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Zerotopia – bounded and unbounded pension adventures

Term structure modelling with and without negative short rates

Samuel Sender

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DESIGN PAPER 73

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Affiliation
Samuel Sender – Tilburg University

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ZEROTOPIA – BOUNDED AND UNBOUNDED PENSION ADVENTURES

Abstract
Pension funds often use Gaussian interest rate models – such as those used and validated by the Dutch central bank – which assign a high probability to rates falling below their current levels, deep into negative territory. However, since 2008, central banks have resorted mostly to quantitative easing instead of deep cuts in short rates to expand monetary policy. The ECB (Draghi, 2014) has even stated that short interest rates could not fall further (than -0.50%), thus suggesting that the Gaussian models used by pension funds lack realism.

If a *lower bound* exists, a *Gaussian* pension fund – one that uses a Gaussian model – will tend to buy bonds or derivatives in order to hedge the risk of negative rates, although this risk is small. The funds used to hedge the risk of negative rates could be best invested elsewhere.

Finally, monetary expansion close to the *lower bound*, realised through quantitative easing, can impact equity prices more than liability prices, thus benefitting pension funds overall. By contrast, higher interest rates would be a bigger risk if liabilities are overhedged. A *Gaussian* pension fund may fail to recognize the unusual risks near the lower bound.

Nederlandse samenvatting

Pensioenfondsen gebruiken meestal door DNB gesanctioneerde *gaussische* rentemodellen, waarin de kans op negatieve rente onder de huidige omstandigheden groot is. De centrale banken echter hebben *quantitative easing* geïmplementeerd in plaats van de rente verder omlaag te brengen toen deze in de buurt van nul kwam. En in de toekomst zal de korte rente niet lager zijn dan nu (dus rond de −0.5%) volgens de ECB (Draghi). De modellen van de pensioenfondsen zijn dus volgens de ECB niet realistisch.

Als er een ondergrens bij een rente van nul is, neigt een gaussisch pensioenfonds – een fonds dat een gaussisch model gebruikt – er naar om obligaties of derivaten te kopen om het risico van negatieve rentes af te dekken (zelfs wanneer dat risico klein is), zodat kapitaal anders ingezet kan worden.

Tot slot heeft, bij een ondergrens voor rente rond de nul, monetaire versoepeling mogelijk meer impact op de aandelenprijzen dan op die van de verplichtingen. Omgekeerd vormen hogere rentes en lagere obligatieprijzen een groter risico wanneer verplichtingen teveel gehedged worden. Deze atypische risico's ziet een *gaussisch* model niet.

1. Introduction

1.1 Empirical Motivation: stylised facts on interest rates

Pension funds often rely on Gaussian affine term structure models in their ALM exercises. In the Netherlands, the Dutch National Bank [DNB] has also validated the KNW (Koijen, Nijman and Werker, 2010) model. These models reproduce the stylised fact – a rule of thumb – that yearly changes in short interest rates have historically followed a normal distribution with a 1% standard deviation.

However, since 2009, short rates have nearly always hovered around zero. In addition, central banks have continued to ease the monetary stance with *non-conventional* tools, *i.e.*, quantitative easing [QE], instead of lowering their rates below zero. The U.S. Federal Reserve has unequivocally changed monetary instruments when interest rates approached the zero level, while the European Central Bank pushed rates to a slightly negative level (see B.2 on page 41 for a discussion), claiming that this is the effective lower bound. We use here the acronym ZLB, which stands for *Zero*[1] *Lower Bound*.

Gaussian models cannot accommodate both a historical standard deviation of approximately 1% and an interest rate volatility that drops to close to 0% for any extended period of time. These models assume that the short rate is the unique policy instrument and ignore the historical use of quantitative easing as a substitute near the zero bound. In technical terms, Gaussian models cannot truncate the distribution of short rates at zero.

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1 US data are consistent with a zero lower bound, while both European data and the ECB suggest -0.5% as a possible lower bound. We use a model calibrated with US data and denote zero as a simplification for the lower bound.

Figure 1: 3-Month Treasury Bills rates since 1934

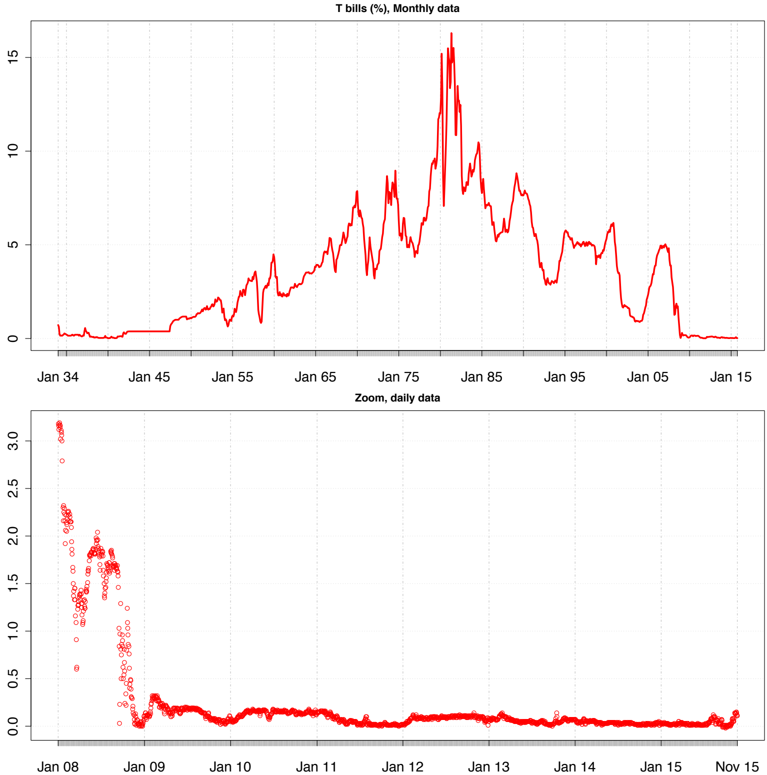

Figure 1 shows the secondary market yield for 3-month U.S. Treasury Bills. The upper panel consists of monthly data since 1934. In the two 'depression' periods 1937-1941 and 2009-2015, 3-month rates were virtually constantly at zero.
The lower panel zooms in on the recent 2008-2015 period, with daily data. The pre-2009 daily volatility is 'absorbed' as short rates approach zero, which is inconsistent with Gaussian models.
We display the 3-month rate because it reflects the short-term expectations of monetary policy. The overnight rates are driven by monetary policy, which is not a continuous-time equilibrium process as described by the model: at the very least, monetary policy does not change between FMOC meetings.

Then, if a lower bound exists, Gaussian models assume a too high probability to rates becoming negative when rates are already very low.

We further discuss in this introduction:

- Models that yield non-negative rates, and how well they capture empirical (historical) facts.
- For the near future, whether the recently observed ZLB-like policy is backed by theory, and whether we can reasonably trust the existence of a (Z)LB.
- Whether profound institutional changes could permit central banks to abstract from the ZLB in a more distant future.

1.2 Interest rate models consistent with stylised facts

Between the end of 2008 and 2015, the Federal Reserve set a target range of $[0\% - 0.25\%]$ for the refinancing short rate, with zero as an explicit lower bound. The bottom panel of Figure 1 shows the daily secondary market 3-month Treasury Bills, which volatility declined to virtually zero as short rates approached zero.

Empirical facts are supportive of interest models with a low bound on interest rates. Figure 2 shows that the volatility of the long and short rates can decouple: after 2009, while the volatility of short rates fell close to zero, that of the 10-year bond yield remained high.

Since Gaussian models are not compatible with the volatility of short term rates falling to zero as rates approach zero, alternative models must be used.

The historical alternatives are the CIR model of Cox et al. (1985) and the family of log-normal models (see for instance Rendleman and Bartter, 1980; Black et al., 1990; Black and Karasinski, 1991). Yet the volatility of long and short yields cannot decouple in CIR and log-normal models: in these models, the volatility of short and long rates fall to zero altogether as short rates fall to zero.

Reproducing such volatility discrepancy requires non-affine short rates dynamics, such as embedded in quadratic term structure and ZLB/shadow models. It has been shown that both ZLB and quadratic models permit a good historical fit (small pricing error for both short and long-term bonds, see references in the text).

Quadratic term structure models make the short rate a quadratic function of a Gaussian process. As desired, the modelled short rate is never negative (or not more than a chosen 'lower bound'), its volatility can shrink to zero, and the volatility of the long rate is non-linear in the short rate. But economic interpretation of this model is difficult: the short rate is not quadratic in inflation, so factors do not relate to the economy. In addition, since the factors in quadratic and Gaussian models differ, the sensitivity to the factors cannot be compared in these two models.

The ZLB model of Krippner (2012) takes the observed short rate $\underline{r} = [r]^+$ as the positive part[2] of a shadow equilibrium rate r, which is assumed to be Gaussian. The future ZLB rate can be thought of as an option on a Gaussian process; this option has zero intrinsic value, and its time value typically increases with the maturity of the option. Bond yields are functions of such options, and the volatility of the term structure changes in a non-linear way.

In addition, a precise comparison can be made with non-ZLB models currently used by pension funds. Away from the zero bound, the impact of the lower bound vanishes, and the ZLB models behaves as the reference shadow Gaussian model. Near the ZLB, the short rate progressively becomes less sensitive to the state of the economy.

2 We use $[x]^+ = \max(x, 0)$ to denote the positive part of a number.

Figure 2: 3-Month Treasury Bills and Bonds rates since 1934

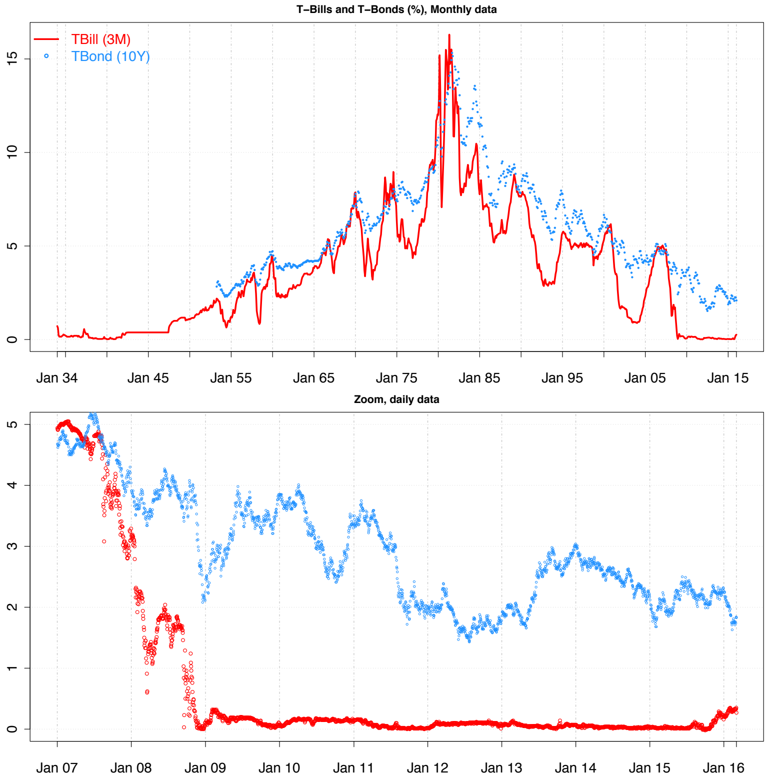

Figure 2 adds the dynamics of 10-year bond rates (the T-Bond, in blue) to Figure 1 on page 9.
At the monthly frequency, the short rates are more volatile than the long rates (upper pane).
The same applies with daily data up to 2009 (lower pane). From 2009, the volatility of the short rate vanishes, while that of the long rate remains high.
The volatility of long rates cannot be higher than that of short rates in CIR and log-normal models.
Data source: The U.S. Federal Reserve

Finally, ZLB models developed at central banks permit interpretation of state variables. This is not possible with quadratic models, with for instance a shadow rate that follows a Taylor-type rule (positive link with inflation) that is only attenuated by the ZLB.

A slightly more mathematical review of interest rate models can be found in Appendix B on page 39.

1.3 Is there an economically motivated lower bound on nominal interest rates?

The ZLB model can be motivated in economic terms if two conditions hold: (a) equilibrium short rates desired by central banks may become negative, but (b) central banks must maintain non-negative interest rates.

Standard theory dictates that (a) is true: the Taylor rule stipulates that for each one percentage point increase in inflation, the central bank should raise the nominal interest rate by at least one percentage point. Equilibrium nominal interest rates should then become negative when inflation falls.

The standard argument for (b) is the availability of currency, *i.e.*, the view that individuals would not put money in the bank if deposit rates are more negative than the cost of storing and securing money.[3] For details, see Goodhart (2013) and other references on shadow rates.

If a lower bound exists, it does not need to be exactly zero because money storage has a cost. ECB president Mario Draghi (Draghi, 2014, 2015) declared that the 'effective lower bound' on

3 Investors can accept negative rates since they would otherwise need to pay intermediary costs, plus the cost of storing and securing physical currency, plus potential insurance costs.

nominal rates 'had been reached', which indicates -0.50% as a possible floor.

It is often thought that institutional investors cannot pass negative rates on to individuals. This means that a negative short rate would lower institutional investors' profits.

To avoid such impact, individual accounts have in practice been exempted from negative rates:

- In 2014, the ECB lowered the deposits facility rates to -0.20% as from September 2014, then to -0.30% from December 2015, and to -0.40% from March 2016; at the same time, the refinancing rate was reduced from 0.05% to 0% in 2016.
- On February 6, 2015, the Danish central bank lowered its currency deposit rate to -0.75%, but simultaneously expanded the cap on current accounts with zero deposit rates. It also maintained the lending rate at 0.05%.
- The Swiss central bank set all base rates to around -0.75% in January 2015. However, like the Danish central bank, it exempts domestic deposits (up to a cap) from negative rates.

It is worth noting that both the larger negative rates in Denmark and Switzerland were designed as a temporarily defense of a quasi-pegged policy, to cool off an upward currency pressure that these central banks could not fight with asset purchases[4] due to their relatively small size: '[this policy move is] a stand against an acute threat to the Swiss economy and to counter the risk of a deflationary trend emanating from the massive overvaluation of the Swiss franc' (Zurbrügg, 2015, p.3).

4 The Swiss central bank discontinued its asset-purchases-based euro-peg policy on January 15, 2015. The Swiss franc thereupon rose by 20%. The non-conventional negative-interest-rates monetary policy was implemented to protect the currency. With negative inflation, this policy could continue.

We argue that deeply negative interest rates designed to force institutional investors to invest in non-domestic assets are only feasible for 'small' economic zones; ZLB models are justified for larger economic zones.

1.4 Can institutional changes remove the ZLB?

Since the main argument underlying the ZLB is the availability of paper money that does not bear interest, digitalization could in theory help remove the lower bound constraint. Yet the combination of digital currency and negative interest rate policies is uncharted territory; some possible implementations issues are:

- If digitalization is perceived as a means of imposing negative interest, consumers may shift money abroad. Capital flight occurred in Ecuador, when it started to digitalise its currency, due to fears of de-dollarization.
- Preventing capital transfers implies restricting international transactions and banning electronic currencies such as bit-coin and other neo-currencies.
- Even if achievable, it is still possible that negative interest in a digitalised currency world generates speculation on the prices of financial and real estate assets. Loose monetary policy is now thought to have contributed to price bubbles and subsequent financial instability since the end of the 1990s.

For the foreseeable future, quantitative easing seems to be the new standard, and a digitalised negative interest policy remains uncharted territory.

2. Set-up and assumptions

2.1 Definitions and notations

In all illustrations/quantifications, the interest rates generating process is the ZLB/shadow model used and calibrated in Krippner (2012)[K12]. It involves a *two-factor Vasicek* model for equilibrium interest rates plus a *ZLB* constraint, which performs a non-linear transformation of the shadow Vasicek model.

We use the two-factor Vasicek model and calibration of Krippner (2012), but interpret the shadow term structure model as having explicit factors, with π as the *inflation*[5] expectation and r^R as the *shadow* real rate.

r denotes the shadow-short rate, a two-factor Vasicek process, which satisfies $r = r^R + \pi$. By contrast $\underline{r} = [r]^+$ denotes the observed short rate, bounded from below.

We use the following definitions: model *parameters* are *constants* that in theory would never need to be updated; real rates are dynamic and reflect the changing state of the economy, so they are *updated* to fit the recent (ideally, current) yield curves.

For simplicity, all examples are with zero coupon bonds (so their par value is one) and liability.

5 In Figure 2, the long and short rates seem to evolve almost parallel beyond the business cycle, which shows a very persistent factor, usually modelled as inflation, and modelled as integrated. It was traditionally modelled as mean-reverting, but developments of the level of interest rates have always surprised, and an integrated or quasi-integrated factor appears a more robust specification than an estimated long-term mean. In recent calibration (see for instance Christensen and Rudebusch, 2014, or Koijen et al., 2010), inflation, the level factor is either integrated or quasi-integrated.

2.2 Focus of the study

We here assume that equilibrium interest rates follow a *Gaussian* shadow model, and that a ZLB constraint exists – *market* prices/yields are thus *ZLB* prices/yields – but that a *Gaussian* pension fund fails to incorporate this constraint in the equilibrium/shadow model[6] that it has otherwise correctly identified. We analyse the impact of this unique source of mis-specification on the interest rate hedging policy, whether it is derivative or bond-based; we give qualitative comments on how interest-rate risk may feed into the equity risk budget; finally we discuss the consequences of further quantitative monetary easing.

2.3 Rationale for main results

When rates are close to the ZLB, Gaussian models forecast a probability of negative rates that is simply too high. Overestimation of the probability of negative rates may result in overestimation of derivative prices: buying insurance against a risk that does not exist is costly. Overestimation of liability risk also tends to lead to bond over-hedging, in turn exposing the pension fund to the risk of a rise in interest rates. Hedges based on duration or on modelled bond sensitivity – the bond sensitivity is the derivative of the model price – are also wrong with the same qualitative implications.

For an investor with a limited total risk budget, overestimation of interest rate risk may reduce the equity risk budget.

Last, failing to take the ZLB into account may also lead to misinterpretation of the monetary policy: the instantaneous real

6 Pension funds did not incorporate a ZLB in their Gaussian models before 2008 because the probability of having negative rates was of second order: short rates were several standard deviations above zero, so the ZLB option was far out of the money and worth little. As a rule of thumb, short rates at 3% were above the ZLB by 3 times the annual interest rate volatility.

rate (nominal rate minus inflation) rises as inflation falls as a sole consequence of the lower bound.

Although our study focuses on pension funds and uses the vocabulary typical of this sector, the results apply to other institutional investors with long-term liabilities as well.

2.4 General lessons from the specific ZLB/Shadow model used in illustrations

The qualitative conclusions of this study are likely to hold with any Gaussian model. Gaussian models assume a state-independent sensitivity of bond yields to inflation,[7] which the ZLB diminishes in a non-linear way, so using any Gaussian bond sensitivity should result in wrong hedging demands.

The K12 model is low dimensional, thus the in-sample fit is lower than that of higher dimensional models. Adding stochastic volatility, other non-linearities in addition to the lower bound, or a third 'curvature' factor, would give more flexibility to fit historical data. Alternatively, infinite-dimensional models such as the Hull-White model would also be able to perfectly fit any observed term structure (thanks to a continuously re-calibrated term structure of forward rates). Note that with any model, valuation errors still could arise for illiquid, long-dated liabilities where market rates are not fully observed.

One motivation to base this study on a two-factor model is that pension funds' risk management models are often based on two-factors.

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7 For instance, the sensitivity of the bond yield to inflation does not depend on the shape of the forward yield curve in a Hull-White model. Inflation is replaced by a persistent factor in models with implicit variables. Similar sensitivities are then expected to arise as in the K12 model.

Although higher dimensional models could better fit historical data in-sample, they do not necessarily make better forecasts. In addition, the main ingredient of an interest rate risk hedging program is the future sensitivity of liability prices to inflation and real rates. Since these sensitivities are unobserved, they are subject to model risk, and over-fitting usually yields poorer prediction of sensitivities. With a limited amount of data near the ZLB, over-fitting is a risk that cannot be well controlled.

Another motivation is that low-dimensional models with explicit factors permit clear interpretations.

In the K12 model, long yields tend to fall because of the convexity effect, *i.e.*, the volatility of inflation. As the volatility of short rates diminishes when these approach the ZLB, so does convexity. Recalibration of a low-dimensional Gaussian model near the lower bound discards historical information and 'assumes' lower inflation volatility, or that nominal short rates barely react to inflation. Such 'assumptions' are not always correct, as the historical inflation volatility and the link with interest rates shows.

Recalibrating higher-dimensional parameter models would generate similar distortions, but with less straightforward interpretation. For instance, in the Hull-White model, forward rates are a free parameter typically recalibrated to match observed yield curves, but which changes cannot be interpreted.

3. ZLB, distribution of future interest rates, and derivatives' prices

3.1 Distribution of state variables in one year

Figure 3 shows $\underline{r}_1$, the observed ZLB rates in one year (at $t = 1$), which is the positive part of the future equilibrium rates r_1. The distribution of $\underline{r}_1$ is the left-truncated distribution of r_1, which is the sum of the expected inflation π_1 and the equilibrium real rate r_1^R, both Gaussian processes. These processes are positively correlated, meaning that deflation is likely to be associated with recession, and inflation with economic growth.

In these two graphs, the real rate is 1.5% while inflation is -1.5%. Here, both the shadow and the ZLB rates are zero at time zero: ($\underline{r}_0 = r_0 = 0\%$), but the real rate is greater than desired by the central bank.

These graphs illustrate that Gaussian models assign an overestimated probability to short rates becoming negative: they predict a normal rather than a truncated distribution for future short rates.

3.2 Impact on derivatives hedging policy

Derivative instruments are often used in ALM to circumvent practical limitations such as the lack of ability to trade complex strategies, the cash/borrowing constraints,[8] or to guarantee of a future minimum income from the investment of future premiums. Receiver swaptions or floors (which pay the difference between a fixed and a market

8 Hedging demands can be close to 100%, in particular when the funding ratio is low. As we will see, they can even exceed 100% if the pension fund hedges its long-term liabilities with shorter-maturity bonds when interest rates are low. With borrowing constraints and without derivatives, liability hedging could not only be difficult to implement but also prevent the pension fund from investing in risky assets.

Figure 3: Distribution of state variables at $t = 1Y$, starting with $\pi_0 = -1.5\%, r_0^R = 1.5\%$

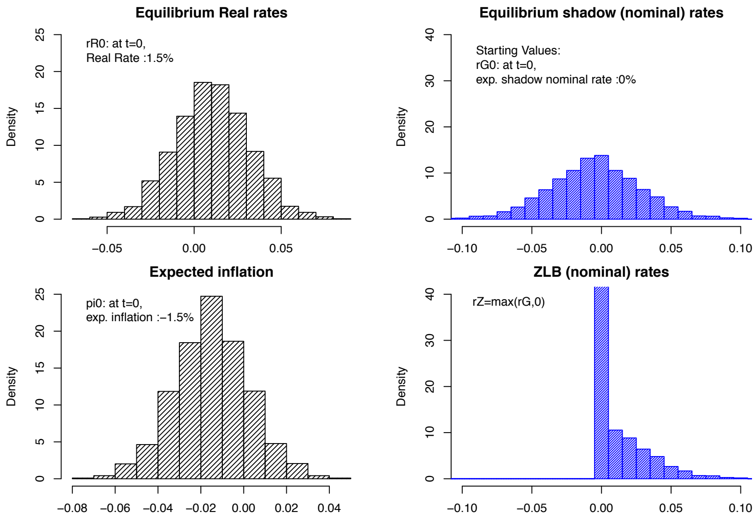


Figure 3 shows the simulated distribution of the two state factors underlying the shadow rate ($r_t = \pi_t + r^R$) and the observed ZLB short rate: $\underline{r_t} = [r_t]^+$, a truncation of the unobserved shadow rate.

rate) are protection against the risk of low yields. These derivatives, often traded over-the-counter, may be overpriced with Gaussian models that attribute a positive probability to rates being negative and to bond prices[9] exceeding par value if these risks are ruled out by a ZLB. Pension funds would be better off not to purchase unnecessary protection.

A call option on a 15-year zero-coupon bond with strike 1 (par value) is typically worth a couple of percentage points (up to 5%) in a Gaussian model with reference starting conditions. One cannot exclude the possibility that a pension fund that wishes to hedge 80% of its liability risk with swaptions will lose a couple of percentage points of funding ratio each year, as long as the short rates remain near the ZLB.

9 A positive probability of negative short rates implies a positive probability of bond prices rising above par. Gaussian models assume that bond prices follow a log-normal distribution, whereas they are bounded from above, at par value.

4. The formation of the modelled yield curve

4.1 Shadow rates overestimate market liability prices

For any shadow model, the ZLB/market bond yield is always at least the shadow bond yield. After all, the ZLB bond yield is the expectation of the ZLB short rates minus a convexity adjustment; ZLB rates are at least equal to the shadow rate, and the convexity adjustment is lower with a ZLB.[10]

Figure 4 shows the importance of these two factors with the Krippner (2012) model with long calibration.[11] Qualitatively, the discrepancy between the ZLB and shadow price grows as inflation and the shadow short rate fall.

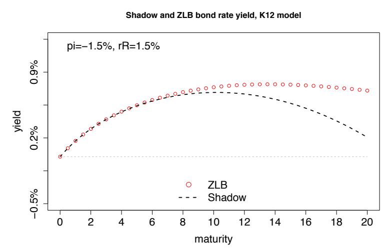
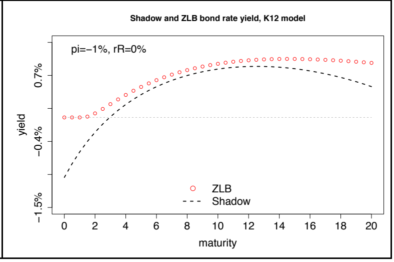
In the left panel, the shadow and ZLB rates are both zero at time zero ($r_0 = \underline{r}_0 = 0$), yet long shadow yields are lower than ZLB yields, entirely because of the convexity adjustment.

In the right panel, shadow short rates are negative at time zero ($\pi_0 = \underline{r}_0 - 1\%, r_0^R = 0$), so the ZLB also impacts the short end of the yield curve. Shadow short yields are below zero, and ZLB short yields are equal to zero.

10 The bond price is the expectation of the exponent of minus the integral of future short rates, and the variance of future short rates pushes bond prices up. In Gaussian models, the distribution of future short rates is normal, so the convexity adjustment is only linked to the variance of short rates.

11 Note that the shadow price is not the price modelled by the pension fund, unless the equilibrium real rate is given by a macro-model (or by the ECB) and inflation is observed.

Figure 4 : Shadow and ZLB yield curves

with $\pi_0 = -1.5\%, r_0^R = 1.5\%$	with $\pi_0 = -1\%, r_0^R = 0\%$
	

In the two panels, the black dotted curves represent the modelled shadow yields (K12 model), where inflation is persistent (the level factor) and the equilibrium real rate mean-revert quicker (the slope factor).
Long shadow yields decrease because of the convexity adjustment: bond prices, the expectation of the exponential of minus average future short-term rates, are more impacted by falls rather than by rises in the future short rate
The red points show the ZLB yield curve. ZLB short rates cannot fall below zero, which limits the convexity adjustment. As inflation falls, forward rates fall closer to the ZLB, and short rates can barely fall further. The convexity adjustment then disappears altogether.

4.2 Updating and recalibration

If the interest rate process follows a ZLB/shadow model, then the interest rate dynamics and the shape of the yield curve are modified near the ZLB.

Recalibration makes it possible to better capture the local dynamics, but not the non-linearities.

The following subsections illustrate the valuation/dynamics trade-off between a 'global fit' (*i.e.*, long-term calibration) and a 'local fit' (*i.e.*, short-term calibration) of a Vasicek model.

As illustrated in Figure 4 on the opposite page, the convexity of the Gaussian yield curve — the extent to which the long-term yields slope downwards — is linked to the long-term volatility of interest rates. With the Vasicek model, the convexity is constant.

If the Gaussian pension fund focuses on the **long-term** properties of the model, it will use panel data — made up of the time series of all relevant bond yields — to calibrate the model. The pension fund will measure correctly both the high historical sensitivity of short rates to inflation and the historical inflation volatility. It will correctly model the average historical convexity, but it will overestimate this near the ZLB. Modelled long yields will have a more negative slope than market yields because the ZLB truncates the distribution of future short rates and flattens the long-term yield curve.

With a low-dimensional model, the Gaussian pension fund updates state variables — here, inflation expectations and the real rate — to fit the yield curve as well as possible, but it has no control over the shape of long-term yields.

Typically, such Gaussian pension funds would match the 10-year bond yield, while underestimating very long-term liabilities, as illustrated in Figure 5.

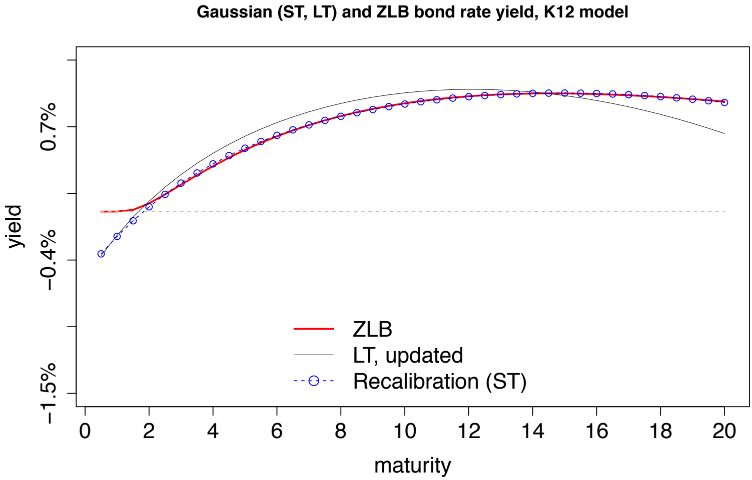
Underestimation of long-term bond yields can go unnoticed if they are illiquid. Such underestimation would lead to overestimation of the funding ratio and to a pension fund that feels poorer than it really is. The funding ratio would also be reported as riskier than it really is.

Recalibration gives more weight to recent history, but as it discards historical information, it becomes irrelevant for any long-term application. Just as long-term calibration of the Vasicek model implies better fits when rates are significantly above zero, short-term calibration generally allows a better fit of the yield curve when rates are close to the ZLB.

Near the ZLB, recalibration reduces convexity by assuming low inflation volatility or a disconnect between interest rates and inflation. This can sometimes allow better approximation of the local, very short-term dynamics.

Such approximated dynamics are only valid locally, for a particular level of inflation. Gaussian models assume a linear constant relationship between short rates and inflation, while this relationship is state dependent, high in normal times (as shown by historical data) and low near the ZLB (which disconnects short rates from state variables). So, a recalibrated Gaussian model would not be able to capture the long-term dynamics and distribution of interest rates.

Figure 5: ZLB, LT- and ST-calibration



The red curve shows the ZLB curve with initial conditions: $\pi_0 = -1\%, r_0^R = 0\%$, as in the right-hand panel of Figure 4 on page 24.
The black curve shows the Gaussian yield curve with long-term calibration. Value of factors minimizes the sum of the price discrepancy of 0 to 20-year bonds, which implies a focus on long yields (the higher the duration, the more price is sensitive to yield). With long-term calibration, the Gaussian yield curve remains convex, and very long-term yields, even though included in the objective function, are poorly fitted. The 15-year yield is underestimated by 12 basis points, the 17-year yield by 20bp, and the 20-year yield by 35bp.
The blue points and curve show a recalibrated Gaussian yield curve. The recalibrated model implicitly assumes a low sensitivity of the short rate to inflation, which diminishes convexity makes it possible to perfectly fit yields (except for maturities of less than two years which have little weight in the optimization program).

Recalibration of high dimensional models — such as the Hull-White model — or of models with implicit state variables is expected to lead to similar but less visible *implicit* changes:[12] all Gaussian models assume linear dynamics instead of non-linear.

Remark 4.1: We aim for — but this is never entirely feasible — interest rate models that fit the yield curve and the interest rate dynamics when rates are either high or low.[13] Model parameters would be calibrated with a large set of historical data and barely need updating; planning to recalibrate a model is the ex-ante recognition that it fails to capture the long-term dynamics of interest rates. In addition, with the low signal-to-noise ratio typical of financial markets, recalibrating a model using a short-time frame generates noise in parameter estimates and can lead to noise trading. Rather than planning to change models randomly, it makes sense to build a model that is valid in every possible situation.

12 In a Hull-White model, the shape of long-term forward rates could become steeper near the ZLB, but such changes can be difficult to notice since they are essentially non-parametric.

13 This is called the conditional distribution of interest rates in all states of the world.

5. Bond over-hedging near the ZLB

Liability hedging programs are usually based on duration or sensitivity analysis. The former is a model-independent rule of thumb, the latter is model-dependent. Both are prospective estimates of the unknown sensitivities of long yields to inflation and to the real rate. Any error in the calculation of sensitivities alters the effectiveness of the hedging program.

All illustrations in this section rely on the K12 model, assuming that a ZLB exists; the base scenario is that of the right-hand panel in Figure 4, with $\pi_0 = -1\%, r_0^R = 0\%$. Keep in mind that ZLB sensitivities are state-dependent and that they depend on the value of inflation and the equilibrium real rate.

We first analyse the sensitivity errors of Gaussian models, then their impact on liability hedging.

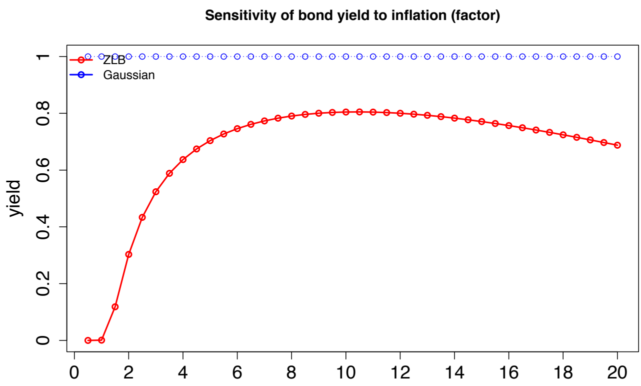
5.1 Sensitivity Analysis

In a Gaussian (shadow) model the sensitivity of bond yields to state variables is independent of their levels; the ZLB implies a non-linear, state-dependent attenuation of the sensitivity.

As inflation is assumed to follow a random walk,[14] the derivatives of bond yields of all maturities *with regard to* inflation are equal and constant across states in the shadow model; the ZLB attenuation is option-like, see Figure 6. Figure 7 shows bond sensitivities to the real rate and their attenuation.

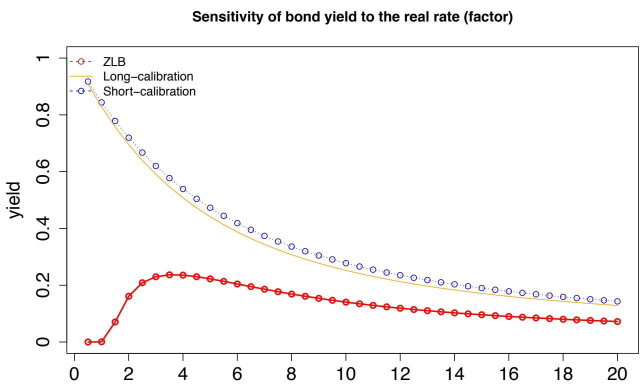
14 If inflation were modelled as a near-unit root process (highly auto-correlated), then the inflation sensitivity in the Gaussian model would decrease marginally with maturity but would still be constant across states and time.

Figure 6: Derivative of yield with regard to inflation



Gaussian yields have a constant sensitivity to inflation, the 'level' factor.

Figure 7: Derivative of yield with regard to real rate



Because of mean reversion, the derivative of the bond yield with respect to the (equilibrium) real rate is a decreasing function of maturity. In a Gaussian model, it is state-independent (blue points for ST and yellow line for LT in the K12 model). It is attenuated by the ZLB model in a non-linear way (red line with points): when equilibrium short rates are negative, the sensitivity to the real rate shrinks all the way to zero for short maturities and partly for others.

5.2 Over-hedging with Gaussian models

Duration hedging:
Pension funds sometimes rely on duration hedging, that is, they match the value-weighted maturity of assets with that of the liability. This concept is strictly valid for a one-integrated factor interest rate model (Merton, 1973, also called the Black model). However, since in our two-factor model the level factor (inflation) predominantly matters for long yields, duration hedging can be thought as a first-order approximation for sensitivity hedging of long-dated liabilities.

Pension funds often invest in bonds with a shorter maturity than that of the liabilities.[15] The rule of thumb[16] is that a Gaussian pension fund would borrow short-term funds which have sensitivity zero, and that it would invest in two 10-year bonds (or a 1.33 unit of a 15-year bond) to hedge the sensitivity of a 20-year liability. However, the sensitivity profile of Figure 6 implies that it suffices to buy 1.55 units of 10-year bonds (or a 1.15 unit of a 15-year bond) to hedge the 20-year inflation risk.

A pension fund that seeks full immunization of interest rate risk with a duration hedge is exposed to the risk of a rise in inflation. A rise of inflation by 1.5% leads to a funding ratio loss[17] of 7% if a 10-

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15 The DNB reported in 2006 an average liability duration of 15 years, with an average asset duration of 5 years.

16 This would be correct if all yields were zero and bonds were worth 1. Then, a fall of yields by 1% raises the price of the liability by 20%, from 1 to 1.2, while the price of a 10-year bond rises by 10%, and the price of the cash account does not instantly change. To be precise, calculations take into account the actual value of the 10- and 20-year bonds.

17 Assume for instance that the initial funding ratio is 100% and that the liability risk is fully hedged; alternatively, hedging weights would be of A/L for any initial funding ratio. Then inflation rises (and the pension fund does not rebalance its portfolio).

year bond is used for hedging, and of 3.6% if a 15-year bond is used for hedging.

Sensitivity hedges:
Two different bonds are needed to hedge both the inflation and real rate risk factors. In the shadow K12 model, inflation risk can be calculated by duration, so the long-short hedging portfolio has the value-weighted duration of the lability.

Suppose long-term (20-year) liabilities are hedged with the combination of a 15-year and a 10-year bond.

As illustrated in table 1, the Gaussian hedge involves approximately two 15-year bonds minus one 10-year bond, for a value-weighted duration of 20 years.

A 1.5% rise in inflation and in equilibrium short rates leads to a funding ratio loss of 2.8% with long-term calibration and of 1.8% with an ideal short-term calibration: even though short-term calibration perfectly captures local dynamics, it does not capture the impact of non-negligible changes in inflation. Sensitivity hedging, however, leads to a lower risk than duration hedging, which involves a 3.8% loss when a 15-year bond is used for hedging.

The pension fund overhedges interest rate risk. This result is qualitatively similar with long- and short-term calibration and is expected to be similar with a wide range of Gaussian models.

Table 1: Hedging with a 10-year and a 15-year bond
('$2 * 15 - 10 = or \neq 20$'?)

Model	ZLB	$G(\infty)$	$G(CS/0)$
Hedging scheme: units in bonds			
10Y bond	-0.61	-1.02	-0.97
15Y bond	1.60	2.06	1.98
(rise of 1.5% in state variables):			
Inflation	0.2%	-2.8%	-1.8%
Equil. real rate	-	-0.1%	-

The top panel shows the hedging weights in the 10- and 15-year bonds. The first column shows the optimal hedging weights, the middle column the Gaussian model hedging weights with long-term calibration, and the right column with short-term calibration.
The Gaussian pension fund replicates a 20-year liability with approximately two 15-year bonds and a short position in a 10-year bond, which matches the duration of the liability (2*15-1*10=20); this holds both for short-term and long-term calibration.
The ZLB hedge, however, requires both less leverage and less duration (in the ZLB, a long position of 160% is needed in the 15-year bond rather than 200%).
The bottom panel shows the residual exposure in the case of a rise by 1.5% in state variables. If inflation rises by 1.5%, the Gaussian pension fund trying to shut interest rates would see its funding ratio fall by 1.8% with short-term and by 2.8% with long-term calibration.

Risk associated with possible valuation issues:
As we have seen, long-dated liabilities may become overpriced near the ZLB if the Vasicek shadow model is calibrated using a long-time span.[18] This overvaluation could increase over-hedging, with minor benefits — the bond risk premium would vanish together with the bond volatility near the ZLB — but with the risk of additional losses for the pension fund if interest rates rise.

A 35 basis point error on a 20-year yield may seem insignificant, but it represents an 8% price error on the 20-year bond or liability. An 8% over-investment, followed by an inflation increase of 1.5%, generates a 6.4% loss, even larger than losses associated with simple duration hedges.[19]

Steep rises of inflation and interest rates rarely recur in a single scenario, and crash risk has a low probability in equilibrium models. Scenarios with repeated small losses (inflation rises by one percent and falls back repeatedly, and short rates remain close to zero) are more frequent and overall costlier in expectation (when losses are weighted by their probability).

Derivatives' losses may arise even without a rise in inflation. They are more frequent and costly.

Risk budgeting implications of interest rate model risk
Possible reasons for resorting to risk budgeting are that regulations require pension funds to be overfunded over the medium term, that

18 The same could arise with calibration using a short-time span if long-dated bonds are illiquid.
19 This error varies with circumstances. It is lower if for instance only a fraction of the liability is hedged; it is higher if for instance inflation is lower.

pension cuts are unpopular among pension participants,[20] and that insurance companies must always be overfunded.

Misspecification of the term structure model prevents the risk-budgeting Gaussian pension fund from securing a minimum funding ratio: as the Gaussian pension fund seeks to shut off interest rate risk, it overhedges interest rate risk and is exposed as a consequence to the risk of an economic recovery.

Overestimation of interest risk may reduce the equity risk budget of investors that have a total risk budget.

When the two sources of error are combined, the distribution of the future funding ratio (in one or five years) can become left-skewed with a high probability of underfunding instead of being right-skewed above 100%. Additionally, investment opportunities can be missed. Simulations of such effects are available on demand.

20 Fiduciary duties require managers to protect pension promises. This is sometimes interpreted as a duty to avoid pension cuts.

6. Conclusion

If a lower bound exists, it modifies the interest rate dynamics in a non-linear way. Gaussian models are unable to describe simultaneously the interest rates dynamics when short rates above zero move at least one on one with inflation, and the dynamics when short rates close to zero disconnect from inflation.

Gaussian models may overestimate the probability of negative rates. A Gaussian pension fund will then allocate funds to hedge the risk that rates would fall deeper in negative territory, even though this risk is very small.

Derivatives may involve a cost that is maintained so long as rates remain close to zero. The use of bonds usually involves overhedging and possible exposure to the one-off risk of a rise in rates. Indeed, with a ZLB, the risk of a rise of interest rates is greater than that of a fall, regardless of the underlying model. Expanding liability hedging programs when bond yields are close to zero may then be risky.

The incorporation of a lower bound in a Gaussian model — which truncates the distribution of rates at zero or possibly -0.50% — helps flatten the long end of the yield curve and could lower the value of long-dated (illiquid) liabilities.

Models without a ZLB make it difficult to read the monetary stance: near the ZLB, real interest rates rise as inflation falls, and a recalibrated Gaussian model signals that the real rate will be permanently higher due to central banking policy. However, the rise in the real rate may only be due to the ZLB constraint and not be permanent. In line with econometric theory, the central bank may indeed seek to expand monetary policy, for instance through quantitative easing.

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Appendices

A) Introduction and organization of the appendices

Pension funds often rely on Gaussian affine term structure models in their ALM exercises. In the Netherlands, the Dutch National Bank [DNB] has also validated the KNW (Koijen, Nijman and Werker, 2010) model. Since 2009, the U.S. Federal Reserve has used non-conventional tools – other than lowering the short term rates below zero – to make its monetary policy more expansive, arguably because of a lower bound on short rates. If a lower bound exists on short rates:

- the assumed dynamic of these models is unrealistic since 2009 (see Section 1 on page 6).
- the assumption that short rates may become very negative may lead to overvaluation of long-term liabilities.

In this study, we have analysed the impact of non-linear interest rate dynamics on pension funds hedging policy, while discussing the impact of different ways for pension funds to fit a ‘wrong’ model to the current yield curve, mainly updating versus recalibration. Of course, all models are wrong. It is, however, convenient to assume that the pension fund will use the correct model, with only one misspecification: the failure to incorporate a zero bound. This makes it easy to analyse the impact of the zero bound.

Note that by assuming that the underlying equilibrium is an extended Vasicek model, similar to what many pension funds use today, we implicitly recognize the benefits of an underlying Gaussian model.

This appendix is organised as follows:

- Section B reviews interest rate models: B.1 gives an overview of interest rate models that yield non-negative interest rates, and B.2 discusses the lower bound in Europe.
- Section C summarizes the pricing framework with ZLB-Shadow Gaussian models, and specifically with the two-factor model of Krippner (2012) in C.2.

B) Review of interest rate models

B.1) ZLB versus alternative approaches

Denoting $W(t)$ as a Brownian motion and $dW(t)$ as its increments, we modify the Ornstein-Uhlenbeck process in (1) to formulate and discuss non-negative interest rate models:

$$dX_t = \kappa\,(\theta - X_t)\cdot\, dt\ + \sigma\, dW(t) \qquad (1)$$

Definition: (1) is called *univariate* if X_t is a scalar (just as $\kappa, \theta, \text{and}\ \sigma$), *independent multivariate* if $X_t{}'$ is a vector and κ is diagonal,[21] and *multivariate* otherwise (then X'_t and θ are vectors while κ and σ are matrices).

The idea of Cox, Ingersoll, and Ross (1985) is to use $r_t = X_t$ and to impose a state-dependent local volatility $\sigma(r_t)$ that goes to zero when the short rate approaches zero (specifically the volatility of the short rate is proportional to the square root of r_t). Then, given $(\theta, \kappa > 0)$, the pull-back $\kappa\,(\theta\ -\ r_t)$ ensures a 'reflection' of the short rate away from zero. In equation (2), the zero level is then called a reflecting barrier.

21 Then each univariate X_i is a univariate OU process which mean reversion dynamics is described by (κ_i, θ_i).

$$dr_t = \kappa\,(\theta - r_t) \cdot dt + \sigma \sqrt{r_t}\, dW(t) \qquad (2)$$

A multivariate Cox Ingersoll Ross requires positive state variables, which exclude economic interpretation since inflation for instance can be negative. In addition, from 2009 to 2015 short-term rates stayed at zero, which is inconsistent with the idea of a reflecting barrier.

The log-normal model (see a *univariate* representation in equation 3) has qualitatively similar dynamics: the volatility of long and short rates shrinks to zero as the short rates falls.

$$dr_t = \kappa \cdot (\theta - r_t) \cdot dt + \sigma\, r_t\, dW(t) \qquad (3)$$

With these models, the volatilities of short and long yields go together to zero. Yet the following non-linear transforms of the Ornstein-Uhlenbeck process in (1) permit non-linear volatility dynamics:

$$dX_t = \kappa\,(\theta - X_t) \cdot dt + \sigma\, dW(t);\;\; r_t = X_t^2 \qquad (4)$$
$$dr_t = \kappa\,(\theta - r_t) \cdot dt + \sigma\, dW(t);\;\; \underline{r_t} = [r_t]^+ \qquad (5)$$

Equation (4) is a simple[22] quadratic term structure model (see Ahn et al., 2002). Volatility in Quadratic term structure models is non-linear, and the simple model in equation (4) has zero volatility at r_t=0. The fact that short rates have a zero mass at precisely zero could be a limitation to reproduce the US yield curves. In addition, factors lack

22 $f(x) = \alpha + \beta\, x + x'\psi\, x$ (using multivariate notations) yields the family of Quadratic (affine) term structure models (Ahn et al., 2002). The conditional and unconditional distributions of the interest rates can be represented as an infinite mixture of noncentral χ^2 distributions.

structural interpretation, since one cannot envisage that central banks set interest rates as a quadratic function of inflation and economic growth.

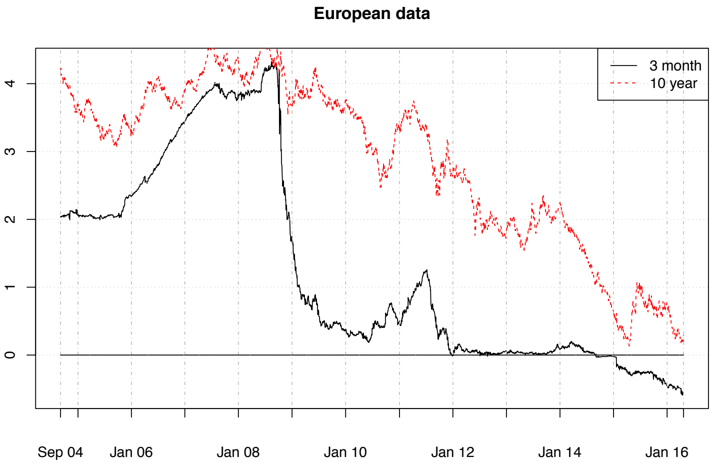
The model in (5) is the ZLB model, where the observed short rate $\underline{r}$ is the positive part of a Gaussian *shadow*-rate in (1). Truncation generates a positive mass at zero, thus making developments experienced in the US explicitly possible. The interpretation of the state variables is the same as in the shadow-model. The ZLB bond price reads $\underline{P}(t,T) = exp(-\int_0^{\tau} \underline{f} \ (t+u)\, du \cdot 1)$, where $\underline{f}(t+u)$, the modified forward rate, can be interpreted as an option on the shadow short rate. As soon as $r_t < 0$, short maturity options have little value and volatility, but deeply negative shadow short rates are needed for the volatility of the long bonds to drop to a level close to zero. The volatility of the term structure increases non-linearly with shadow and ZLB short rates.

Overall, in the category of models that yield non-negative interest rates, we choose the ZLB model developed at central banks. These score high on three important criteria: economic interpretation (they permit negative state variables that are arbitrarily correlated, negative equilibrium interest rates, and yet positive observed ones), a good fit (short-term rates can remain low for protracted periods of time), and tractability. In addition, they permit a direct comparison with Gaussian models commonly used by pension funds, since they can share the same underlying dynamics and calibration.

B.2) The Lower bBound in the European Monetary Easing

The switch from conventional to unconventional monetary policy as rates approach zero has been less clear-cut with the European Central Bank than with the U.S. Federal Reserve.

Figure 8: European 3-month and 10-year yields



Quantitative easing has been more limited in Europe. Purchase of government bonds has mainly been implemented after the promise of reform. The requirement that the ECB buys only high-quality non-government bonds has been seen as a limitation to 'full-blown QE', although securitised loans (*e.g.*, mortgages) were eligible.

Contrary to the Fed, the ECB has instituted slightly negative rates for banks (but not for customers). However, there are indications that current rates are close to a lower bound: the ECB (Draghi, 2015) made an explicit statement to that effect. Following its commitment to give virtually unlimited support and to make a big policy move, the ECB recently cut rates by 10 basis points — much lower than the usual 50bp cut in normal times or 100bp in case of distress.

The Financial Times (FT, 2016) states that some banks are considering stashing cash in costly deposit boxes instead of keeping it with the European Central Bank. This also suggests that we are indeed not far from the lower bound.

C) Pricing with ZLB models

C.1) Bond pricing formulas

Liability price in the shadow model

The pension liability is modelled by a zero-coupon payment at time T as a proxy for a more complicated distribution of cash flows.

Denote $\mathbb{E}$ as the expectation under $\mathbb{Q}$, with $\mathbb{Q}$ as the risk-neutral measure, a virtual measure where risk is not rewarded. r is the (modelled) Gaussian shadow rate and $\underline{r} = r^+$ the ZLB rate.

The shadow price of the liability is given by (6):

$$L_t = \mathbb{E}^{\mathbb{Q}}[exp(-\int_t^T r_\tau \, d\tau)] \qquad (6)$$

Liability price in the ZLB model

The market-consistent (ZLB) price at time t is given by (7)

$$\underline{L\_t} = \underline{P(t,T)} := \mathbb{E}^{\mathbb{Q}}[exp(-\int_t^T \underline{r_\tau} d\tau)] \quad (7)$$

Pricing error with the shadow model

Because $r \leq \underline{r}$, the shadow price L_t is at least equal to $\underline{L_t}$, the ZLB market price. Equation (9) shows that the overestimation of the liability price is at least equal to the expectation of a cash account that only accrues when short rates are negative (with their absolute value). Writing $r = r^+ + r^-$ we have:[23]

$$L_t \geq \underline{L_t} \cdot \mathbb{E}^{\mathbb{Q}}\left[exp\left(\int_t^T |r^-|_\tau d\tau\right)\right] \quad (9)$$

The overestimation from the shadow model grows as either state variable falls, i.e., as the shadow rate becomes negative and the nominal interest rate falls to zero.

ZLB pricing

The ZLB rate models were introduced by Black (1995) with a univariate $f(r) = r^+$ for any underlying/shadow r model. In his original method, bonds are priced by simulation as in equation (7), since even with a Gaussian r, the expectation in (7) does not have a closed form solution.

23 This is easily seen because:

$$\hat{P}(t,T) = \mathbb{E}^{\mathbb{Q}}\left[exp\left(-\int_t^T r_\tau d\tau\right)\right] = \mathbb{E}^{\mathbb{Q}}\left[exp\left(-\int_t^T (r_\tau^+ + r_\tau^-) d\tau\right)\right]$$

$$\geq \mathbb{E}^{\mathbb{Q}}\left[exp\left(-\int_t^T r_\tau^+ d\tau\right)\right] \cdot \mathbb{E}^{\mathbb{Q}}\left[exp\left(-\int_t^T r_\tau^- d\tau\right)\right] \quad (8)$$

$$\geq \underline{P(t,T)} \cdot \mathbb{E}^{\mathbb{Q}}\left[exp\left(-\int_t^T r_\tau^- d\tau\right)\right]$$

Closed form solutions for bond prices are sometimes useful. Krippner (2012, 2013) obtained bond prices as an integral of options with closed form values by rewriting the bond price in terms of forward rates. Denoting $T = t + \tau$, we have:

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[exp\left(- \int_0^\tau r(t+u)du \cdot 1\right)\right] = exp\left(- \int_0 f^\tau (t+u)du \cdot 1\right) \quad (10)$$

The ZLB condition is transposed to the forward rate, and the availability of money also implies that the future yield of any current bond will always be non-negative. Taking the limit at the time prior to maturity, this condition applies to the forward rate, which yields (11) and (12).

$$\underline{f}(t+u) = [f(t+u)]^+ \quad (11)$$

$$\underline{L_t} = \underline{P}(t,T) = exp\left(- \int_0^\tau \underline{f}\ (t+u)du \cdot 1\right) \quad (12)$$

Since the distribution of the forward rate is also known in closed form for Gaussian affine term structure models, the ZLB-floored forward rate $\underline{f}$ can also be expressed as an option on the shadow-forward f:

$$\underline{f}(t,t+u) = f(t,t+u) + z(t,t+u) \quad (13)$$

where $z(t,t+u)$ is an American option on the shadow forward rate, i.e., a derivative of an American option on the shadow bond price. Krippner (2012) approximates this American option by a European option with a closed form solution. The approximation error has been shown to be negligible in recent research (Christensen et al., 2013; Christensen and Rudebusch, 2014). This yields (14)

$$z(t,\tau) = \lim_{\delta \to 0} \left[\frac{\partial}{\partial \delta} \left\{ \frac{C^E(t,\tau,\tau+\delta;1)}{P(t,\tau,\tau+\delta)} \right\} \right] \quad (14)$$

where $C^E(\dots)$ is a European option price that serves as an empirical proxy for the American version which is harder to value. With Gaussian affine term structure models, forward rates being Gaussian, C^E is known in closed form, yielding for z and $\underline{f}$:

$$z(t,\tau) = -f(t,\tau) \cdot \left(1 - N\left[\frac{f(t,\tau)}{\omega(\tau)}\right]\right) + \omega(\tau) \frac{1}{\sqrt{2\pi}}\ exp\left(-\frac{1}{2}\left[\frac{f(t,\tau)}{\omega(\tau)}\right]^2\right)$$

$$\underline{f}(t,\tau) = f(t,\tau) \cdot\ N\left[\frac{f(t,\tau)}{\omega(\tau)}\right] + \omega(\tau) \frac{1}{2\pi}\ exp\left(-\frac{1}{2}\left[\frac{f(t,\tau)}{\omega(\tau)}\right]^2\right) \quad (15)$$

where N is the Gaussian cumulative distribution function and $w()$ the volatility of the forward rate. The ZLB bond price is then computed as in (12) as a simple integral of options.

See C.3 on page 49 for computation details for Gaussian affine term structure models.

C.2) Krippner (2012) – a ZLB with a 2-factor Vasicek shadow model

The K12 model is a special case of a ZLB/shadow-Vasicek model, where one of the two factors of the Vasicek shadow model, namely inflation, is integrated. It can be compared to KNW08, a two-factor Gaussian model, obtained by setting $n = 2$ in the general model of Appendix C.3. The state variables of KNW08 have the following dynamics:

– The real rate r^R follows an OU process:[24]

$$dr_t^R = -\kappa_r \,(r_t^R \; - \delta_r)\, dt \; + \sigma_R \, dW_t^R$$

– The expected[25] inflation π follows

$$d\,\pi_t = -\kappa_\pi \,(\pi_t \; - \delta_\pi)\, dt \; + \sigma_\pi \, dW_t^\pi$$

Proof: From KNW08 to K12: sketch of the proof and notations

Since inflation is integrated in K12, the K12 shadow model is the limit of the KNW08 model when the mean-reversion of inflation tends to zero: $\kappa_\pi \to 0$.

To obtain the forward rate in equation (16) on the next page we use the detailed formulas of Appendix C.3 and use $\lim_{\kappa \to 0} \mathrm{G}(\kappa, \tau) = \tau$, and, by the Continuous Mapping Theorem, $\lim_{\kappa \to 0} \mathrm{G}^2(\kappa, \tau) = \tau^2$.

Notations for the K12 Vasicek shadow model are similar to those for KNW08, but abstract $\kappa_\pi = 0$, as well as $\delta_\pi = \delta_R = 0$.[26]

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24 Note that if $r = \delta + X$ and $dX = -\kappa\, X\, dt \; + \sigma\, dW$,
then $dr = -\kappa\,(r - \delta)\, dt \; + \sigma\, dW$.

25 Since no risk premia are assumed on unexpected inflation risk, expected inflation is on average equal to realised inflation.

26 We interpret X_1 as inflation and X_2 as the real rate, but these are unobserved state variables in K12.
And the mean-reversion cannot be identified for each state variable when these are not observed. In such models, it is customary to write $r = \delta + \sum X_n$,where X_n have no mean reversion. But when one factor is integrated, δ cannot be identified either. If we really calibrated K12 to observed real rates and inflation, then $\delta, \delta_R, and\ \delta_\pi$ would of course be identified.

K12 shadow forward yield:

$$f(t,\tau) = \pi(t) + r^R(t)exp(-\kappa_r \cdot \tau) + \sigma_\pi \lambda_\pi \tau + \sigma_R \lambda_R G(\kappa_r,\tau) \\ -\tfrac{1}{2}\sigma_\pi^2 \tau^2 - \tfrac{1}{2}\sigma_R^2 [G(\kappa_r,\tau)]^2 - \rho\, \sigma_\pi\, \sigma_R \cdot \tau \cdot G(\kappa_r,\tau) \qquad (16)$$

where λ_π is the risk premium attached to inflation risk, λ_R to the real interest rate risk, and the function $G(\kappa\,,\tau) = \frac{1}{\kappa}[1 - exp(-\kappa\ \cdot \tau)]$.

Shadow forward yield volatility:

$$\omega(\tau) = \sqrt{\sigma_\pi^2 \cdot \tau + \sigma_R^2 \cdot G(2\kappa_r,\tau) + 2\,\rho\, \sigma_\pi \sigma_R \cdot \tau \cdot G(\kappa_r,\tau)} \qquad (17)$$

where $\sigma_\pi(t)$ is the volatility of the level factor in the shadow forward rate and $\sigma_R(t) \cdot exp(-\kappa_r \cdot \tau)$ the volatility of the slope factor. These elements permit the construction of the ZLB rate $\underline{f}^B(t,\tau)$ as in (19).

K12 shadow interest rate yield:

writing $r(t,\tau) = \frac{1}{\tau}\int_0^\tau f(t,u)du$ we have for the shadow bond-yield:

$$\begin{aligned} r(t,\tau) = {} & \pi(t) + r^R(t) \cdot \tfrac{1}{\tau} G(\kappa_r,\tau) \\ & + \sigma_\pi \lambda_\pi \tfrac{1}{2}\,\tau + \sigma_R\, \lambda_R \tfrac{1}{\kappa_r}\left[1 - \tfrac{G(\kappa_r,\tau)}{\tau}\right] \qquad (18) \\ & - \sigma_\pi^2 \frac{1}{6}\,\tau^2 - \sigma_R^2 \cdot \frac{1}{\kappa_r^2}\left[\frac{1}{2} - \frac{1}{\tau} G(\kappa_r,\tau) + \frac{1}{2\tau} G(2\kappa_r,\tau)\right] \\ & - \rho\, \sigma_\pi\, \sigma_R \cdot \frac{1}{\kappa_r^2}\left[\frac{1}{2}\kappa_r \tau - \frac{1}{\tau} G(\kappa_r,\tau) + exp(-\kappa_r\, \tau)\right] \end{aligned}$$

Calibration:

The calibration in Krippner (2012) is: $\kappa_r = .3884, \lambda_\pi = .1435, \lambda_R = .2895, \sigma\_\pi = .0172, \sigma_R = .0250,\ \rho = .4098$ and, for the risk premia, $\lambda_\pi = 0.1435$ and $\lambda_R = 0.2895$

C3) Computation details for a general shadow-Gaussian model

Krippner (2012, p15) obtains the following formulae in the case of the *independent* multivariate Gaussian affine term structure model. The notation is inspired by Chen (1995), where

$$r(t) = \sum_{i=1}^{N} s_n(t)$$

and the $s_n(t), n \in [1{:}\,N]$ are state variables that follow correlated Ornstein–Uhlenbeck processes under the physical $\mathbb{P}$ measure:

$$ds_n(t) = \kappa_n[\mu_n - s_n(t)]dt + \sigma_n \, dW_n(t)$$

with $\kappa_n, \mu_n, \sigma_n$ being positive numbers representing the long-run level, mean reversion rate, and volatility of the s_n and $W_n(t)$ correlated Wiener processes. The constant market prices of risk are denoted λ_n, usually positive numbers which lift forward rates. The bond price of maturity $\tau = T - t$ reads at time t:

$$P(t,\tau) = exp\left[-H(t) - \sum_{n=1}^{N} s_n(t) \cdot \; G(\kappa_n, \tau)\right]$$

where the function $H(\tau)$ for Gaussian affine term structure models is

$$H(\tau) = -\frac{1}{2} Tr[\Xi(\tau)\Psi] + \sum_{n=1}^{N} \left[\mu_n + \sigma_n \frac{\lambda_n}{\kappa_n}\right] [\tau - G(\kappa_n, \tau)]$$

with

$Tr[\cdot]$ as the matrix operator, Ψ as $\Psi_{ij} = 1/\kappa_i \kappa_j$, and

$$\Xi_{\mathrm{ij}}(\tau) = \rho_{ij}\sigma_i\,\sigma_j \cdot [\tau - G(\kappa_i,\tau) - G(\kappa_j,\tau) + G(\kappa_i+\kappa_j,\tau)].$$

The shadow forward rate

A single numerical integration of the ZLB forward rate in (19) is necessary to obtain the bond price. The ZLB forward rate reads

$$\begin{aligned}\underline{f}^B(t,\tau) &= f^B(t,\tau)\ +\ z^B(t,\tau) \\ &= f(t,\tau)\cdot\ N\left[\frac{f(t,\tau)}{\omega(\tau)}\right] + \omega(\tau)\frac{1}{\sqrt{2\pi}}exp\left(-\frac{1}{2}\left[\frac{f(t,\tau)}{\omega(\tau)}\right]^2\right)\end{aligned} \tag{19}$$

with the shadow forward rate

$$f(t,\tau) = \sum_{n=1}^{N}\mu_N\ +\ [s_n(t)-\mu_n]exp(-\kappa_n\,\tau)\ + \sum_{n=1}^{N}\sigma_n\ \lambda_n\,G(\kappa_n,\tau) - \frac{1}{2}Tr[\Theta(\tau)\Psi] \tag{20}$$

where $\Theta_{i,j}(\tau) = \rho_{ij}\sigma_i\sigma_j \cdot \kappa_i\kappa_j\,G(\kappa_i,\tau)G(\kappa_j,\tau)$,

$Tr[\cdot]$ as the trace operator, $N[\cdot]$ as the standard normal *cdf* and $\omega(\tau)$ as the annualised instantaneous option volatility:

$$\omega(\tau)\ = \lim_{\delta\to 0}\left\{\frac{1}{\delta}\sum(\tau,\tau+\delta)\right\} \tag{21}$$

$$= \sqrt{\sum_{i=1}^{N}\sigma_n^2\cdot\ G(2\kappa_n,\tau)\ +\ 2\sum_{m=1}^{N}\sum_{n=m+1}^{N}\rho_{mn}\ \sigma_m\sigma_n\cdot\ G(\kappa_m+\kappa_n,\tau)}$$

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Zerotopia – bounded and unbounded pension adventures

This paper analyses pension funds hedging policy when interest rates are near zero, focusing on interest rate models with and without negative interest rates.
The author compares the use of a Gaussian interest rate model – typical of those used and validated by the Dutch central bank – as well as an extended version that incorporates a zero lower bound (ZLB).
ZLB models better fit historical developments, and Samuel Sender examines whether a ZLB is backed by theory, can reasonably be trust for the near future, and which lessons can be drawn for pension funds interest rates hedging policy.

Dit is een uitgave van:
Netspar
Postbus 90153
5000 LE Tilburg
Telefoon 013 466 2109
E-mail info@netspar.nl
www.netspar.nl

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