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Term structures with converging forward rates

Michel Vellekoop

Jan de Kort

DESIGN 54

Michel Vellekoop and Jan de Kort

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DESIGN PAPER 54

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Frans Kooymans
Netspar

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Affiliations

Michel Vellekoop – University of Amsterdam
Jan de Kort – University of Amsterdam

TERM STRUCTURES WITH CONVERGING FORWARD RATES

Summary

The risk-free term structure of interest rates is used by financial institutions to determine how much money needs to be invested today to receive a given amount of money on a later date. Michel Vellekoop and Jan de Kort (both UvA) have investigated inter- and extrapolation techniques that can be used to create discount curves from observed market data under different assumptions on asymptotic forward rates. They discuss the methods proposed by EIOPA and the Commission UFR and suggest some new alternatives.

1. Introduction

The risk-free term structure of interest rates is used by financial institutions to determine how much money needs to be invested today to receive a given amount of money on a later date. More specifically, if we write $p(0, T)$ for the investment needed today (time zero) to receive 1 euro T years later, then the curve $T \to p(0, T)$ is called the discount curve and different interest rates such as yields, spot rates and forward rates, can be directly derived from it. Clearly, the times T for which information is given in fixed income products are limited to a finite number of discrete maturity points and not available beyond a certain largest maturity $T_{\max}$. This means that the definition of the whole discount curve cannot just be based on a dataset of fixed income products alone, but that structural assumptions are required[1].

One possibility is to define a parametric form of the discount curve, yield curve or forward curve and then choose the parameters in such a way that market prices for risk-free fixed income products are fitted as well as possible. A popular choice is the Nelson-Siegel-Svensson specification (Nelson and Siegel 1987, Svensson 1994) which uses six parameters that all have a clear interpretation. One of the parameters involved in that parametrization can be interpreted as an *ultimate forward rate* i.e. the limiting value for forward rates (and thus for yields) when the maturity goes to infinity. One of the other parameters describes the *speed of convergence* to this limiting value for higher maturities.

But since only six parameters are involved in the Nelson-Siegel-Svensson model, a given set of five or more prices for fixed income products can in general not be fitted perfectly. If one insists on a perfect fit for market data, instead of allowing a certain aggregate measurement error to be minimized, such functional families for term structures which involve only a few parameters can therefore not be used. Many more parameters are needed and it then becomes most practical to use functions that are piecewise polynomial between the maturity points that are given by the market data. The order of these polynomials has to be chosen beforehand, and this choice will determine the

[1] Notice that we only consider the current discount curve in this paper; we do not specify its future dynamics. One could do this, for example, using a Hull-White model which takes the current curve as its starting point. However, this does not guarantee that choosing the initial term structure in a certain class of parameterized interpolation curves will lead to term structures at later times which remain in that class.

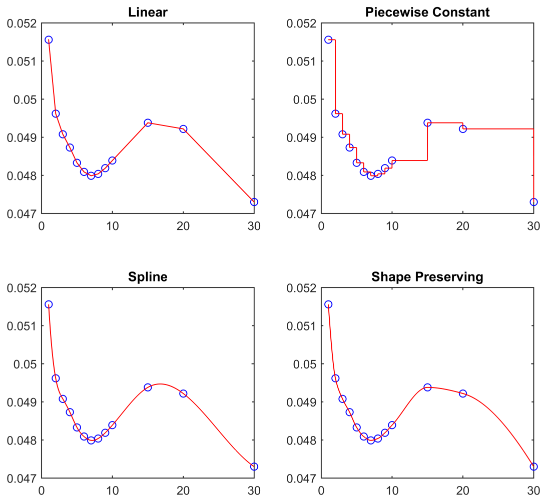

Figure 1: Inter- and extrapolation methods. DNB Curve July 31, 2008.

smoothness of the interpolating curves.

We give some examples in Figure 1, which interpolates points on the Dutch central bank curve of July 31, 2008. The piecewise constant interpolation (top right) is not continuous; the linear interpolation (top left) is continuous but not differentiable; the shape-preserving spline (bottom right) has continuous derivative but the second derivative is not continuous; a property which is true for the cubic spline (bottom left). Note that the cubic spline introduces a local maximum between maturity 10 and 20 and that we may see only a small difference of the two bottom curves between maturity 20 and 30. But if we were to use these curves to extrapolate beyond the 30 year maturity point, this would still have a big effect.

An overview of these and other possible interpolation methods is given in a paper by Hagan & West (Hagan and West 2006). Smoothness of the curve is mentioned as an important criterion to judge the results of the different methods, but other desirable properties of interpolating functions are mentioned, such as preserving local monotonicity or convexity in the given dataset. The authors also investigate whether changes in the input data which are restricted to the vicinity of a particular maturity might influence the values of the interpolated discount curve at maturities far away from it. Strong non-local sensitivities in an interpolation method make it less suitable for risk management purposes since it complicates the use of fixed income market instruments to mitigate the interest rate risk in a portfolio of liabilities.

Similar considerations play a role when deciding on a method to extrapolate the discount curve beyond the last maturity for which reliable market information is available. In the new Solvency II regulatory framework for European insurers (EIOPA 2010) term structure inter- and extrapolation is based on a method that was proposed by Smith & Wilson in earlier work (Smith and Wilson 2000). This method uses exponential tension splines with a given tension parameter[2]. Exponential tension splines were introduced by Schweikert (Schweikert 1966) and first proposed for term structure modelling by Barzanti & Corradi (Barzanti and Corradi 1998a, Barzanti and Corradi 1998b). Andersen (Andersen 2007) rewrote the exponential tension spline fitting problem in terms of a more convenient basis of functions and showed how non-local sensitivities are controlled by the tension parameter.

The Smith-Wilson method implements the explicit constraint in the Solvency II regulatory framework that forward rates must converge to a value, the *Ultimate Forward Rate* (UFR) of 4.2%, which is specified by EIOPA. However, there is no theoretical justification or empirical evidence for such an assumption[3]. The Dutch central bank (DNB) and the Dutch Bureau for Economic Policy (CPB) have voiced their concern about this assumption. Moreover, the discount curve for pension funds used by the Dutch regulator now explicitly incorporates the fact that ultimate forward rates vary over time. That curve is built according to an algorithm proposed by the *Commission UFR*, a commission which was asked by the Dutch government to investigate the ultimate forward rate. The commission defines the UFR as the running 120-month historical average of the forward interest rate between maturities 20 and 21. Extrapolation beyond the maturity of 20 years (the *first smoothing point*) is then performed using an exponential function.

[2]This parameter was taken $\alpha = 0.10$ by the European insurance regulator EIOPA.

[3]For a detailed discussion on this issue, see the Netspar Opinion paper *The Ultimate Forward Rate: time for a step backwards ?*

Both the EIOPA proposal and the proposal by the Commission UFR thus apply an extrapolation method for forward rates beyond maturity 20 towards a given value, which is a constant (4.2%) according to EIOPA and a value which only depends on (historical) 20-year forward rates according to the Commission. If forward rates for the relevant higher maturities (say 20 to 60 years) would be very stable over time and almost constant over the different maturities, then taking them constant or taking an average over historical values would be justified. But there is clear empirical evidence that long term rates are in fact highly uncertain. That is, for example, the conclusion of a recent paper by Balter, Pelsser and Schotman (Balter, Pelsser, and Schotman 2014) in which a Bayesian approach is used to describe parameter uncertainty in a Vasicek short rate model.

We would therefore like to estimate the asymptotic value of forward rates directly from the available market data. In this paper we therefore extend the Smith-Wilson method to the case where the limit of forward rates is assumed to exist, but not given a certain value beforehand. This not only makes it possible to construct discount curves without such a restrictive assumption but it also provides information on the behavior of forward rates at high maturities in historical data, without making specific model assumptions. This provides a way to establish whether the values that are imposed in regulatory frameworks are consistent with what is found in market data.

The structure of the paper is as follows. First we define the term structure inter- and extrapolation problem and the solution proposed by Smith & Wilson on the one hand and the Commission UFR on the other hand in section 2. We then show in section 3 how we can transform the constrained optimization problem into an unconstrained optimization problem. Section 4 presents numerical results and we formulate policy implications in section 5.

We have attempted to make the sections 4 and 5 self-contained so readers who are less interested in the technical details can skip sections 2 and 3.

2. Constrained Extrapolation Methods

Term structures can be represented in many equivalent ways . We will first discuss a few of these formulations and fix our notation, before we discuss the extrapolation approaches of EIOPA and the Commission UFR.

2.1. Coordinates to Express Interest Rates

As is customary, we use the notation $p(t, T)$ for the amount to be paid at a time $t \geq 0$ (in years) to receive a certain single unit of currency at a later time $T \geq t$, i.e. the zero-coupon bond price at time t for maturity T. This makes the function $T \to p(t, T)$ the *discount curve* at time t. The continuous-time *yield curve* $T \to y(t, T)$ and *forward curve* $T \to f(t, T)$ are then defined by[4]

$$f(t, T) = -\frac{\partial}{\partial T} \ln p(t, T), \qquad y(t, T) = -\frac{\ln p(t, T)}{T - t},$$

so we get the following equivalent representations

$$p(t, T) \; = \; e^{-(T-t)y(t,T)} \; = \; e^{-\int_t^T f(t,u)du}.$$

We only treat the static case in this paper, so we fit today's term structure and do not consider interest rate dynamics at later times; this means $t = 0$ in the expressions above. If one wants to use discrete rates instead of rates in continuous time then these can be found by noticing that a discrete rate δ between times S and $T > S$ must satisfy $p(0, T)(1 + \delta)^{T-S} = p(0, S)$.

Since we do not have a continuum of bond prices available we must use interpolation methods and it is necessary to use extrapolation methods for times beyond the maximal bond maturity that is available in the market. We will assume that at the current time $t = 0$ we know the market prices m_i of certain fixed income instruments that pay given amounts c_{ij} at given times $u_j \geq 0$. Here the index $i \in \mathcal{I} = \{1, 2, ..., n_{\mathcal{I}}\}$ refers to a certain instrument and $j \in \mathcal{J} = \{1, 2, .., n_{\mathcal{J}}\}$ to all possible times when payments may occur. If asset i does not pay anything at time u_j the corresponding value c_{ij} is simply set to zero.

---

[4]Notice that when we talk of yield we always mean zero-coupon yield, which is also known as the *spot rate* for the matuirty under consideration. In the documents of the Commission UFR the notation $z_c(t, T)$ is used for such rates.

An interpolating discount curve $\overline{p}(0, t)$ for these coupon bonds must thus satisfy

$$(\forall i \in \mathcal{I}) \qquad m_i = \sum_{j \in \mathcal{J}} c_{ij} \overline{p}(0, u_j). \tag{1}$$

A specific criterion is needed to decide how the interpolation is defined between the times u_j, since there are many possible ways to do this. Often one chooses a criterion which regulates the smoothness of the interpolating curve. One could for example require the first and second order derivatives of an interpolating curve g to stay small in a quadratic sense. This means that one tries to minimize

$$\mathcal{L}(g) \;:=\; \int_0^\infty [g''(s)^2 + \alpha^2 g'(s)^2]\, ds \tag{2}$$

over all g in a certain set of possible interpolating functions. Notice that a parameter $\alpha > 0$ is needed for the trade-off between the requirements that both slope and curvature remain small. We will see later that this parameter has a natural interpretation as a *speed of convergence* when we consider forward rates that converge to a specific value.

2.2. Smith-Wilson Approach

We thus want to define a discount curve $p(0, t)$ which is smooth but also market-consistent in the sense that it matches quoted market prices of certain fixed income instruments. In the formulation of Smith & Wilson (Smith and Wilson 2000) there is the additional requirement that the forward rates converge to an a priori given value f_∞. This is implemented by taking the following form for the interpolation function:

$$p(0, t) = (1 + g(t))e^{-f_\infty t}. \tag{3}$$

We thus look for a sufficiently smooth function g which minimizes $\mathcal{L}(g)$ under the constraints that $g(0) = 0$ and $\lim_{t \to \infty} g'(t) = 0$ while satisfying

$$m_i = \sum_{j \in \mathcal{J}} c_{ij}(1 + g(u_j))e^{-f_\infty u_j} \tag{4}$$

for a given $n_\mathcal{I} \times n_\mathcal{J}$ matrix $\mathbf{C}$ of cashflows and a $n_\mathcal{I} \times 1$ vector $\mathbf{m}$ of market prices. We will always assume that the rows of C are linearly independent vectors so there are no superfluous instruments in our set. We define

$\overline{c}_{ij} = c_{ij} e^{-f_\infty u_j}$ and $\overline{m}_i = m_i - \sum_{j \in \mathcal{J}} \overline{c}_{ij}$, to ease the notation of (4) into $\overline{m}_i = \sum_{j \in \mathcal{J}} \overline{c}_{ij} g(u_j)$.

The solution to this problem can be written in terms of certain basis functions $SW(t, u_j)$ which depend on time t and the maturities u_j. These functions SW introduced by Smith and Wilson (Smith and Wilson 2000) are scaled versions of other functions W that we will find more convenient to work with:

$$\begin{aligned} SW(t, u) &= e^{-f_\infty (t+u)} W(t, u), \\ W(t, u) &= \alpha \min(t, u) - \tfrac{1}{2} e^{-\alpha |t-u|} + \tfrac{1}{2} e^{-\alpha (t+u)}, \end{aligned} \tag{5}$$

for[5] $(t, u) \in \mathbb{R}^+ \times \mathbb{R}^+$.

The functions W are called *exponential tension splines* and they were introduced by Schweikert in 1966 (Schweikert 1966). Their properties have been shown to have certain advantages which are useful for modelling yield curves and for other applications, see for example (Andersen 2007, Andersen and Piterbarg 2010, Pruess 1976).

In EIOPA documentation about the Smith-Wilson interpolation method (EIOPA 2010) it is mentioned that the function $W(t, u)$ is related to the covariance function of an integrated Ornstein-Uhlenbeck process. However, a paper by Andersson and Lindholm (Andersson and Lindholm 2013) shows that W does not exactly correspond to the covariance function of an integrated Ornstein-Uhlenbeck process unless some further restrictive assumptions are imposed. But one may show[6] that the functions W represent the covariance functions of a certain stochastic process that can be constructed explicitly and this ensures that a matrix with entries $W_{jk} = W(u_k, u_j)$ is invertible as long as the u_k represent different maturities.

Theorem 2.1. *If there is a discount curve (3) such that the smoothness criterion (2) is minimized and such that all the market prices (4) are fitted perfectly, then g*

[5] In our notation $\mathbb{R}^+ = [0, \infty[$ i.e. zero is included.

[6] We will not give proofs for the technical results in this Design Paper. They can be found in (de Kort and Vellekoop 2016).

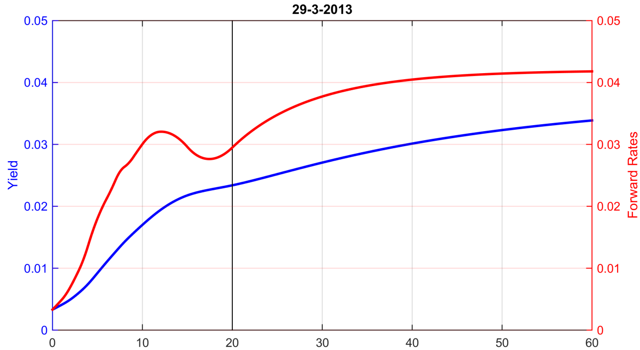

Figure 2: Inter- and extrapolation of Dutch Central Bank data for 29-3-2013 up to maturity 20. Lower curve = Yields, Upper curve = Forwards.

must take the form[7]

$$g(t) = \sum_{i \in \mathcal{I}} \zeta_i \sum_{j \in \mathcal{J}} \overline{c}_{ij} W(t, u_j) \tag{6}$$

with $(\zeta_i)_{i \in \mathcal{I}}$ *a vector of weights and the functions* W *as defined in (5).*

This result has the interpretation of the best possible interpolation under a criterion based on the operator $\mathcal{L}$, since the functions SW that they propose are simply scaled versions of the exponential tension splines W.

The Smith-Wilson method may result in rather unnatural shapes for the interpolated yield and forward curves beyond maturity 20 since forward rates and yields are forced to converge to 4.2%. The curve beyond maturity 20 is therefore not consistent with interest rate markets. This leads to a direct mismatch between values for assets and liabilities on the balance sheet for all cashflows between maturities 20 and 50, since fixed income assets are still priced by the markets for those maturities but the constructed discount curve is not consistent with the market prices.

[7]Here and in the sequel we will always assume that certain technical conditions are imposed on the candidate functions g for the inter- and extrapolation problem: g must have square integrable first and second order derivatives with limit zero and the second derivative $g''(0)$ must equal zero as well. The value of $g(0)$ equals zero unless stated otherwise.

In Figure 2 we show the forward rates and yields produced by EIOPA's inter- and extrapolation method. We notice a very unnatural kink in the term structure of forward rates around the last liquid point (the "LLP" at maturity 20) which is clearly the result of the requirement that forward rates converge to 4.2%. If we were to use the usual interpolation optimization criteria without that requirement the forward curve and yield curve would propagate more smoothly, as we will see later.

2.3. Commission UFR Approach

The Commission UFR defines the UFR as the 120-month average of 20-year (continuous) forward rates (as observed at the *end* of the month)[8]:

$$\widehat{f}(t,\infty) \;=\; \ln\left(\frac{1}{120}\sum_{i=0}^{119}\left(\frac{p(t-\frac{i}{12},t-\frac{i}{12}+20)}{p(t-\frac{i}{12},t-\frac{i}{12}+21)}\right)\right).$$

When the Commission proposed this definition in the fall of 2013 it led to a UFR of 3.9%; it was 3.3% by early July 2015. The 20-year forward rate was around 4% ten years ago and is now around 2%. If it stays around 2% the UFR thus drops by $\frac{1}{120}(4\%-2\%)=1.6$ basis points per month in the coming months since a new datapoint of approximately 2% is added every month while a point around 4% is removed.

The first smoothing point, the forward at maturity 20, is defined as a weighted average over (current !) forward rates between maturity 20 and later maturities[9]. This means that $\widehat{f}(t,20)$ equals

$$\frac{\left(\frac{1}{5}\ln\frac{p(t,t+25)}{p(t,t+20)}\right)+\frac{1}{2}\left(\frac{1}{10}\ln\frac{p(t,t+30)}{p(t,t+20)}\right)+\frac{1}{4}\left(\frac{1}{20}\ln\frac{p(t,t+40)}{p(t,t+20)}\right)+\frac{1}{8}\left(\frac{1}{30}\ln\frac{p(t,t+50)}{p(t,t+20)}\right)}{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}$$

which can be rewritten in terms of yields:

$$\frac{120}{45}y(t,t+25)+\frac{36}{45}y(t,t+30)+\frac{12}{45}y(t,t+40)+\frac{5}{45}y(t,t+50)$$
$$-\frac{128}{45}y(t,t+20).$$

Note that this means that information in the discount curve for maturities as high as 50 years is assumed to be available from market data but that this

[8] In the notation of the Commission, $\widehat{f}(t,\infty)=UFR_c(t)$ and $\widehat{f}(t,20)=f_c^*(t)$.

[9] The Commission also introduced an extra smoothing over time of this quantity but DNB removed this from the final implementation of the method so we present here the method after DNB's small modification.

information is not influencing the estimate of the ultimate forward rate. In fact, the ultimate forward rate only depends on market data for the 20-year forward while the virtual forward rate at maturity 20 is assumed to depend on market data for maturities far beyond 20.

Extrapolation beyond the first smoothing point at maturity 20 is implemented by taking for $h \geq 0$:

$$\widehat{p}(t, 20+h) = p(t, 20)e^{-\widehat{f}(t,\infty)h-(\widehat{f}(t,20)-\widehat{f}(t,\infty))\frac{1-e^{-ah}}{a}}.$$

so

$$\widehat{f}(t, 20+h) = \widehat{f}(t, \infty) + e^{-ah}(\widehat{f}(t, 20) - \widehat{f}(t, \infty)).$$

This extrapolation function is consistent with a Vasicek interest rate model (after maturity 20) under the additional assumption that there is no volatility in interest rates, i.e. that rates for any maturity in the future can be perfectly predicted today[10]. The other parameters such as the convergence parameter ($a = 1/10$), the maturity where extrapolation starts ($LLP = 20$) and the number of monthly historical rates on which the UFR is based ($H = 120$) have also been fixed by the Commission.

In the next section we will propose alternative methods in which the UFR is estimated using market data without the implicit a priori assumption by EIOPA and the Commission UFR that the uncertainty associated with the value of cashflows that will be paid very far into the future is much smaller than the uncertainty for cashflows that must be paid sooner.

[10]In terms of the original Vasicek model, the volatility parameter σ of the short rate has been chosen to be equal to zero. This makes all interest rates deterministic in this one-factor model.

3. Unconstrained Alternatives

3.1. A Smoother Discount Curve

The Smith-Wilson procedure defines a method to obtain an extrapolated discount curve based on given market data. The asymptotic value of the forward rate is given a priori so it is an input parameter for the methodology. Smoothness, as measured by the functional $\mathcal{L}$, is used as a criterion for the interpolation between maturities for which data are available. But the requirement of a fixed and given ultimate forward rate means that the discount curve may be less smooth around the last maturity for which liquid market data are available. It would be more consistent to take the asymptotic forward rate as a free parameter, which is *chosen* in such a way that we get the smoothest possible discount curve for all relevant maturities. Mathematically this would mean that we propose to optimize the functional $\mathcal{L}$ for a given value of α over all possible functions g in a certain class but also over all possible values of f_∞. This will not only give a more natural continuation of the discount curve, but also provides an objective estimation method for the ultimate forward rate in terms of market quotes for bond and swap data.

Problem 1. (Smoothest discount curve for converging forward curve)
Find the minimizer for f_∞ in

$$\min_{f_\infty} \min_{g \in \mathcal{H}(f_\infty)} \mathcal{L}[g] \tag{7}$$

where $\mathcal{H}(f_\infty)$ is the set of all interpolating functions that fit the market prices perfectly for the chosen value of f_∞ i.e. (4) must hold. □

It turns out that the asymptotic forward rate can be characterized explicitly as the solution of a matrix equation.

Theorem 3.1. *For an $n_\mathcal{I} \times n_\mathcal{J}$ cashflow matrix $\mathbf{C}$ and price vector $\mathbf{m} \in \mathbb{R}^{n_\mathcal{I}}$ the optimized asymptotic forward rate $f = f_\infty$ of Problem 1 solves*

$$(\mathbf{m} - \mathbf{C}\mathbf{D}^f\mathbf{e})^T(\mathbf{C}\mathbf{D}^f\mathbf{W}\mathbf{D}^f\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{D}^f\mathbf{U}\left(\mathbf{e} + \mathbf{W}\mathbf{D}^f\mathbf{C}^T(\mathbf{C}\mathbf{D}^f\mathbf{W}\mathbf{D}^f\mathbf{C}^T)^{-1}(\mathbf{m} - \mathbf{C}\mathbf{D}^f\mathbf{e})\right) = 0$$

where

$$\mathbf{W}_{ij} = W(u_i, u_j), \qquad \mathbf{D}^f_{ij} = e^{-f\,u_j}\mathbf{1}_{i=j}, \qquad \mathbf{U}_{ij} = u_j\mathbf{1}_{i=j}$$

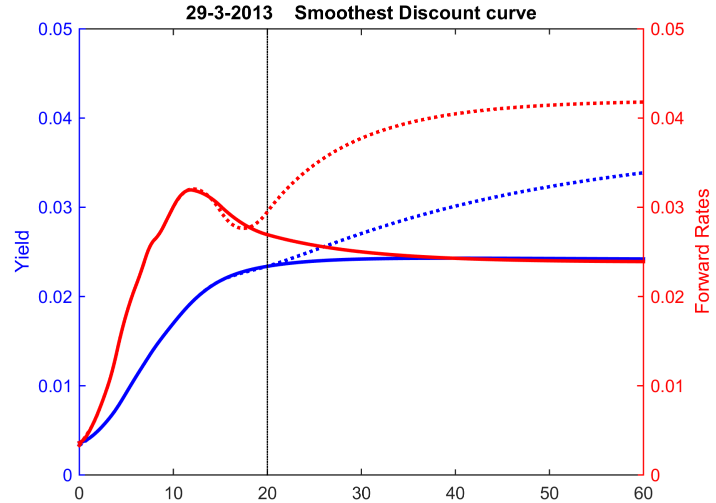

Figure 3: Inter- and extrapolation of Dutch Central Bank data for 29-3-2013 up to maturity 20. Blue/Black = Yields, Red/Grey = Forwards.

and with $\mathbf{e}$ *a vector full of ones in* $\mathbb{R}^{n_{\mathcal{J}}}$. *If the cashflow matrix* $\mathbf{C}$ *is invertible this simplifies to*

$$\sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{J}}\left(u_j\pi_j e^{fu_j}\right)\mathbf{W}_{jk}^{-1}\left(\pi_k e^{fu_k}-1\right) \quad = \quad 0$$

with $\pi = \mathbf{C}^{-1}\mathbf{m}$.

This equation can be solved rather easily using numerical algorithms since only the matrix $\mathbf{D}^f$ contains the unknown parameter f_∞. Calculating the optimized asymptotic forward rate therefore typically takes only a fraction of a second. Once the value has been determined, the usual Smith-Wilson procedure can be applied.

In Figure 3 we show the results of the inter- and extrapolation procedure under assumed convergence of forward rates to a value of 4.20% (dashed lines) and to the value which is consistent with the Smith-Wilson smoothness criterion beyond 20 years (solid). Both yields and forwards are shown. The unnatural kink at the last liquid point disappears and both the yield and forward values converge to 2.39% now that they are no longer constrained.

3.2. Smoother Forward or Yield Curves

The Smith-Wilson interpolation method is a transparent method that is easy to implement because its fitting procedure can be directly applied to bond and swap data. But it has important disadvantages. The discount curve (and hence certain zero coupon bond prices) can become negative. We also note that the Smith-Wilson method with an UFR of 4.2% results in a kink around the last liquid point in the forward and yield curves which suggests that it is more natural to apply a smoothness criterion to those curves instead of the discount curve.

To overcome this problem, we now propose two alternative extrapolation methods. Where the Smith-Wilson approach takes the the first and second order derivatives of the (transformed) discount curve as a criterion for smoothness, we will now minimize the weighted integrals over first and second order derivatives of the forward or yield curves. We still do this under the conditions that forward rates converge and that the market prices of a set of given fixed income instruments are fitted perfectly. This means we require that

$$\sum_{j=1}^{J} c_{ij}\, e^{-\int_0^{u_j} g(s)\, ds} \quad = \quad m_i\,, \qquad \text{for } i = 1, \ldots, I \tag{8}$$

if we solve the problem for the forward curve g and

$$\sum_{j=1}^{J} c_{ij}\, e^{-u_j g^y(u_j)} \quad = \quad m_i\,, \qquad \text{for } i = 1, \ldots, I \tag{9}$$

if we solve it for the yield curve g^y. Since

$$p(0,t) = \exp\left(-\int_0^t f(0,s)ds\right) = \exp(-ty(0,t))$$

this will also guarantee that p, and hence all coupon bond prices, will always be strictly positive. If we apply the criterion to the forward curve, it will also lead to increased smoothness of the discount curve. If we apply the criterion to the yield curve the smoothness of the discount curve does not change.

We formalize the new optimization problem as follows.

Problem 2. (Smoothest forward or yield curve which is constrained to converge)
Find

$$\min_{g \in \mathcal{H}^f} \mathcal{L}[g], \qquad \min_{g \in \mathcal{H}^y} \mathcal{L}[g] \tag{10}$$

where $\mathcal{H}^f$ consists of all functions g satisfying (8) and $\mathcal{H}^y$ consists of all functions g satisfying (9) with $g(0) = a$ for a given value $a > 0$ in both cases.

Note that $g(0) = a$ for functions g in $\mathcal{H}^f$ or $\mathcal{H}^y$ so the initial short rate r_0 equals a given constant a. This implies that we assume that the short rate is given. If this is not the case, we may determine its value by including it in the optimization procedure. We will show later in this section how to do this.

This problem can be solved explicitly in terms of the earlier defined functions W and a different class of interpolating and extrapolating functions:

$$\overline{W}(t,u) = 1 - e^{-\alpha t}\,\frac{\cosh(\alpha u) - 1}{\frac{1}{2}\alpha^2 u^2} + \mathbf{1}_{t \le u}\left(\frac{\cosh(\alpha(u-t)) - 1 - \frac{1}{2}\alpha^2 (u-t)^2}{\frac{1}{2}\alpha^2 u^2}\right)$$

These functions can be written as affine combinations of integrals of Smith-Wilson functions and are therefore smoother than the Smith-Wilson functions themselves. For every $u > 0$ we have $\overline{W}(0,u) = 0$ and they all converge to $\lim_{t\to\infty} \overline{W}(t,u) = 1$. They become linear for very small positive values of t in the sense that $\partial_1^2 \overline{W}(0,u) = 0$.

Theorem 3.2. *The solutions of Problem 2 must take the form*

$$\begin{aligned} g^f(t) &= g(0) + \sum_{i\in\mathcal{I}} \zeta_i^f \sum_{j\in\mathcal{J}} \pi_j^f c_{ij} u_j^2\, \overline{W}(t,u_j)\,, \\ g^y(t) &= g(0) + \sum_{i\in\mathcal{I}} \zeta_i^y \sum_{j\in\mathcal{J}} \pi_j^y c_{ij} u_j W(t,u_j) \end{aligned} \tag{11}$$

with the functions $\overline{W}$ as defined above. for certain weights the $(\zeta_i^f)_{i\in\mathcal{I}}$, $(\pi_j^f)_{j\in\mathcal{J}}$, $(\zeta_i^y)_{i\in\mathcal{I}}$ and $(\pi_j^y)_{j\in\mathcal{J}}$.

Figures 4 and 5 show the results for the same data as used in the previous figure. Notice that the forward curve flattens compared to the results of the procedure in the previous section since we now take the smoothness of forward rates as our optimization criterion.

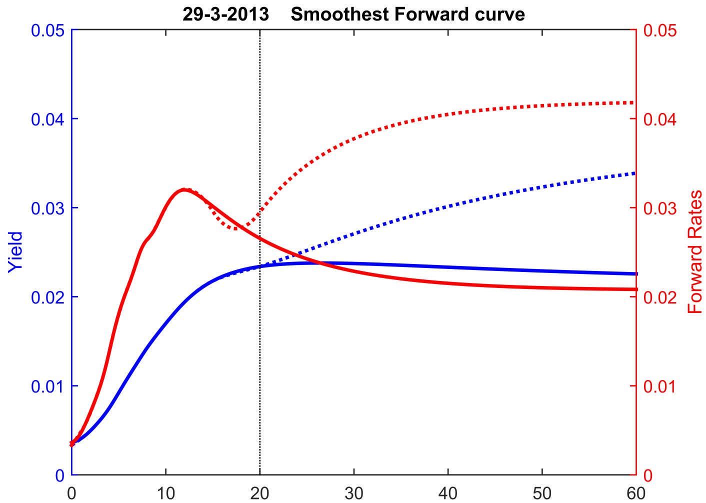

Figure 4: Inter- and extrapolation of Dutch Central Bank data for 29-3-2013 up to maturity 20. Blue/Black = Yields, Red/Grey = Forwards. Shown if $g^f(t)$ i.e. criterion based on smoothness of forward curve.

From this construction we immediately get expressions for the ultimate forward rate. Since $\lim_{t\to\infty} \overline{W}(t,u) = 1$ and $\lim_{t\to\infty} W(t,u) = \alpha u$ for all $u > 0$, this follows from (11).

Moreover, we can derive rather simple and intuitive expressions for these two estimated ultimate forward rates in terms of a linear combination of yields at different maturities. Let

$$y_k = y(u_k) = -\ln p(0, u_k)/u_k$$

denote the yield for maturity u_k when $1 \leq k \leq n$. We take $u_0 = 0$ so we can write $y_0 = f(0,0) = y(u_0)$ for the short rate.

Corollary 3.1. *The optimized asymptotic forward rate equals a linear combination of the yields:*

$$f_\infty = \sum_{k=0}^{n} v_k y_k.$$

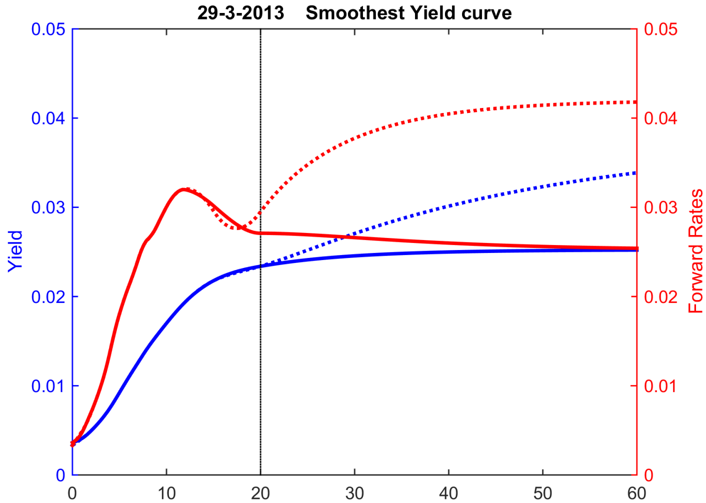

Figure 5: Inter- and extrapolation of Dutch Central Bank data for 29-3-2013 up to maturity 20. Blue/Black = Yields, Red/Grey = Forwards. Shown if $g^y(t)$ i.e. criterion based on smoothness of yield curve.

The weights $\{v_k,\ k = 0..n\}$ *are* $v_k = \sum_{j=1}^{n} [G^{-1}]_{jk}$ *with* $v_0 = 1 - \sum_{k=1}^{n} v_k$ *where the matrices* $\mathbf{G}^f$ *and* $\mathbf{G}^y$ *for the forward curve based method and the yield curve based method are*

$$G^f_{kj} = \frac{1}{u_k} \int_0^{u_k} \overline{W}(s, u_j) ds, \qquad G^y_{kj} = \frac{1}{\alpha u_j} W(u_k, u_j) ds.$$

respectively.

As expected, we see that if all forward rates (and thus all yields) have the same value, f_∞ must also equal that value. Moreover, when the cashflow matrix $\mathbf{C}_{ij}$ is invertible and the short rate is known we can observe the yields directly since $\pi = \mathbf{C}^{-1}\mathbf{m}$ and $\mathbf{y}_k = -(\ln \pi_k)/u_k$. This means that applying the whole optimization procedure to find ultimate forward rates is reduced to taking a simple weighted sum of directly observable yields.

When α goes to infinity one can easily show that the interpolating functions $\overline{W}(t,u)$ will converge to $\frac{t}{u}(2-\frac{t}{u})$ for $t \leq u$ and to 1 for $t > u$. The interpolating forward function $g^f(t)$ is thus constant after the last maturity point u_n and since g is continuous this means that the ultimate forward rate equals the forward rate for the last maturity: $f_\infty = g(\infty) = g(u_n)$ which seems a natural choice[11].

If the short rate is not given a priori it can be chosen as a free parameter in the optimization problems. As indicated earlier, an explicit formula for this value of the short rate which gives the smoothest curve for small maturities can be derived.

Corollary 3.2. *If $n_\mathcal{I} = n_\mathcal{J} = n$, $(c_{ij})_{i\in\mathcal{I},j\in\mathcal{J}}$ is an invertible matrix and the initial short rate is not known then the previous formula still holds if we substitute for the unknown short rate the optimized value*

$$y_0 = \frac{\displaystyle\sum_{j=1}^{n} \frac{1}{u_j} \sum_{k=1}^{n} G^{-1}_{jk} \frac{y(u_j)+y(u_k)}{2}}{\displaystyle\sum_{j=1}^{n} \frac{1}{u_j} \sum_{k=1}^{n} G^{-1}_{jk}}$$

where $\mathbf{G}$ *equals* $\mathbf{G}^f$ *or* $\mathbf{G}^y$ *as defined above.*

We now have four ways to inter- and extrapolate the risk-free term structure: the Smith-Wilson method to construct the discount curve with an UFR of 4.2%, the Smith-Wilson method for the discount curve with a free UFR parameter, a method based on the Smith-Wilson optimization criterion which is applied to the forward curve instead of the discount curve and a method where that criterion is applied to the yield curve. In the next section we will investigate which asymptotic values of forward rates are found for the last three methods and how they compare to EIOPA's value of 4.2%.

[11] Some institutions, such as the Dutch Central Bank, used this extrapolation before the Smith-Wilson method was introduced.

4. Results: Asymptotic Forward Estimates

In this section we present the results of an empirical study of extrapolation methods for the term structure. We consider the extrapolation method of Smith and Wilson that forces forward rates to converge to 4.2%, as advocated by EIOPA, but also an alternative formulation in which the asymptotic value is not restricted beforehand. Both methods require that we specify how quickly convergence to the limit takes place (i.e. we need a *speed of convergence*) and at which maturity we want convergence to start (i.e. we need a *last liquid point* or *first smoothing point*). We will specify the values that have been chosen for all cases discussed below.

Input data consisted of historical EURIBOR swap rates on all trading days in the last ten years, which were obtained from Datastream. All rates were middle rates and quotes were given with an accuracy that varied from 0.50 basis point (in the beginning of the dataset) up to 0.01 basis point (for the most recent data). Data for swaps with maturities $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 20, 30, 40$ and 50 years were downloaded and we always used all swap rates up to and including maturity 20 but sometimes included maturity 30 as well and sometimes included all rates, as will be indicated below.

An example of inter- and extrapolated yield and forward curves is shown in Figure 6 which shows the results for the market data of December 19, 2011. We used all swap data up to and including maturity 20 or 50 years and took $\alpha = \frac{1}{10}$ or $\alpha = \frac{1}{2}$ so four different graphs are shown. In all of them, the solid lines represent the Smith-Wilson method in which the UFR is a free parameter, whereas the dashed lines correspond to Smith-Wilson extrapolation to the ultimate forward rate of 4.2%.

We see that the restriction that the forward rate must converge to that asymptotic value causes the forward curve to show a kink around maturity 20, i.e. at the last maturity date for which market data was included in the construction of the curves. When the speed of convergence α is larger, this kink becomes more pronounced. The solid lines in the three figures show that in all three methods proposed in this paper, where such a restriction is no longer imposed, the kink disappears. As a result we find asymptotic values for forward rates that are much lower than 4.2%. Moreover, the asymptotic values found seem to be reasonably consistent when we vary α or the last liquid point. If $\alpha = \frac{1}{10}$ we find 2.27% and 2.26% for last liquid points at 20 and 50 years respectively, while for $\alpha = \frac{1}{2}$ they become 2.64% and 2.45%. Inclusion of data beyond maturity 20 thus barely changes our estimate of the asymptotic forward rate and we see that all four yield curves (the solid blue lines) hardly differ. The yield curves generated by the original Smith-Wilson approach (the dashed blue curves) show dramatic changes when long maturity data is included. Indeed, one sees that when swap data beyond maturity 20 is included, those forward curves (the dashed red lines) are now hardly distinguishable from the forward curves for the new method.

In Figure 6 we always use the criterion for the discount curve; plots for the method based on yield or forward curves give results that hardly differ. Indeed, we usually find asymptotic values that are relatively close to each other. Figure 7 can be used to examine this consistency between the three proposed methods, as well as the influence of the maturities that are included to construct the curves. It shows monthly updates for the estimated asymptotic forward rate generated by the methods which smooth the discount curve (top graph), the yield curve (middle graph) and the forward curve (bottom graph). We took the original speed of convergence $\alpha = \frac{1}{10}$ in all cases and estimates are displayed for the cases where instruments are interpolated up to a last liquid point of 20 years, 30 years or 50 years (purple line). The three graphs show that the first two methods provide estimates which are more stable over time than the last method. Indeed, the third method only gives good results when the last swap with maturity 50 is included; if this not the case then the estimates are too volatile to be of practical use.

In Figure 8, the same estimates of the asymptotic forward rate are plotted but now for a daily update frequency. Again we find that estimates are too volatile when only maturities up to 20 or 30 years are included so only results for maturity 50 are shown. This suggests that extrapolation of yields is possible beyond maturity 20 but that market-implied forward rates are not smooth enough to allow extrapolation if information from the far end of the curve is not used. This seems to be caused by the relatively low speed of convergence that EIOPA prescribes. If we use the value of $\alpha = \frac{1}{2}$ instead of $\alpha = \frac{1}{10}$ then we get a more stable pattern, as shown in Figure 9.

Even though estimates differ a bit depending on the speed of convergence and the last liquid point, the overall pattern is similar. In the years before the financial crisis of 2008 asymptotic rates were close to the assumed value of 4.2%, but they have dropped substantially since. After 2012 the first two methods show values which do not differ much when market data beyond maturity 20 is included or not and values are much lower than 4.2%. Between 2009 and 2012 there is a more marked effect but the difference between results up to and beyond maturity 30 are very small for both methods. We conclude that extrapolation methods which implement an explicit trade-off between first and second order derivatives of the yield curve give stable estimates when data up to maturity 20 are used. But inclusion of the data for maturity 30 seems preferable since this does not lead to more volatile estimates. Swap rates of maturities up to 30 or up to 50 give roughly the same estimates so it does not make much of a difference whether they are included or not.

We thus see that if extrapolation is applied to the discount curve or the yield curve then the choice of the last used maturity only mildly affects the asymptotic estimates that are found after 2011. This supports the use of such extrapolation methods to determine a proxy for long-term yields.

5. Policy Implications & Outlook

We have investigated inter- and extrapolation techniques that can be used to create discount curves from observed market data under different assumptions on asymptotic forward rates. We have shown that an existing approach, which requires the a priori specification of the asymptotic value, can be extended in such a way that this limit is implied by market data. Moreover, we show how the optimal smoothness criterion of the original problem can also be applied to the forward or yield curve instead of the discount curve. This leads to a market-consistent asymptotic forward rate which can be written as a weighted combination of yields at earlier maturities, with weights that can be calculated beforehand. This provides an intuitive characterization of asymptotic forward rates in terms of well understood market information. On days when reliable market prices are available for high maturities these can be easily incorporated and when there is reduced liquidity for the highest maturities the same method can be used with a restricted set of maturities.

An empirical study using swap data from the last ten years leads to the following conclusions:

- Asymptotic forward rates were close to 4.2% in the years before the financial crisis of 2008 and EIOPA's value was therefore not too far off. They are estimated to be close to 2% today. This means that a 120-month average of such rates would result in a decrease of 1.6 basis points per month at the moment.

- Inclusion of data beyond maturity 20 makes the estimates of asymptotic forward rates more stable. For methods which smooth the forward curve, inclusion of such later maturities is necessary since forward rates before maturity 20 do not seem to lead to good predictions of forward rates at later maturities.

- For methods which smooth the yield or discount curve there is hardly any difference in volatility when higher maturities are included or not. This suggests that one could use all available maturities up to 50 years if such methods were to be applied.

- Since 2012 the estimates of all methods are roughly the same: there seems to be hardly any influence from the choice of the convergence speed α or the last liquid point.

Summarizing, we conclude that there is no indication that data beyond maturity 20 is not reliable enough to include in the design of the discount curve. Policymakers can therefore not use a perceived lack of market information on long term rates to defend their current optimistic view on the value of long term liabilities. In fact, estimates of long term rates are quite stable over time. They just happen to have dramatically decreased since the financial crisis. The methods proposed by EIOPA and the Commission UFR suggest that uncertainty about the value of liabilities which lie very far in the future is much less than for liabilities which have to be paid soon. This is counterintuitive and hard to defend for financial institutions which see it as one of their main tasks to manage interest rate risks over long horizons.

As a final remark, we note that we have always assumed that market prices must be fitted exactly. If one allows some mispricing of market instruments in return for smoother curves, a different optimization problem must be solved. One could keep the same optimization function $\mathcal{L}_\alpha$, but one would have to impose, for example, that the weighted sum of pricing errors does not exceed a certain threshold value. Taking that value equal to zero would give back our old solutions but by varying it we can implement a trade-off between smoothness and pricing accuracy. Such problems have been studied for the case where there are no constraints on the asymptotic behavior of forward rates and when forward rates are observed directly (see for example (Andersen and Piterbarg 2010)). Finding the solution under our additional requirements is an interesting topic for further research.

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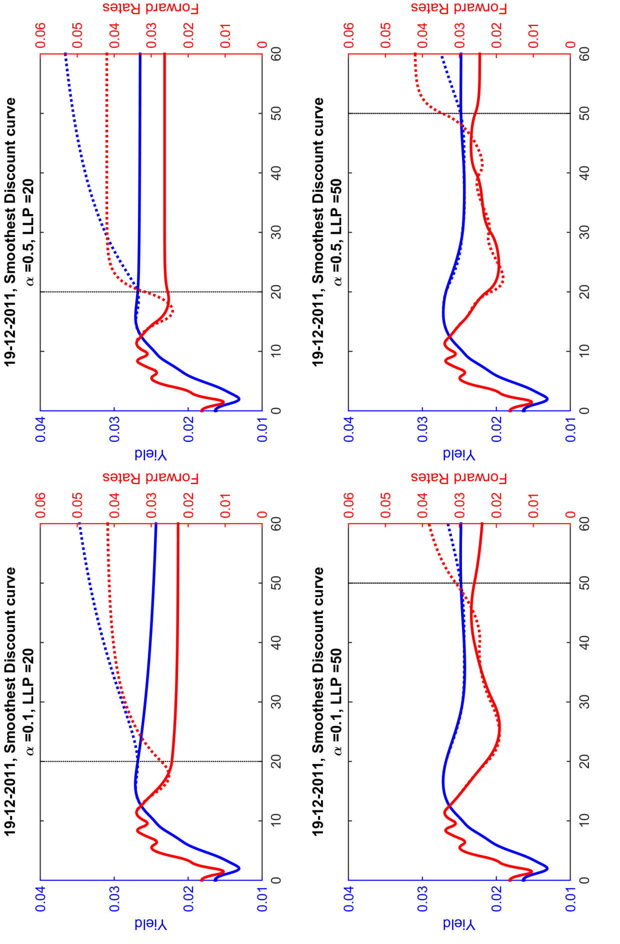

Figure 6: Inter- and extrapolation of Datastream swap data of December 19, 2011. Blue/Black lines show yield curves, red/grey lines forward curves. Dashed lines correspond to the EIOPA forward rate limit of 4.2% while solid curves are based on forward rates with a limit that is not specified a priori. We vary the speed of convergence parameter α (which is 0.10 on the left and 0.50 on the right) and the last liquid point (which is 20 in the top graphs and 50 for the bottom graphs).

Figure 7: Inter- and extrapolation based on Datastream swap data up to maturity $T = 20, 30, 50$ when applying our objective functional $\mathcal{L}$ to the discount curve (top figure), yield curve (middle figure) and forward curve (bottom curve).

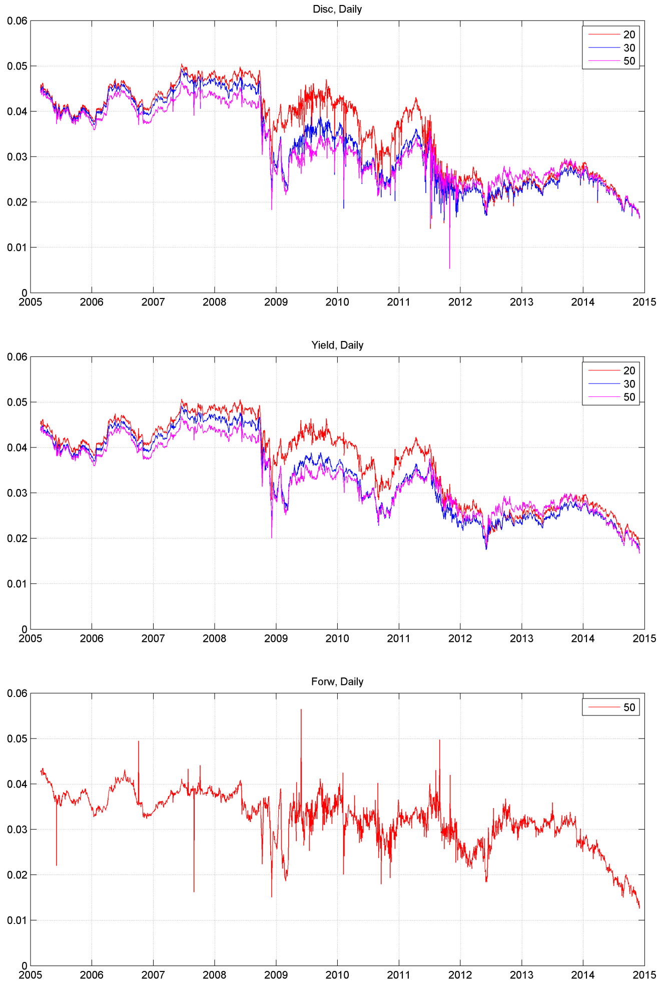

Figure 8: The same quantities are shown as in the previous figure but now on a daily basis. The forward curve method in the bottom graph gave very volatile results after 2009 when the last included maturity was taken to be 20 or 30, so these results have been omitted.

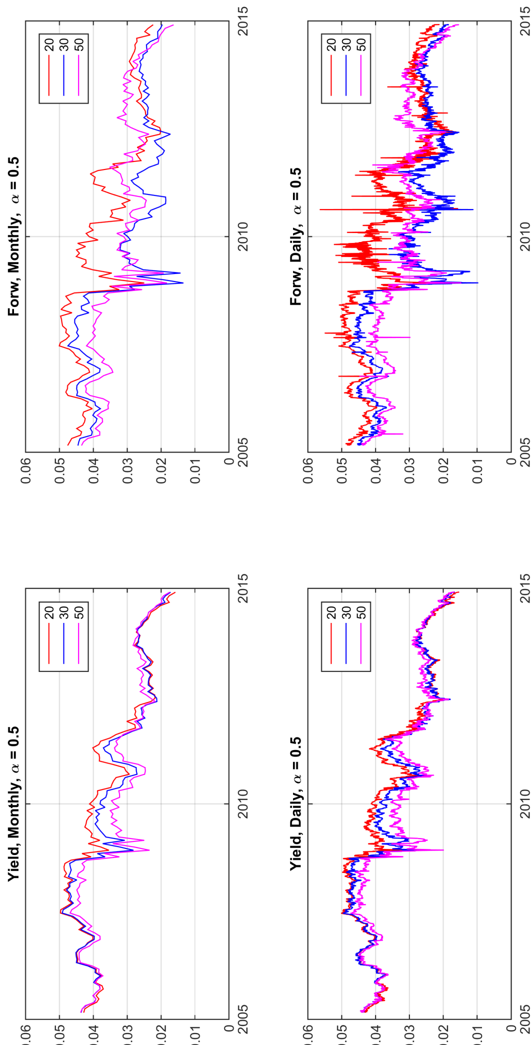

Figure 9: The same quantities are shown as in the previous figure but with different values for α.

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Term structures with converging forward rates

The risk-free term structure of interest rates is used by financial institutions to determine how much money needs to be invested today to receive a given amount of money on a later date. Michel Vellekoop and Jan de Kort (both UvA) have investigated inter- and extrapolation techniques that can be used to create discount curves from observed market data under different assumptions on asymptotic forward rates. They discuss the methods proposed by EIOPA and the Commission UFR and suggest some new alternatives.

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Netspar
P.O. Box 90153
5000 LE Tilburg
the Netherlands
Phone +31 13 466 2109
E-mail info@netspar.nl
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