scenario, including the penalty term, ensures that she considers "pessimistic" probability measures (which result in low expected utility), but at the same time she only considers "reasonable" probability measures (that are not too different from the base measure).

Following Anderson, Hansen and Sargent (2003), we quantify how different probability measures are by their *Kullback-Leibler* divergence, which is also known as the relative entropy. The reason why we use the Kullback-Leibler divergence as the penalty function lies not only in its intuitive interpretation (see, e.g., Cover and Thomas (2006), Chapter 2), but also in its mathematical tractability.

We now rewrite Problem 1 and, following Maenhout (2004), introduce a concrete specification for Υ_t. In view of (6) and Girsanov's theorem, we obtain

$$\begin{aligned}\frac{\partial \mathrm{E}^{\mathbb{U}}\left[\log\left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{B}}\right)_t\right]}{\partial t} &= \frac{\partial}{\partial t}\mathrm{E}^{\mathbb{U}}\left[\frac{1}{2}\int_0^t \|\boldsymbol{u}(s)\|^2 \mathrm{d}s - \int_0^t \boldsymbol{u}(s)\mathrm{d}\boldsymbol{W}_s^{\mathbb{U}}\right] \qquad (9)\\ &= \frac{1}{2}\|\boldsymbol{u}(t)\|^2, \qquad (10)\end{aligned}$$

where $\|\boldsymbol{u}(t)\|$ denotes the Euclidean norm of $\boldsymbol{u}(t)$. Furthermore, in order to ensure homotheticity of the investment rule[5], we use the following specification of Υ_t, introduced in Maenhout (2004)[6].

$$\Upsilon_t = \exp(\delta t) \frac{1-\gamma}{\theta} V_t(X_t), \tag{11}$$

where X_t denotes optimal wealth at time t. Substituting (10) and (11) into (7), the value function becomes

$$\begin{aligned} V_0(x) = \inf_{\mathbb{U}} \sup_{X_T} \mathrm{E}^{\mathbb{U}} \Bigg\{ & \exp(-\delta T) \frac{X_T^{1-\gamma}}{1-\gamma} \\ & + \int_0^T \frac{(1-\gamma)\, \|\boldsymbol{u}(t)\|^2}{2\theta} V_t(X_t) \mathrm{d}t \Bigg\}. \end{aligned} \tag{12}$$

This expression of the value function is recursive, in the sense that the right-hand side contains future values of the same value function. The following theorem gives a non-recursive expression. All proofs are in the appendix.

Theorem 1. *The solution to (12) with initial wealth x is given by*

$$V_0(x) = \inf_{\mathbb{U}} \sup_{X_T} \mathrm{E}^{\mathbb{U}} \left\{ \exp\left(-\delta T + \frac{1-\gamma}{2\theta} \int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t \right) \frac{X_T^{1-\gamma}}{1-\gamma} \right\}. \tag{13}$$

subject to the budget constraint

$$\mathrm{E}^{\mathbb{Q}} \left[\exp\left(-\int_0^T r_s \mathrm{d}s \right) X_T \right] = x. \tag{14}$$

Theorem 1 gives an alternative interpretation to the robust investment

[5]Homotheticity of the investment rule means that the optimal ratio of wealth to be invested in a particular asset at time t does not depend on the wealth at time t itself.

[6]This specification has been criticized in, e.g., Pathak (2012) for its recursive nature. Alternatively we could have specified the robust investment problem directly as in Theorem 1.

Problem 1, with parameterization (11). Effectively, the investor maximizes her expected discounted utility of terminal wealth, under the least-favorable physical measure $\mathbb{U}$, using an adapted subjective discount factor

$$\delta - \frac{1-\gamma}{2\theta}\frac{1}{T}\int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t. \tag{15}$$

As $\theta > 0$, we obtain, for $\gamma > 1$, that the subjective discount rate increases in the time-average of $\|\boldsymbol{u}(t)\|^2$, i.e., in deviations of the least-favorable physical measure $\mathbb{U}$ from the base measure $\mathbb{B}$. The investor thus becomes effectively more impatient. However, this is not the reason why the robustness affects the asset allocation, as the subjective time preference does not affect the asset allocation in this standard terminal wealth problem. Rather, the effect of robustness is that the prices of risk that the robust investor uses are affected by the least-favorable measure $\mathbb{U}$ in the value function in (13). We show this formally in the following section.

2.1 Optimal terminal wealth

We now solve Problem 1 using the reformulation in Theorem 1. In the present literature, dynamic robust investment problems are mostly solved by making use of a "twisted" Hamilton-Jacobi-Bellman (HJB) differential equation. Solving this HJB differential equation determines both the optimal investment allocation and the optimal final wealth. However, we propose to use the so-called martingale method. This approach has not only mathematical advantages (one does not have to solve higher-order partial differential equations), but it also provides economic intuition and insights into the decision-making of the investor. We provide this intuition at the end of this section, directly after Theorem 2. The martingale method was developed by Cox and Huang (1989) for complete markets, and a detailed description can be found in Karatzas and Shreve (1998). The basic idea is to first determine the optimal terminal wealth X_T (Theorem 2) and to subsequently determine the asset allocation that the investor has to choose in order to achieve that optimal terminal wealth (Corollary 1).

In our setting of robust portfolio choice, we can still follow this logic. It is important to note that the budget constraint (8) is, obviously, not subject to uncertainty, i.e., $\mathbb{Q}$ is given. The value function in Theorem 1 contains an inner (concerning X_T) and an outer (concerning $\mathbb{U}$) optimization. Solving these in turn leads to the following result, whose proof is again in the appendix.

Theorem 2. *The solution to the robust investment Problem 1 under (11) is given by*

$$\hat{X}_T = x \frac{\exp\left(\frac{1}{\gamma}\int_0^T (\hat{\boldsymbol{u}}(t)+\boldsymbol{\lambda})'\,\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} + \frac{1}{\gamma}\int_0^T r_t \mathrm{d}t\right)}{\mathrm{E}^{\mathbb{Q}}\exp\left(\frac{1}{\gamma}\int_0^T (\hat{\boldsymbol{u}}(t)+\boldsymbol{\lambda})'\,\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} + \frac{1-\gamma}{\gamma}\int_0^T r_t \mathrm{d}t\right)}, \tag{16}$$

with the least-favorable distortions

$$\hat{\boldsymbol{u}}(t)_F = -\frac{\theta}{\gamma+\theta}\left[\boldsymbol{\lambda}_F + \boldsymbol{\sigma}_F' \boldsymbol{B}(T-t)'\boldsymbol{\iota}\right], \tag{17}$$

$$\hat{\boldsymbol{u}}(t)_{N+1} = -\frac{\theta}{\gamma+\theta}\boldsymbol{\lambda}_{N+1}, \tag{18}$$

where $\boldsymbol{B}(\cdot)$ is defined in (54).

Equation (16) shows the stochastic nature of the optimal terminal wealth. The denominator is a scaling factor, and the numerator can be interpreted as the exponential of a (stochastic) yield on the investment horizon T. The investor achieves this yield on her initial wealth x if she invests optimally throughout her life-cycle.

The absolute value of the least-favorable distortions (17) and (18) increase as θ increases. This is in line with the intuition that a more uncertainty-averse investor considers alternative measures that are "more different" from the base measure. If the investor is not uncertainty-averse, i.e., $\theta = 0$, the least-favorable distortions are all zero. This means that the investor considers only the base measure and she makes her investment decision based on that measure. On the other hand, if the investor is infinitely uncertainty-averse, i.e., $\theta = \infty$, the least-favorable distortions are

$$\tilde{\boldsymbol{u}}(t)_F = -\left[\boldsymbol{\lambda}_F + \boldsymbol{\sigma}'_F \boldsymbol{B}(T-t)'\boldsymbol{\iota}\right], \tag{19}$$

$$\tilde{\boldsymbol{u}}(t)_{N+1} = -\boldsymbol{\lambda}_{N+1}. \tag{20}$$

An infinitely uncertainty-averse investor thus uses $-\boldsymbol{\sigma}'_F \boldsymbol{B}(T-t)'\boldsymbol{\iota}$ as the market price of risk induced by $\boldsymbol{W}^{\mathbb{U}}_{F,t}$ and 0 as the market price of risk induced by $W^{\mathbb{U}}_{N+1,t}$.

2.2 Optimal portfolio strategy

The final step in our theoretical analysis is to derive the investment strategy that leads to the optimal final wealth $\hat{X}_T$ derived in Theorem 2. The following theorem gives the optimal exposures to the driving Brownian motion

$\boldsymbol{W}_t^{\mathbb{Q}}$. The proof can be found in the appendix.

Theorem 3. *Under the conditions of Theorem 2, the optimal final wealth $\hat{X}_T$ is achievable using the wealth evolution*

$$\mathrm{d}\hat{X}_t = \ldots \mathrm{d}t + \left[\frac{\boldsymbol{\lambda}_F'}{\gamma+\theta} + \frac{1-(\gamma+\theta)}{\gamma+\theta} \boldsymbol{\iota}' \boldsymbol{B}(T-t)\boldsymbol{\sigma}_F;\ \frac{\lambda_{N+1}}{\gamma+\theta} \right] \hat{X}_t \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}}, \quad (21)$$

starting from $\hat{X}_0 = x$.

Introducing the notation

$$\boldsymbol{\mathcal{B}}(\tau) = \left[\boldsymbol{B}(\tau_1)\boldsymbol{\iota}; \ldots; \boldsymbol{B}(\tau_N)\boldsymbol{\iota} \right], \quad (22)$$

where τ_j denotes the maturity of bond fund j, we can state the following corollary (proved in the appendix).

Corollary 1. *Under the conditions of Theorem 2, the optimal investment is a continuous re-balancing strategy where the fraction of wealth invested in the constant maturity bond funds is*

$$\begin{aligned} \hat{\boldsymbol{\pi}}_{B,t} = &- \frac{1}{\gamma+\theta} \boldsymbol{\mathcal{B}}(\tau)^{-1} \left(\boldsymbol{\sigma}_F'\right)^{-1} \left(\boldsymbol{\lambda}_F - \frac{\lambda_{N+1}}{\sigma_{N+1}} \boldsymbol{\sigma}_{FS} \right) \\ &- \frac{1-\gamma-\theta}{\gamma+\theta} \boldsymbol{\mathcal{B}}(\tau)^{-1} \boldsymbol{B}(T-t) \end{aligned} \quad (23)$$

and the fraction of wealth invested in the stock market index is

$$\hat{\pi}_{S,t} = \frac{\lambda_{N+1}}{(\gamma+\theta)\,\sigma_{N+1}}. \quad (24)$$

Equations (23) and (24) provide closed-form solutions for the optimal fractions of wealth to be invested in the bond and stock markets. For the latter one, this fraction is time independent and it is equal to the market price of the idiosyncratic risk of the stock market (i.e., λ_{N+1}) divided by $(\gamma + \theta)$ and by the volatility of the unspanned stock market risk σ_{N+1}. So even though the return on the stock market is influenced by all of the $N+1$ sources of risk, only the stock market specific risk matters when the investor decides how much to invest in the stock market. This investment policy closely resembles the solution to Merton's problem (Merton (1969)), the main difference being that γ is replaced by $\gamma + \theta$ in the denominator.

The optimal fraction to be invested in the bonds has two components (similarly to Merton's intertemporal consumption model (Merton (1973)). The first component is the myopic demand, and its form is similar to that of (24). The second component represents the intertemporal hedging demand, and it is present due to the stochastic nature of the investment opportunity set: the investor holds this component in order to protect herself against unfavorable changes of the N factors. The main difference to the solution to Merton's problem is again γ being replaced by $\gamma + \theta$. This means that if we compare a robust and a non-robust investor's optimal investment policy (with the same level of relative risk-aversion), the only difference between them is that in the robust investor's case the risk-aversion parameter γ is replaced by the sum of the risk-aversion and the uncertainty aversion parameters, $\gamma + \theta$. In (23) we find that robustness increases the intertemporal hedging demand of the investor. This finding is in accordance with Maenhout (2006).

3 Model calibration

In this section we calibrate our model to market data. We use weekly observations from 5 January 1973 to 29 January 2016. We use continuously compounded zero-coupon yields with maturities of 3 months, 1 year, 5 years, and 10 years. The zero-coupon yields for maturities of 1 year, 5 years, and 10 years were obtained from the US Federal Reserve Data Releases, while the yields for maturity of 3 months are the 3-month T-bill secondary market rates from the St. Louis Fed Fred Economic Data.[7] The spot rates of 1 and 2 months of maturity sometimes show extremely large changes within one period, so the shortest maturity that we used is 3 months. Regarding the sample period we did not go back further than 1973, because yield curve estimations from earlier years contain relatively high standard errors due to the many missing values. Moreover, the monetary policy before the 1970s was very different from the period afterwards. As stock market index we use the continuously compounded total return index of Datastream's US-DS Market.[8]

To utilize both cross-sectional and time-series bond market data, we follow the estimation methodology of de Jong (2000) based on the Kalman filter and Maximum Likelihood. The error terms in the observation equation are allowed to be cross-sectionally correlated, but they are assumed to be serially uncorrelated. We initialize the Kalman filter recursions by conditioning on the first observations and the initial MSE matrix equals the covariance

[7]These 3-month yields assumed quarterly compounding, so we manually transformed them into continuously compounded yields.

[8]Datastream code: TOTMKUS(RI).

matrix of the observation errors. The initial factor values are set equal to their long-term mean. For identification purposes $\boldsymbol{\mu}_F$ is assumed to be a zero vector. The standard errors of the estimated parameters are obtained as the square roots of the diagonal elements of the inverted Hessian matrix. Table 1 contains our estimates[9].

Table 1. Parameter estimates and standard errors
Estimated parameters and standard errors using Maximum Likelihood with weekly observations. At each time we observed four points on the U.S. zero-coupon, continuously compounded yield curve, corresponding to maturities of 3 months, 1 year, 5 years and 10 years; and the total return index of Datastream's US-DS Market. The observation period is from 5 January 1973 to 29 January 2016.

	Estimated parameter	*Standard error*
$\hat{\kappa}_1$	0.0763***	*0.0024*
$\hat{\kappa}_2$	0.3070***	*0.0108*
$\hat{A}_0$	0.0862***	*0.0013*
$\hat{\lambda}_{F,1}$	−0.1708	*0.1528*
$\hat{\lambda}_{F,2}$	−0.5899***	*0.1528*
$\hat{\lambda}_{N+1}$	0.3180**	*0.1528*
$\hat{\sigma}_{F,11}$	0.0208***	*0.0009*
$\hat{\sigma}_{F,21}$	−0.0204***	*0.0012*
$\hat{\sigma}_{F,22}$	0.0155***	*0.0003*
$\hat{\sigma}_{FS,1}$	−0.0035	*0.0038*
$\hat{\sigma}_{FS,2}$	−0.0121***	*0.0035*
$\hat{\sigma}_{N+1}$	0.1659***	*0.0025*

One of the two factors exhibits reasonably strong mean reversion as $\hat{\kappa}_2$ is higher than 0.3, while the mean-reversion parameter of the other factor is quite small (around 0.08), however still statistically significantly different

[9]As a robustness check, we also estimated the parameters using monthly and quarterly data. Our estimates are very similar to the ones we obtained using weekly data, which verifies that our model is indeed a good description of the behaviour of returns. All our observations are on the last trading day of the particular period.

from zero. The long-term mean of the short rate, under the risk-neutral measure, A_0 is estimated to be around 9%. The negative sign of $\hat{\boldsymbol{\sigma}}_{F,21}$ shows that there is a negative correlation between the two factors. The two elements of $\hat{\boldsymbol{\sigma}}_{FS}$ are economically not significant and only one of them is statistically significantly different from zero. $\hat{\sigma}_{N+1}$, on the other hand, is both statistically and economically significant: it is around 17% per annum. The market prices of the two risk sources that influence the factors, $\boldsymbol{\lambda}_F$, are both negative.

Table 2. Model implied instantaneous excess returns, volatilities and Sharpe ratios

The model implied instantaneous expected excess returns, volatilities of returns and Sharpe ratios using the estimates in Table 1.

	Expected excess return	Volatility of return	Sharpe ratio
Constant maturity (1 y.) bond fund	0.83%	1.36%	0.61
Constant maturity (5 y.) bond fund	2.93%	5.25%	0.56
Constant maturity (10 y.) bond fund	4.25%	9.53%	0.45
Stock market index	6.05%	16.64%	0.36

Using these estimates, we calculate the model-implied instantaneous expected excess returns, volatilities and Sharpe ratios for the constant maturity bond funds with maturities of 1 year, 5 years, and 10 years and for the stock market index fund (Table 2). The stock market index fund has the highest expected excess return (6.05%), but also the highest volatility (16.64%). Regarding the constant maturity bond funds, the expected excess returns are lower than for the stock market index fund: for 10 years of maturity, the expected excess return is slightly higher than 4%. At the same time, the lower expected excess returns come with lower volatilities, which are 1.36%, 5.25% and 9.53% for the three bond funds, respectively. The highest Sharpe ratio is produced by the constant maturity bond fund with 1 year of maturity: its volatility is relatively low (only 1.36%) given its expected excess return of 0.83%.

For comparison purposes, we also estimated the expected excess returns, the volatilities and the Sharpe ratios directly using excess return data. We used the 3 month T-bill rate as a proxy for the risk-free rate[10]. We used the same stock market index data that we used when estimating the parameters, with weekly observations. To directly estimate the returns, the volatilities and the Sharpe ratios of constant maturity bond funds, we assumed that the investor buys the bond fund at the beginning of the quarter and sells it at the end of the quarter, when its maturity is already 3 months less. To this end we used the same continuously compounded, zero-coupon yield curve observations that before, now with quarterly frequency, plus we observed three additional points on the yield-curve, namely for maturities of 6 months, 4 years and 9 years. These three additional maturities were used – together with the maturities of 1 year, 5 years and 10 years – to obtain our estimates by linear interpolation for the continuously compounded, zero coupon yields for 9 months, 4.75 years and 9.75 years. Our direct estimates of the expected excess returns, the volatilities, and the Sharpe ratios can be found in Table 3.

The directly estimated Sharpe ratio of the stock market index fund is 0.37, which is practically the same as the model-implied value. The estimated expected excess returns of the constant maturity bond funds with

[10]We compared the model implied risk-free rates and the 3 month T-bill rates for the entire estimation period, and they are very close to each other. So the 3 month T-bill rate is a good proxy for the risk-free rate.

Table 3. Directly estimated excess returns, volatilities and Sharpe ratios

The directly estimated excess returns, volatilities and Sharpe ratios, using the same observation period as for estimating the parameters in Table 1, assuming 3 months holding period.

	Expected excess return	Volatility	Sharpe ratio
Constant maturity (1 y.) bond fund	1.53%	1.64%	0.94
Constant maturity (5 y.) bond fund	2.85%	7.08%	0.40
Constant maturity (10 y.) bond fund	4.40%	12.86%	0.34
Stock market index	6.07%	16.58%	0.37

maturities of 5 and 10 years are very close to their model implied counterparts: they are 2.85% and 4.40%. The directly estimated expected excess return for the constant maturity bond fund with 1 year of maturity is somewhat higher than its model-implied counterpart. The directly estimated volatilities of the bond funds are slightly higher than what the model implies, which leads to lower values of the directly estimated Sharpe ratios for maturities of 5 years and 10 years.

We can clearly recognize the bond premium puzzle in Tables 2 and 3. The Sharpe ratios of all of the constant maturity bond funds are higher than the Sharpe ratio of the stock market index, the only exception being the directly estimated Sharpe ratio of the constant maturity bond fund with 10 years of maturity. Even without considering the supply side, these Sharpe ratios of long-term bonds seem very high given a reasonable level of risk aversion. Interestingly, the puzzle is stronger in the case of the shorter maturity bond funds: the Sharpe ratio of the 1-year constant maturity bond fund is 0.61, while that of the 10-year constant maturity bond fund is 0.45. We now proceed to show that ambiguity aversion can explain this bond-premium puzzle to a large extent.

4 Robust general equilibrium

In Section 2 we obtained the optimal investment policy of the representative investor: if she would like to maximize her value function (7) subject to her budget constraint (8), she has to invest according to (23) and (24). In this section we introduce the concept of robust general equilibrium and we estimate the sum of the risk-aversion and uncertainty-aversion parameters $\gamma+\theta$. Section 5 will be devoted to separately identifying γ and θ using the concept of detection error probabilities.

Definition 1. *The market is in robust general equilibrium if the following conditions are satisfied:*

1. *The representative investor solves Problem 1 with parameterization (11).*
2. *All of the security markets (the bond markets, the stock market and – as a consequence – the money market) clear continuously, i.e., for all $t \in [0, T]$,*

$$\hat{\boldsymbol{\pi}}_{B,t} = \boldsymbol{\pi}^*_B \qquad \forall\, t \in [0, T]\,, \tag{25}$$

$$\hat{\pi}_{S,t} = \pi^*_S \qquad \forall\, t \in [0, T]\,, \tag{26}$$

where $\boldsymbol{\pi}^*_B$ and π^*_S denote the exogenously given supply of the N constant maturity bond funds and of the stock market index as a fraction of the total wealth of the economy.

If we substitute the optimal investment ratios for the bonds and the stock into the market clearing equations (25)-(26), we obtain that the market is in robust equilibrium if, for all $t \in [0, T]$, we have

$$\begin{aligned}\boldsymbol{\pi}_B^* =& -\frac{1}{\gamma+\theta}\boldsymbol{\mathcal{B}}(\tau)^{-1}\left(\boldsymbol{\sigma}_F'\right)^{-1}\left(\boldsymbol{\lambda}_F - \frac{\lambda_{N+1}}{\sigma_{N+1}}\sigma_{FS}\right)\\ &-\frac{1-\gamma-\theta}{\gamma+\theta}\boldsymbol{\mathcal{B}}(\tau)^{-1}\boldsymbol{B}(T-t)\,\boldsymbol{\iota}, \qquad (27)\\ \pi_S^* =& \frac{\lambda_{N+1}}{(\gamma+\theta)\,\sigma_{N+1}}. \qquad (28)\end{aligned}$$

We perform the calibration assuming different exogenous supply sides and several different sets of market clearing conditions, which will be subsets of the market clearing conditions (25)-(26). To calibrate $\gamma + \theta$, we use our estimates from Table 1. In (27)-(28), γ and θ appear only as a sum, so they cannot be separately identified. Moreover, since the system (27)-(28) is overidentified, an exact solution of $\gamma + \theta$ does not exist, hence we minimize the sum of squared differences to estimate $\gamma + \theta$. I.e., we calculate $\|\boldsymbol{\pi}_B^* - \hat{\boldsymbol{\pi}}_B\|^2 + (\pi_S^* - \hat{\pi}_S)^2$ for several $T - t$ and $\gamma + \theta$ combinations, and then, for each $T - t$, we select that value of $\gamma + \theta$ that minimizes this sum of squared differences.

Since our model has two factors driving the short rate, we need two constant maturity bond funds with different maturities to assure market completeness. To determine the maturities of these two bond funds, we use the data of U.S. government debt (Table 4). Since we have data available only for the clustered maturities as shown in Column 1, we assume that the distribution of maturities within the clusters is uniform. This way we can calculate the average maturities for each cluster (Column 4). For

Table 4. U.S. government debt by maturities
The government debt of the U.S. by maturities as of 1 September 2015 and the average maturities of the maturity-clusters. When calculating the average maturities, we assumed that the distribution of the debts within the clusters is uniform. Moreover, we assumed that the average maturity of debts with more than 20 years of maturity is 25 years. Source: Datastream.

Maturity	Debt outstanding (million USD)	Ratio of total debt outstanding	Average maturity
<1 year	2,890,796	0.2824	-
1-5 years	4,335,287	0.4235	3 years
5-10 years	2,035,095	0.1988	7.5 years
10-20 years	187,318	0.0183	15 years
>20 years	789,260	0.0771	25 years

bonds with more than 20 years of maturity we assume that their average maturity is 25 years. Because the shortest maturity cluster is less than one year, we treat this cluster of bonds as the money market account. Besides the money market account our model needs two zero-coupon bonds with different maturities in order to have a complete market. Since bonds with 1-5 years of maturity make up more then 42% of all government debt, we assume in the calibration that one of the available zero-coupon bonds has a maturity of 3 years. To determine the maturity of the other zero-coupon bond, we calculate the weighted average of the maturities of bonds with maturities of more than 5 years, where the weights are determined by their amount outstanding. This way we obtain an average 12.55 years maturity for the coupon bonds. As the duration of these coupon bonds is shorter than their maturity, in the calibration we assume that a zero-coupon bond with a maturity of 10 years is available.

Table 5 shows the calibrated $\gamma+\theta$ values for different investment horizons

Table 5. Calibrated values of the sum of the risk and uncertainty aversion parameters

Calibrated $\gamma + \theta$ values for several investment horizons and market clearing conditions. Estimates for all of the other parameters are from Table 1 and the maturities of the available zero-coupon bonds are $\tau_1 = 3$ years and $\tau_2 = 10$ years. The market is assumed to be in equilibrium according to Definition 1.

$T - t$	$\gamma + \theta$			
	$\pi_S = 1$	$\boldsymbol{\pi}_{B_{\tau_1}} = 0$ $\boldsymbol{\pi}_{B_{\tau_2}} = 0$	$\pi_S = 1$ $\boldsymbol{\pi}_{B_{\tau_1}} = 0$ $\boldsymbol{\pi}_{B_{\tau_2}} = 0$	$\pi_S = 0.67$ $\boldsymbol{\pi}_{B_{\tau_1}} = 0.14$ $\boldsymbol{\pi}_{B_{\tau_2}} = 0.10$
10	1.92	154.20	97.78	69.14
15	1.92	19.17	18.09	16.90
20	1.92	11.85	11.48	11.00
25	1.92	9.44	9.23	8.92
30	1.92	8.32	8.16	7.92
50	1.92	6.96	6.86	6.70
100	1.92	6.68	6.59	6.44
∞	1.92	6.67	6.58	6.43

and market clearing conditions. As a first experiment, we only impose that the stock market is in equilibrium, $\pi_S^* = 1$. In this case, the required $\gamma + \theta$ parameter is reasonable, 1.92, regardless of the investment horizon. Second, we only impose the restrictions on the bond markets, assuming it is in zero net supply, $\boldsymbol{\pi}_B^* = \mathbf{0}$. The calibrated $\gamma+\theta$ value now depends on the assumed investment horizon $T - t$. The values for $\gamma + \theta$ are more than 150 for $T - t = 10$ years and gradually decrease to 6.67 as $T - t$ approaches infinity.

In the next experiment, we combine both restrictions on the stock and bond market and assume the following exogenous stock and bond supply

$$\pi_S^* = 1, \tag{29}$$

$$\boldsymbol{\pi}_B^* = \mathbf{0}. \tag{30}$$

The reasoning behind (29)-(30) is that if an agent in the economy borrows money either by buying a long-term bond or by investing in the money-market account, another player has to lend that money to her. As a result, in general equilibrium the bond holdings of the representative agent will be zero and she keeps all of her wealth invested in the stock market. Thus in general equilibrium the representative investor's holdings of long-term bonds is zero, and all of her wealth is invested in the stock market. The results of this experiment are shown in Column 4 of Table 5. If the investor's investment horizon is 10 years, she needs a risk-aversion plus uncertainty-aversion parameter of nearly 100 in order to match market data. As the investment horizon increases, the calibrated $\gamma + \theta$ value decreases to 6.58. The first paper which described the bond premium puzzle, Backus, Gregory and Zin (1989), calibrates the risk-aversion parameter to be around 8, which is what we find using a reasonable investment horizon of 30 years. Hence, our results confirm the bond premium puzzle: if we calibrate our our model in the absence of ambiguity aversion (i.e., setting $\theta = 0$), we need an unreasonably high risk-aversion parameter.

Although the assumption that long-term government bonds are in zero net supply is customary in the literature[11], it is by no means innocuous,

[11]Note that several authors, e.g., Gomes and Michaelides (2008), assume that long-term

as noted in Donaldson and Mehra (2008). One may argue that the bond supply is exogenously determined by the government (since we only consider government bonds). Then the market clearing conditions are (25)-(26) with $\boldsymbol{\pi}_B^*$ and π_S^* equal to the weights in a value-weighted market portfolio of stocks and bonds. The U.S. stock market capitalization on 1 September 2015 was $20,885,920$ million USD (using Datastream's TOTMKUS index). Adding up the amounts of debt outstanding in Table 4, we find that the total amount of government debt of the U.S. is $10,237,756$ million USD. According to our model the financial market constitutes of the stock market, the bond market, and the money market (assumed to be in zero net supply), thus the total financial market capitalization is $31,123,676$ million USD. This leads to actual aggregate supply portfolio weights equal to

$$\pi_S^* = \frac{20,885,920}{31,123,676} = 0.67, \tag{31}$$

$$\pi_{B_{\tau_1}}^* = \frac{4,335,287}{31,123,676} = 0.14, \tag{32}$$

$$\pi_{B_{\tau_2}}^* = \frac{3,011,673}{31,123,676} = 0.10. \tag{33}$$

Using these market clearing conditions, the calibrated values of $\gamma + \theta$ are very close to the previous calibration. For $T - t = 10$, the calibrated value of $\gamma + \theta$ is 69.14, and it decreases gradually for longer investment horizons. At $T - t = 30$ it is 7.92 and as the investment horizon increases to infinity, $\gamma + \theta$ converges to 6.43. Hence, also if we use the more realistic market clearing conditions of (32)-(31) instead of (30)-(29), the bond-premium puzzle

government bonds are in strictly positive net supply.

remains with similar magnitude.

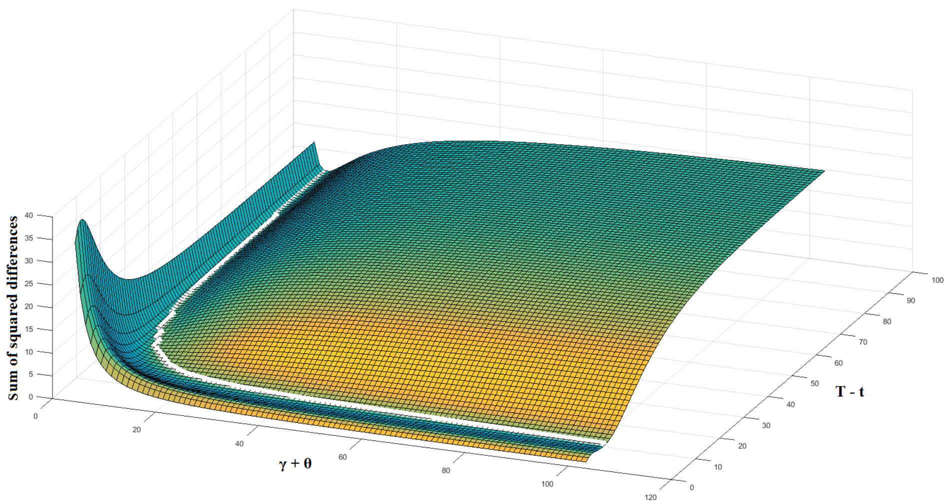
Figure 1 illustrates the behavior of the calibrated $\gamma + \theta$ values as the time horizon changes. We assume the market clearing condition (31)–(33). The white line shows the minimized sum of squared differences for given investment horizons up to 100 years. As $T - t$ approaches infinity, the minimizing $\gamma+\theta$ converges to 6.43. If we decrease the investment horizon, the minimizing $\gamma + \theta$ is increasing. So the presence of the bond premium puzzle is robust to the investment horizon: given any reasonable investment horizon for our representative investor, if we calibrate our model in the absence of uncertainty aversion (i.e., setting $\theta = 0$), the risk-aversion parameter γ is large. For example, for a 15 years investment horizon, the value of γ is 16.90 and for a 25 years horizon it is 8.92.

We have illustrated above that the bond risk-premium puzzle is also present in the sample period we consider, in the absence of uncertainty aversion, i.e., for $\theta = 0$. In the next section, we will identify the risk-aversion parameter γ separately from the uncertainty-aversion parameter θ.

5 Separating risk and uncertainty aversion

In Section 4 we calibrated the sum of the risk and uncertainty aversion parameters $(\gamma+\theta)$ to market data, but without further assumptions we cannot identify them separately. In this section we recall the concept of detection error probability (Anderson, Hansen and Sargent (2003)) in order to identify the risk-aversion parameter γ and the uncertainty-aversion parameter θ separately.

Figure 1. Sum of squared differences between supply and demand of the securities
Sum of squared differences between supply and demand of the two long-term bonds and the stock for different investment horizons $T-t$. The maturities of the two long-term bonds are 3 years and 10 years. The white line shows the minimized sum of squared differences for given investment horizons $T-t$ up to 100 years.



We assume that the investor observes the prices of N constant maturity bond funds and of the stock market index. As assumed in Section 2, the investor can observe these stock and bond prices continuously. The observation period is from $t-H$ to t, where t is the moment of observation and $H>0$. We assume that she then performs a likelihood-ratio test to decide whether the true physical probability measure is $\mathbb{B}$ or the least-favorable $\mathbb{U}$ as derived in Theorem 2. To be more precise: we assume that she calculates the ratio of the likelihoods of the physical probability measure being $\mathbb{B}$ and $\mathbb{U}$, respectively, and accepts the measure with the larger likelihood. If the true measure is $\mathbb{B}$, then the probability that the investor will be wrong is

$$P^{\mathbb{B}}\left(\log\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{U}}<0\right). \tag{34}$$

Similarly, if the true measure is $\mathbb{U}$, the probability that the investor will be wrong when determining the probability measure given a sample of data is

$$P^{\mathbb{U}}\left(\log\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{U}}>0\right). \tag{35}$$

We now define the detection error probability, following Anderson, Hansen and Sargent (2003).

Definition 2. *The detection error probability (DEP) is defined as*

$$DEP=\frac{1}{2}P^{\mathbb{B}}\left(\log\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{U}}<0\right)+\frac{1}{2}P^{\mathbb{U}}\left(\log\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{U}}>0\right). \tag{36}$$

In the following theorem we give the detection error probability for any arbitrary $\mathbb{U}$. Then, in Corollary 2 we give the detection error probability for the least favorable $\mathbb{U}$ as derived in Theorem 2. Both proofs again are in the appendix.

Theorem 4. *Assume that the investor (continuously) observes the prices of N constant maturity bond funds and of the stock market index. The observation period lasts from $t-H$ to the moment of observation, t. Then, the detection error probability of the investor for given $\mathbb{U}$ is*

$$DEP=1-\Phi\left(\frac{1}{2}\sqrt{\int_{t-H}^{t}\|\boldsymbol{u}\,(s)\,\|^2\mathrm{d}s}\right), \tag{37}$$

where $\boldsymbol{u}(\cdot)$ is defined in (6).

Plugging in the least-favorable $\mathbb{U}$ as derived in Theorem 2 leads to the following corollary.

Corollary 2. *Assume that the conditions of Theorem 4 hold. Then, the detection error probability of the investor for the least-favorable $\mathbb{U}$ is*

$$DEP = 1 - \Phi\left(\frac{\theta}{2(\gamma+\theta)}\sqrt{H\left(\lambda_{N+1}^2 + \|\boldsymbol{\lambda}_F\|^2\right) + \Delta_1 + \Delta_2}\right), \tag{38}$$

with

$$\begin{aligned}\Delta_1 &= 2\boldsymbol{\lambda}_F'\boldsymbol{\sigma}_F'\left[\left(\exp\left(-\boldsymbol{\kappa}\left(T-t+H\right)\right) - \exp\left(-\boldsymbol{\kappa}\left(T-t\right)\right)\right)\boldsymbol{\kappa}^{-2}\right. \\ &\qquad\qquad \left. + H\boldsymbol{\kappa}^{-1}\right]\boldsymbol{\iota}, \qquad (39) \\ \Delta_2 &= \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\left(\boldsymbol{\sigma}_F\boldsymbol{\sigma}_F'\right)_{ij}}{\kappa_i\kappa_j}\left[\boldsymbol{B}_{ii}\left(T-t\right) + \boldsymbol{B}_{jj}\left(T-t\right) - \boldsymbol{B}_{ii}\left(T-t+H\right)\right. \\ &\qquad \left. - \boldsymbol{B}_{jj}\left(T-t+H\right) - \boldsymbol{C}_{ij}\left(T-t\right) + \boldsymbol{C}_{ij}\left(T-t+H\right) + H\right] \qquad (40)\end{aligned}$$

If the investor is not uncertainty-averse at all, her θ parameter is 0, thus her detection error probability is $\frac{1}{2}$. This means that when solving Problem 1, she chooses the alternative measure $\mathbb{U}$ which is exactly the same as the base measure $\mathbb{B}$, hence it is impossible to distinguish between them based on a sample of data. This is reflected in her detection error probability as well: in 50% of the cases she will be wrong when basing her decision on the likelihood-ratio test described above, so she might as well just flip a coin instead of basing her decision on a sample of data.

The more uncertainty-averse the investor is, that is, the larger θ, the larger the distance between the chosen least-favorable $\mathbb{U}$ measure and the base $\mathbb{B}$ measure. And the larger this distance between $\mathbb{U}$ and $\mathbb{B}$, the easier it is to determine (based on a sample of data) which of the two measures is the true one. This is reflected in the detection error probability of the investor: if θ is larger, $\Phi(\cdot)$ is larger, and her detection error probability is lower. That is, she will make the correct decision when distinguishing between the two measures in a higher proportion of the cases.

The detection error probability also depends on the observation span H: the longer time period the investor observes, the higher H will be and, thus, ceteris paribus $\Phi(\cdot)$ will be higher and the detection error probability will be lower. If the observation span goes to infinity, the detection error probability goes to zero.

Anderson, Hansen and Sargent (2003) argue that the largest amount of uncertainty that should be entertained as reasonable corresponds to a detection error probability of approximately 10%. Since we have 42 years of data available to estimate our model, it is reasonable to assume that our representative investor also has access to 42 years of data to distinguish between the base model and the alternative model. Given that her observation period of 42 years is relatively long and that she can observe the data in continuous time, we can reasonably assume that the detection error probability equals 10%. Table 6 presents the induced calibrated values for the γ and θ separately, based on the estimates in Table 5. Besides the γ and θ values, Column 5 of Table 6 shows by how much the risk-aversion parameter decreases by the introduction of an uncertainty-averse representative agent.

Table 6. Disentangled risk- and uncertainty-aversion parameters Calibrated $\gamma + \theta$ values and their disentangled γ and θ components, based on the estimates in Table 5. The detection error probability is 10%. The observation period is 42 years. The last column shows by how much robustness decreases the risk-aversion parameter that we need to match market data compared to the non-robust model.

$T - t$	$\gamma + \theta$	γ	θ	Decrease in γ due to robustness
10	69.1	25.5	43.6	63%
15	16.9	6.2	10.7	63%
20	11.0	4.1	6.9	63%
25	8.9	3.3	5.6	63%
30	7.9	2.9	5.0	63%

As Table 6 shows, regardless of the investment horizon of the representative investor, the required risk-aversion parameter to match market data is decreased by about 63% as a result of the introduction of robustness in our model. The relative risk aversion of the robust investor is much more reasonable than that of a non-robust investor: for investment periods between 15 and 30 years, it is between about 2.9 and 6.2, compared to it being between 7.9 and 16.9 in the case of a non-robust investor.

As a robustness check of our results, we also disentangle the sum of risk- and uncertainty aversion parameters using higher (i.e., more conservative) detection error probability values, concretely 15% and 20%. The results are shown in Table 7 and Table 8, respectively. We see that assuming a detection error probability of 15% still reduces the originally required risk-aversion parameter by more than half of its original value, and even a rather conservative assumption of 20% detection error probability results in a decrease of 41.5% in the required risk-aversion parameter. This reinforces the robustness of our findings that parameter uncertainty can explain a large fraction of the bond premium puzzle.

Table 7. Disentangled risk- and uncertainty-aversion parameters
Calibrated $\gamma + \theta$ values and their disentangled γ and θ components, based on the estimates in Table 5. The detection error probability is 15%. The observation period is 42 years. The last column shows by how much robustness decreases the risk-aversion parameter that we need to match market data compared to the non-robust model.

$T-t$	$\gamma+\theta$	γ	θ	Decrease in γ due to robustness
10	69.1	33.9	35.2	51%
15	16.9	8.3	8.6	51%
20	11.0	5.4	5.6	51%
25	8.9	4.4	4.5	51%
30	7.9	3.9	4.0	51%

Table 8. Disentangled risk- and uncertainty-aversion parameters
Calibrated $\gamma + \theta$ values and their disentangled γ and θ components, based on the estimates in Table 5. The detection error probability is 20%. The observation period is 42 years. The last column shows by how much robustness decreases the risk-aversion parameter that we need to match market data compared to the non-robust model.

$T-t$	$\gamma+\theta$	γ	θ	Decrease in γ due to robustness
10	69.1	40.5	28.6	41%
15	16.9	9.9	7.0	41%
20	11.0	6.4	4.6	41%
25	8.9	5.2	3.7	41%
30	7.9	4.6	3.3	41%

6 Conclusion

We have shown that the introduction of uncertainty aversion in a standard financial market offers a potential solution to the bond-premium puzzle. In the presence of uncertainty aversion, the risk aversion of the representative agent decreases to levels consistent with both economic intuition and experiments. At the same time, our paper offers the methodological contribution to formulate and solve the robust investment problem an uncertainty-averse investor faces using the martingale method.

To disentangle risk aversion from uncertainty aversion, we assumed a detection error probability of 10%, and we used 15% and 20% detection error probability values as robustness checks of our findings. Assuming detection error probability values between 10% and 20% is common in the literature, but there is little research on what determines the level of the detection error probability. This can be a fruitful line of future research. We also assumed that the investor is not uncertain about the volatility. Relaxing this assumption leads us out of the realm of the framework of Anderson, Hansen and Sargent (2003). Extending the penalty approach of dynamic robust asset allocation in a direction that allows for uncertainty about the volatility is another potential area of future research.

Appendix

Proof of Theorem 1. From (12) the value function at time t satisfies

$$\begin{aligned} V_t(X_t) &= \mathrm{E}_t^{\mathbb{U}}\left\{\exp(-\delta T)\frac{X_T^{1-\gamma}}{1-\gamma} + \int_t^T \frac{(1-\gamma)\,\|\boldsymbol{u}\,(s)\,\|^2}{2\theta} V_s(X_s)\mathrm{d}s\right\} \\ &= \mathrm{E}_t^{\mathbb{U}}\left\{\exp(-\delta T)\frac{X_T^{1-\gamma}}{1-\gamma}\right\} + \mathrm{E}_t^{\mathbb{U}}\left\{\int_0^T \frac{(1-\gamma)\,\|\boldsymbol{u}\,(s)\,\|^2}{2\theta} V_s(X_s)\mathrm{d}s\right\} \\ &\quad - \int_0^t \frac{(1-\gamma)\,\|\boldsymbol{u}\,(s)\,\|^2 V_s(X_s)}{2\theta}\mathrm{d}s, \end{aligned} \tag{41}$$

where X_T and $\mathbb{U}$ denote the optimal terminal wealth and least-favorable physical measure, respectively. Introduce the square-integrable martingales, under $\mathbb{U}$,

$$\begin{aligned} M_{1,t} &= \mathrm{E}_t^{\mathbb{U}}\left\{\exp(-\delta T)\frac{X_T^{1-\gamma}}{1-\gamma}\right\}, && (42) \\ M_{2,t} &= \mathrm{E}_t^{\mathbb{U}}\left\{\int_0^T \frac{(1-\gamma)\left\|\boldsymbol{u}\,(s)\right\|^2}{2\theta} V_s(X_s)\mathrm{d}s\right\}. && (43) \end{aligned}$$

According to the martingale representation theorem (see, e.g., Karatzas and Shreve (1991), pp. 182, Theorem 3.4.15), there exist square-integrable stochastic processes $\boldsymbol{Z}_{1,t}$ and $\boldsymbol{Z}_{2,t}$ such that

$$\begin{aligned} M_{1,t} &= \mathrm{E}_0^{\mathbb{U}}\left\{\exp(-\delta T)\frac{X_T^{1-\gamma}}{1-\gamma}\right\} + \int_0^t \boldsymbol{Z}_{1,s}' \mathrm{d}\boldsymbol{W}_s^{\mathbb{U}}, && (44) \\ M_{2,t} &= \mathrm{E}_0^{\mathbb{U}}\left\{\int_0^T \frac{(1-\gamma)\left\|\boldsymbol{u}\,(s)\right\|^2}{2\theta} V_s(X_s)\mathrm{d}s\right\} + \int_0^t \boldsymbol{Z}_{2,s}' \mathrm{d}\boldsymbol{W}_s^{\mathbb{U}}. && (45) \end{aligned}$$

Substituting in (41), we can express the dynamics of the value function as

$$\mathrm{d}V_t(X_t) = -\frac{(1-\gamma)\left\|\boldsymbol{u}\,(t)\right\|^2}{2\theta} V_t(X_t)\mathrm{d}t + \left(Z_{1,t} + Z_{2,t}\right)' \mathrm{d}\boldsymbol{W}_t^{\mathbb{U}}. \tag{46}$$

This is a linear backward stochastic differential equation, that (together with the terminal condition $V_T(X_T) = \exp(-\delta T)X_T^{1-\gamma}/(1-\gamma))$ can be solved explicitly; see Pham (2009), pp. 141-142. The unique solution to (46) is given by

$$\Gamma_t V_t(X_t) = \mathrm{E}_t^{\mathbb{U}}\left\{\Gamma_T \exp(-\delta T)\frac{X_T^{1-\gamma}}{1-\gamma}\right\}, \tag{47}$$

where Γ_t solves the linear differential equation

$$\mathrm{d}\Gamma_t = \Gamma_t \frac{(1-\gamma)\,\|\boldsymbol{u}\,(t)\,\|^2}{2\theta}\mathrm{d}t; \qquad \Gamma_0 = 1, \tag{48}$$

i.e.,

$$\Gamma_t = \exp\left(\int_0^t \frac{(1-\gamma)\,\|\boldsymbol{u}\,(s)\,\|^2}{2\theta}\mathrm{d}s\right). \tag{49}$$

Substituting into (47), we obtain the value function in closed form as

$$V_t(X_t) = \mathrm{E}_t^{\mathbb{U}}\left\{\exp\left(\int_t^T \frac{(1-\gamma)\,\|\boldsymbol{u}\,(s)\,\|^2}{2\theta}\mathrm{d}s - \delta T\right)\frac{X_T^{1-\gamma}}{1-\gamma}\right\}. \tag{50}$$

Recall that X_t and $\mathbb{U}$ represent the optimal wealth and least-favorable physical measure. As a result, we obtain (13). □

Before continuing with the proofs of the remaining theorems, we recall some well-known results.

Lemma 1. *Under (1) and (2), we have*

$$\begin{aligned} r_t =& A_0 + \boldsymbol{\iota}'\left[\boldsymbol{\mu}_F + \exp\left(-\boldsymbol{\kappa}t\right)\left(\boldsymbol{F}_0 - \boldsymbol{\mu}_F\right)\right] \\ &+ \int_0^t \boldsymbol{\iota}'\exp\left(-\boldsymbol{\kappa}(t-s)\right)\boldsymbol{\sigma}_F\mathrm{d}\boldsymbol{W}_{F,s}^{\mathbb{Q}}. \end{aligned} \tag{51}$$

Proof. The proof directly follows from the solution of the stochastic differential equation of the Ornstein-Uhlenbeck process, for which see, e.g., Chin, Nel and Ólafsson (2014), pp. 132–133. □

Lemma 2. *Under (1) and (2), we have*

$$\int_0^t r_s \mathrm{d}s = \left[A_0 + \boldsymbol{\iota}'\boldsymbol{\mu}_F\right] t + \boldsymbol{\iota}'\boldsymbol{B}(t)\left(\boldsymbol{F}_0 - \boldsymbol{\mu}_F\right) + \int_0^t \boldsymbol{\iota}'\boldsymbol{B}(t-v)\boldsymbol{\sigma}_F \mathrm{d}\boldsymbol{W}_{F,v}^{\mathbb{Q}}, \tag{52}$$

with $\boldsymbol{B}$ defined as the matrix integral

$$\boldsymbol{B}(t) = \int_0^t \exp\left(-\boldsymbol{\kappa}s\right) \mathrm{d}s. \tag{53}$$

Since $\boldsymbol{\kappa}$ is a matrix whose elements are constants, $\boldsymbol{B}(t)$ can be expressed in the more compact form

$$\boldsymbol{B}(t) = \left(\boldsymbol{I} - \exp\left\{-\boldsymbol{\kappa}t\right\}\right)\boldsymbol{\kappa}^{-1}. \tag{54}$$

Proof. From Lemma 1 we find

$$\begin{aligned}
\int_0^t r_s \mathrm{d}s &= \left[A_0 + \boldsymbol{\iota}'\boldsymbol{\mu}_F\right] t + \boldsymbol{\iota}'\boldsymbol{B}(t)\left(\boldsymbol{F}_0 - \boldsymbol{\mu}_F\right) + \int_0^t \int_0^s \boldsymbol{\iota}' \exp\left(-\boldsymbol{\kappa}(s-v)\right) \boldsymbol{\sigma}_F \mathrm{d}\boldsymbol{W}_{F,v}^{\mathbb{Q}} \mathrm{d}s \\
&= \left[A_0 + \boldsymbol{\iota}'\boldsymbol{\mu}_F\right] t + \boldsymbol{\iota}'\boldsymbol{B}(t)\left(\boldsymbol{F}_0 - \boldsymbol{\mu}_F\right) + \int_0^t \int_v^t \boldsymbol{\iota}' \exp\left(-\boldsymbol{\kappa}(s-v)\right) \boldsymbol{\sigma}_F \mathrm{d}\boldsymbol{W}_{F,v}^{\mathbb{Q}} \mathrm{d}s \\
&= \left[A_0 + \boldsymbol{\iota}'\boldsymbol{\mu}_F\right] t + \boldsymbol{\iota}'\boldsymbol{B}(t)\left(\boldsymbol{F}_0 - \boldsymbol{\mu}_F\right) + \int_0^t \boldsymbol{\iota}'\boldsymbol{B}(t-v)\boldsymbol{\sigma}_F \mathrm{d}\boldsymbol{W}_{F,v}^{\mathbb{Q}}.
\end{aligned}$$

This completes the proof. □

Lemma 2 immediately leads to the price of bonds in our financial market. We briefly recall this for completeness. The price at time t of a nominal bond with remaining maturity τ is given by

$$
\begin{aligned}
P_t(\tau) &= \mathrm{E}_t^{\mathbb{Q}} \exp\left(-\int_t^{t+\tau} r_s \mathrm{d}s\right) \\
&= \mathrm{E}_t^{\mathbb{Q}} \exp\left(-\left[A_0 + \boldsymbol{\iota}'\boldsymbol{\mu}_F\right]\tau - \boldsymbol{\iota}'\boldsymbol{B}(\tau)\left(\boldsymbol{F}_t - \boldsymbol{\mu}_F\right) - \int_t^{t+\tau} \boldsymbol{\iota}'\boldsymbol{B}(t+\tau-s)\boldsymbol{\sigma}_F \mathrm{d}\boldsymbol{W}_{F,s}^{\mathbb{Q}}\right) \\
&= \exp\left(-\left[A_0 + \boldsymbol{\iota}'\boldsymbol{\mu}_F\right]\tau - \boldsymbol{\iota}'\boldsymbol{B}(\tau)\left(\boldsymbol{F}_t - \boldsymbol{\mu}_F\right) + \frac{1}{2}\int_0^{\tau} \|\boldsymbol{\iota}'\boldsymbol{B}(\tau-s)\boldsymbol{\sigma}_F\|^2 \mathrm{d}s\right). \qquad (55)
\end{aligned}
$$

As a result, the exposure of a constant τ-maturity bond fund to the factors $\boldsymbol{F}_t$ is given by $-\boldsymbol{\iota}'\boldsymbol{B}(\tau)$.

We now continue with the proofs of Theorem 2, 3, and 4.

Proof of Theorem 2. The first step of the optimization is to determine the optimal terminal wealth, given the budget constraint. In order to determine the optimal terminal wealth, we form the Lagrangian from (13) and (14). This Lagrangian is given by

$$
\begin{aligned}
L(x) &= \inf_{\mathbb{U}} \sup_{X_T} \Bigg\{ \mathrm{E}^{\mathbb{U}} \exp\left(\frac{1-\gamma}{2\theta}\int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t - \delta T\right) \frac{X_T^{1-\gamma}}{1-\gamma} \\
&\quad - y\left[\mathrm{E}^{\mathbb{Q}} \exp\left(-\int_0^T r_t \mathrm{d}t\right) X_T - x\right]\Bigg\} \\
&= \inf_{\mathbb{U}} \sup_{X_T} \Bigg\{ \mathrm{E}^{\mathbb{Q}} \exp\Bigg(\int_0^T (\boldsymbol{u}(t)+\boldsymbol{\lambda})' \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} - \frac{1}{2}\int_0^T \|\boldsymbol{u}(t)+\boldsymbol{\lambda}\|^2 \mathrm{d}t \\
&\qquad + \frac{1-\gamma}{2\theta}\int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t - \delta T\Bigg) \frac{X_T^{1-\gamma}}{1-\gamma} \\
&\quad - y\left[\mathrm{E}^{\mathbb{Q}} \exp\left(-\int_0^T r_t \mathrm{d}t\right) X_T - x\right]\Bigg\}, \qquad (56)
\end{aligned}
$$

where y is the Lagrange-multiplier and the second equality uses the Girsanov transformation (6).

We first consider the inner optimization, i.e., the optimal choice of the final wealth X_T given $\mathbb{U}$. The first order condition for optimal final wealth, denoted by $\hat{X}_T$, is

$$\begin{aligned}
&\exp\left(\int_0^T (\boldsymbol{u}(t)+\boldsymbol{\lambda})' \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} - \frac{1}{2}\int_0^T \|\boldsymbol{u}(t)+\boldsymbol{\lambda}\|^2 \mathrm{d}t + \frac{1-\gamma}{2\theta}\int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t - \delta T\right) \hat{X}_T^{-\gamma} \\
&= \quad y \exp\left(-\int_0^T r_t \mathrm{d}t\right). \qquad (57)
\end{aligned}$$

As the Lagrange multiplier y is still to be determined by the budget constraint, we can subsume all deterministic terms in a new Lagrange multiplier y_1 and solve

$$\exp\left(\int_0^T (\boldsymbol{u}(t)+\boldsymbol{\lambda})' \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}}\right) \hat{X}_T^{-\gamma} = y_1 \exp\left(-\int_0^T r_t \mathrm{d}t\right), \qquad (58)$$

i.e.,

$$\hat{X}_T = y_1^{-1/\gamma} \exp\left(\frac{1}{\gamma}\int_0^T r_t \mathrm{d}t + \frac{1}{\gamma}\int_0^T (\boldsymbol{u}(t)+\boldsymbol{\lambda})' \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}}\right), \qquad (59)$$

We can now substitute this optimal terminal wealth into the budget constraint (14) in order to obtain the optimal value of the Lagrange multiplier (denoted by $\hat{y}_1$).

$$\hat{y}_1^{-1/\gamma} = \frac{x}{\mathrm{E}^{\mathbb{Q}} \exp\left(\frac{1-\gamma}{\gamma}\int_0^T r_t \mathrm{d}t + \frac{1}{\gamma}\int_0^T (\boldsymbol{u}(t)+\boldsymbol{\lambda})' \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}}\right)}. \qquad (60)$$

Substituting (60) into (59), the optimal terminal wealth given $\mathbb{Q}$ is obtained explicitly. This yields (16).

We proceed by solving the outer optimization in (13), i.e., we find the least-favorable distortions $\boldsymbol{u}$. Substituting the optimal final wealth (59) into the value function leads to

$$
\begin{aligned}
V_0(x) &= \inf_{\mathbb{U}} \mathrm{E}^{\mathbb{U}} \left\{ \exp\left(-\delta T + \frac{1-\gamma}{2\theta} \int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t \right) \frac{\hat{X}_T^{1-\gamma}}{1-\gamma} \right\} \\
&= \frac{x^{1-\gamma}}{1-\gamma} \inf_{\mathbb{U}} \mathrm{E}^{\mathbb{U}} \left\{ \exp\left(-\delta T + \frac{1-\gamma}{2\theta} \int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t + \frac{1-\gamma}{\gamma} \int_0^T r_t \mathrm{d}t \right.\right. \\
&\qquad \left.\left. + \frac{1-\gamma}{\gamma} \int_0^T (\boldsymbol{u}(t) + \boldsymbol{\lambda})' \,\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} \right) \right\} \\
&\quad \times \left[\mathrm{E}^{\mathbb{Q}} \exp\left(\frac{1-\gamma}{\gamma} \int_0^T r_t \mathrm{d}t + \frac{1}{\gamma} \int_0^T (\boldsymbol{u}(t) + \boldsymbol{\lambda})' \,\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} \right) \right]^{\gamma-1}. \qquad (61)
\end{aligned}
$$

We now use the Girsanov transforms (5)–(6) to obtain

$$
\begin{aligned}
V_0(x) &= \frac{x^{1-\gamma}}{1-\gamma} \inf_{\mathbb{U}} \mathrm{E}^{\mathbb{Q}} \left\{ \exp\left(-\delta T + \frac{1-\gamma}{2\theta} \int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t + \frac{1-\gamma}{\gamma} \int_0^T r_t \mathrm{d}t \right.\right. \\
&\qquad \left.\left. + \frac{1-\gamma}{\gamma} \int_0^T (\boldsymbol{u}(t) + \boldsymbol{\lambda})' \,\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} \right) \frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{Q}} \right\} \\
&\quad \times \left[\mathrm{E}^{\mathbb{Q}} \exp\left(\frac{1-\gamma}{\gamma} \int_0^T r_t \mathrm{d}t + \frac{1}{\gamma} \int_0^T (\boldsymbol{u}(t) + \boldsymbol{\lambda})' \,\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} \right) \right]^{\gamma-1} \\
&= \frac{x^{1-\gamma}}{1-\gamma} \inf_{\mathbb{U}} \mathrm{E}^{\mathbb{Q}} \left\{ \exp\left(-\delta T + \frac{1-\gamma}{2\theta} \int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t + \frac{1-\gamma}{\gamma} \int_0^T r_t \mathrm{d}t \right.\right. \\
&\qquad \left.\left. + \frac{1}{\gamma} \int_0^T (\boldsymbol{u}(t) + \boldsymbol{\lambda})' \,\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} - \frac{1}{2} \int_0^T \|\boldsymbol{u}(t) + \boldsymbol{\lambda}\|^2 \mathrm{d}t \right) \right\} \\
&\quad \times \left[\mathrm{E}^{\mathbb{Q}} \exp\left(\frac{1-\gamma}{\gamma} \int_0^T r_t \mathrm{d}t + \frac{1}{\gamma} \int_0^T (\boldsymbol{u}(t) + \boldsymbol{\lambda})' \,\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} \right) \right]^{\gamma-1} \\
&= \frac{x^{1-\gamma}}{1-\gamma} \inf_{\mathbb{U}} \exp\left(-\delta T + \frac{1-\gamma}{2\theta} \int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t - \frac{1}{2} \int_0^T \|\boldsymbol{u}(t) + \boldsymbol{\lambda}\|^2 \mathrm{d}t \right) \\
&\quad \times \left[\mathrm{E}^{\mathbb{Q}} \exp\left(\frac{1-\gamma}{\gamma} \int_0^T r_t \mathrm{d}t + \frac{1}{\gamma} \int_0^T (\boldsymbol{u}(t) + \boldsymbol{\lambda})' \,\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} \right) \right]^{\gamma}. \qquad (62)
\end{aligned}
$$

Using Lemma 2 we find

$$\begin{aligned}
V_0(x) &= \frac{x^{1-\gamma}}{1-\gamma}\inf_{\mathbb{U}}\exp\left(-\delta T+\frac{1-\gamma}{2\theta}\int_0^T\|\boldsymbol{u}(t)\|^2\mathrm{d}t-\frac{1}{2}\int_0^T\|\boldsymbol{u}(t)+\boldsymbol{\lambda}\|^2\mathrm{d}t\right)\\
&\quad\times\exp\big((1-\gamma)\left[A_0+\boldsymbol{\iota}'\boldsymbol{\mu}_F\right]T+(1-\gamma)\boldsymbol{\iota}'\boldsymbol{B}(T)\left(\boldsymbol{F}_0-\boldsymbol{\mu}_F\right)\big)\\
&\quad\times\left[\mathrm{E}^{\mathbb{Q}}\exp\left(\frac{1-\gamma}{\gamma}\int_0^T\boldsymbol{\iota}'\boldsymbol{B}(T-t)\boldsymbol{\sigma}_F\mathrm{d}\boldsymbol{W}_{F,t}^{\mathbb{Q}}\mathrm{d}t+\frac{1}{\gamma}\int_0^T(\boldsymbol{u}(t)+\boldsymbol{\lambda})'\,\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}}\right)\right]^{\gamma}\\
&= \frac{x^{1-\gamma}}{1-\gamma}\inf_{\mathbb{U}}\exp\left(-\delta T+\frac{1-\gamma}{2\theta}\int_0^T\|\boldsymbol{u}(t)\|^2\mathrm{d}t-\frac{1}{2}\int_0^T\|\boldsymbol{u}(t)+\boldsymbol{\lambda}\|^2\mathrm{d}t\right)\\
&\quad\times\exp\big((1-\gamma)\left[A_0+\boldsymbol{\iota}'\boldsymbol{\mu}_F\right]T+(1-\gamma)\boldsymbol{\iota}'\boldsymbol{B}(T)\left(\boldsymbol{F}_0-\boldsymbol{\mu}_F\right)\big)\\
&\quad\times\exp\left(\frac{1}{2\gamma}\int_0^T\|(1-\gamma)\boldsymbol{\iota}'\boldsymbol{B}(T-t)\boldsymbol{\sigma}_F+(\boldsymbol{u}_F(t)+\boldsymbol{\lambda}_F)'\,\|^2\mathrm{d}t\right.\\
&\qquad\left.+\frac{1}{2\gamma}\int_0^T(u_{N+1}(t)+\lambda_{N+1})^2\,\mathrm{d}t\right). \qquad (63)
\end{aligned}$$

Recall that $\boldsymbol{u}_F(t)$ is an N-dimensional column vector containing the first N elements of $\boldsymbol{u}(t)$ and $u_{N+1}(t)$ is its last element. Similarly, $\boldsymbol{\lambda}_F$ is the N-dimensional column vector containing the first N elements of $\boldsymbol{\lambda}$ and λ_{N+1} denotes its last element.

This expression of the value function allows to perform the outer minimization with respect to $\mathbb{U}$, i.e., with respect to $\boldsymbol{u}$. The first-order condition of this minimization reads, with respect to $\boldsymbol{u}_F(t)$,

$$\begin{aligned}
0 &= \frac{1-\gamma}{\theta}\boldsymbol{u}_F(t)-\boldsymbol{u}_F(t)-\boldsymbol{\lambda}_F+\frac{1}{\gamma}\big((1-\gamma)\boldsymbol{\sigma}_F'\boldsymbol{B}(T-t)'\boldsymbol{\iota}+\boldsymbol{u}_F(t)+\boldsymbol{\lambda}_F\big)\\
&= \frac{1-\gamma}{\theta}\boldsymbol{u}_F(t)+\frac{1-\gamma}{\gamma}\left(\boldsymbol{u}_F(t)+\boldsymbol{\lambda}_F\right)+\frac{1-\gamma}{\gamma}\boldsymbol{\sigma}_F'\boldsymbol{B}(T-t)'\boldsymbol{\iota}\\
&= (1-\gamma)\frac{\gamma+\theta}{\gamma\theta}\boldsymbol{u}_F(t)+\frac{1-\gamma}{\gamma}\left[\boldsymbol{\sigma}_F'\boldsymbol{B}(T-t)'\boldsymbol{\iota}+\boldsymbol{\lambda}_F\right]. \qquad (64)
\end{aligned}$$

This proves (17). Finally, minimizing the value function $V_0(x)$ with respect to $u_{N+1}(t)$ we find the first-order condition

$$\begin{aligned}
0 &= \frac{1-\gamma}{\theta} u_{N+1}(t) - u_{N+1}(t) - \lambda_{N+1} + \frac{1}{\gamma}\left(u_{N+1}(t) + \lambda_{N+1}\right) \\
&= \left[\frac{1-\gamma}{\theta} + \frac{1-\gamma}{\gamma}\right] u_{N+1}(t) + \frac{1-\gamma}{\gamma}\lambda_{N+1}. \qquad (65)
\end{aligned}$$

This proves (18) and completes the proof. □

Proof of Theorem 3. Theorem 2 has actually a fairly simple form in the driving Brownian motion $\boldsymbol{W}_t^{\mathbb{Q}}$. Using Lemma 2 we find

$$\begin{aligned}
&\exp\left(-\int_0^T r_t \mathrm{d}t\right)\hat{X}_T \\
&\propto \exp\left(\int_0^T \frac{\hat{\boldsymbol{u}}(t)' + \boldsymbol{\lambda}'}{\gamma}\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} + \frac{1-\gamma}{\gamma}\int_0^T \left[\boldsymbol{\iota}'\boldsymbol{B}(T-t)\boldsymbol{\sigma}_F; 0\right]\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}}\right) \\
&= \exp\left(\int_0^T \left[\frac{\boldsymbol{\lambda}_F'}{\gamma+\theta} + \frac{1-(\gamma+\theta)}{\gamma+\theta}\boldsymbol{\iota}'\boldsymbol{B}(T-t)\boldsymbol{\sigma}_F; \frac{\boldsymbol{\lambda}_{N+1}}{\gamma+\theta}\right]\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}}\right). \qquad (66)
\end{aligned}$$

As $\exp\left(-\int_0^t r_s \mathrm{d}s\right)\hat{X}_t = \mathrm{E}_t^{\mathbb{Q}}\exp\left(-\int_0^T r_t \mathrm{d}t\right)\hat{X}_T$, the wealth evolution (21) is easily obtained. □

Proof of Corollary 1. This is a direct consequence of Theorem 3 and the martingale representation theorem (see, e.g., Karatzas and Shreve (1991), pp. 182, Theorem 3.4.15). □

Proof of Theorem 4. The investor observes the prices of N constant maturity bond funds and of the stock market index in continuous time from

$t-H$ to t. The moment of observation is t. Using (6) we have

$$\log\left(\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{U}}\right)_H = -\frac{1}{2}\int_{t-H}^{t}\|\boldsymbol{u}(s)\|^2\mathrm{d}s + \int_{t-H}^{t}\boldsymbol{u}(s)\,\mathrm{d}W_s^{\mathbb{U}}$$
$$\overset{\mathbb{U}}{\sim}\mathcal{N}\left(-\frac{1}{2}\int_{t-H}^{t}\|\boldsymbol{u}(s)\|^2\mathrm{d}s, \int_{t-H}^{t}\|\boldsymbol{u}(s)\|^2\mathrm{d}s\right). \tag{67}$$

This implies

$$P^{\mathbb{U}}\left(\log\left(\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{U}}\right)_H > 0\right) = 1-\Phi\left(\frac{1}{2}\sqrt{\int_{t-H}^{t}\|\boldsymbol{u}(s)\|^2\mathrm{d}s}\right). \tag{68}$$

Due to symmetry, we have

$$P^{\mathbb{B}}\left(\log\left(\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{U}}\right)_H < 0\right) = P^{\mathbb{B}}\left(\log\left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{B}}\right)_H > 0\right)$$
$$= P^{\mathbb{U}}\left(\log\left(\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{U}}\right)_H > 0\right). \tag{69}$$

Thus, using Definition 2, the detection error probability is

$$DEP = 1-\Phi\left(\frac{1}{2}\sqrt{\int_{t-H}^{t}\|\boldsymbol{u}(s)\|^2\mathrm{d}s}\right). \tag{70}$$

□

Proof of Corollary 2. Substituting (19) and (20) into (70) and working out the integrals, (38) follows directly. □

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