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Optimal Capital Structure for Insurance Companies*

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Abstract

This paper analyzes the capital structure decision that insurance companies face. A structural microeconomic model is constructed and solved by means of dynamic optimization. The model allows for a careful analysis of various aspects pertaining to the basic economic trade-off between increasing the level of surplus capital on the one hand, incurring high costs in imperfect capital markets, and decreasing the surplus level on the other, eroding the quality and value of insurance protection offered.

Keywords: Risk management; Insolvency risk; Surplus capital; Insurance premium; External financing; Capital market imperfections; Charter value.

JEL Classification: D81; G10; G20.

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1 Introduction

The recent financial crisis has highlighted dramatically the potential extent of risk-taking by financial intermediaries. Materialized losses have exceeded by far the stress test scenarios, leading to eroded solvency positions for many financial intermediaries. The debate on regulatory adjustment has emphasized the need of strong capital buffers, to contain default as well as its propagation.

In this paper we study the case of insurance capital. Figure 1 plots the solvency capital dynamics for Netherlands-based insurance companies over the period 2005-2009. The figure shows that the average solvency ratio, defined as the fraction of available solvency capital over required solvency capital, while exceeding 300% before the crisis, dropped sharply in the years 2007 and 2008, with various individual insurance companies hitting, or even falling below, the bare minimum amount of solvency capital as required by the Dutch Central Bank, the regulatory authority.

Just as in the case of banks, insurer shareholders can choose to let the firm default under protection of limited liability, shifting losses to policyholders. A recent example, prevented from actual default by third party government intervention, is that of the American International Group (AIG), a large US insurance firm. In the early fall of 2008 it became apparent that the private sector was not willing to refinance AIG, which was at that time severely suffering from liquidity problems due to the financial crisis, eventually leading to a bail-out of the firm: “Over the weekend, federal officials had tried to get the private sector to pony up some funds. But when that effort failed, Fed Chairman Bernanke, New York Fed President Timothy Geithner and Treasury Secretary Paulson concluded that federal assistance was needed to avert an AIG bankruptcy, which they feared could have disastrous repercussions.” (Wall Street Journal, September 16, 2008.)

Higher capital buffers help solvency by increasing expected payouts to policyholders in the event of large losses, as recognized by the risk theory literature on capital choice by insurers.[1] This benefits risk averse policyholders, who accordingly may accept higher policy prices, but increases potential shareholder loss. At the same time, higher target capital levels may provide weaker incentives for shareholders to compensate ex post for a particular range of losses. The reason is that equity raising costs empirically increase with the amount raised, especially after losses.[2] Higher and more costly capital buffers therefore make it less attractive ex post to reinvest equity in the company. Insurance

[1]See e.g., the seminal work by Borch (1974).

[2]Reasons for this underpricing are adverse market conditions, asymmetric information and insider signalling (Leland & Pyle, 1977, Ritter, 1984), adverse selection (Rock, 1986), and price pressures (Aggarwal & Rivoli, 1990).

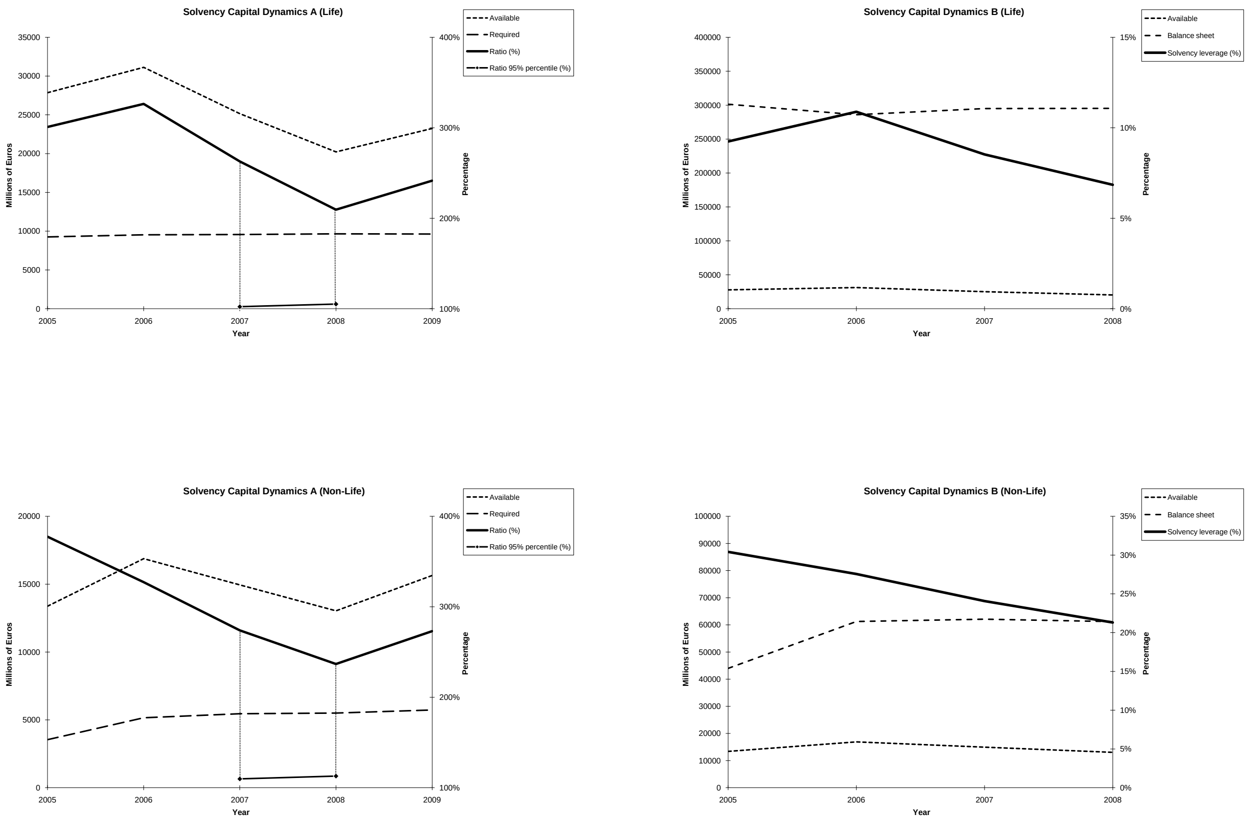

Figure 1: Solvency Position of Insurance Companies based in The Netherlands.
This figure plots the solvency position of Dutch life and non-life insurance companies prior to and during the 2007-2008 financial crisis. The two panels on the left plot the average available solvency capital (small dashes) as reported to (approved by) the Dutch Central Bank, the regulatory authority; the average minimum solvency capital (wide dashes) as required by the regulator; the average solvency ratio (bold solid line; in percentages), defined as the fraction of available solvency capital over required solvency capital; and the lower 95% percentile of this ratio (solid line; in percentages). A distinction is made (by law) between legal entities registered as life insurance companies (top) and legal entities registered as non-life insurance companies (bottom). The sample period is 2005-2009. The two panels on the right plot the average size of the balance sheet (wide dashes); the average available solvency capital (wide dashes); and the average solvency leverage (bold solid line; in percentages), defined as the fraction of available solvency capital over the size of the balance sheet. Again a distinction is made between legal entities registered as life insurance companies (top) and legal entities registered as non-life insurance companies (bottom). The sample period is 2005-2008.

companies which raise more equity funding will increase ex post loss absorption costs. This increases the probability of default and therefore limits the willingness to pay higher policy prices.

The financial crisis has urged the insurance industry (and other financial intermediaries alike) to reconsider the solvency capital architecture. On the one hand, the regulatory authorities, some of which already in a process of developing new standards for capital adequacy[3] have experienced the necessity of scrutinizing their (proposed) regulatory systems. On the other hand, insurance companies have become aware of the importance of reviewing and updating their procedures for internal risk management.

Such a development calls for discussion on and answers to some fundamental questions on solvency capital and regulation, such as those asked and analyzed in this paper. In the abstract, insurance companies hold surplus (or, solvency) capital to serve as a buffer in times of financial distress, upon the occurrence of adverse shocks to their balance sheets. But what exactly are the economic incentives for doing so? How much capital should the institution hold optimally? And what are the dominant economic determinants underlying the optimal amount of surplus capital? Furthermore, what is the economic rationale for capital requirements imposed by a regulatory authority, are they effective and how should they be combined with other regulatory tools?[4,5]

The private choice of capital for insurers trades off the gains due to possibly increased premia against greater potential losses, as well as greater capital costs. A feature that is specific to insurance is the extent of sensitivity of risk averse policyholders to the financial standing of the insurance company, given the high impact of potential losses on the individual policyholder situation. The capital structure decision for insurance companies is a problem of optimal risk-taking and market imperfection. This paper analyzes this

[3]The European Union (EU) is developing Solvency II, which is likely due to be effective as of December 31, 2012. It will apply to EU-based insurance and reinsurance companies with gross premium income exceeding EUR 5 million, or gross technical provisions exceeding EUR 25 million.

[4]Though both insurance companies and pension funds sell insurance products, the capital structure decision problem is not identical for both types of financial institutions. This is due to significant differences between the governance structures of insurance companies and pension funds. In the case of an insurance company these stakeholders are the policyholders and regulatory authority on the one hand, and the company's management and shareholders on the other. The stakeholders of a pension fund are the pension plan participants and regulatory authority on one side and the fund's management and sponsors on the other. The precise governance structure of pension funds depends to a large extent on the formal pension arrangements between the sponsor and the participants. We restrict ourselves in this paper to the case of insurance companies, while the case of pension funds will be investigated further.

[5]Different types of solvency capital need to be distinguished: (i) regulatory (or, required) capital which is the bare minimum amount of capital an insurance company is required to hold; (ii) available (or, book; approved by supervisory authority) capital; (iii) economic (or, management or risk) capital, which arises (at least implicitly) as the result of economic trade-offs and reflects strategic choices.

optimal risk-taking problem in insurance as well as the related optimal insurance regulation problem by explicitly modeling the underlying economic trade-offs. Surprisingly, relatively little is known about this fundamental problem.[6]

A key element of the economic analysis will be the trade-off between the costs of holding and raising solvency capital on the one hand, and the willingness to pay for insurance provided by a financially healthy insurance or pension institution on the other. In imperfect capital markets with taxes and agency concerns, holding surplus capital on the balance sheet is costly for the shareholders of an insurance company. As a result the shareholders experience natural incentives to limit the amount of surplus capital on the balance sheet. At the same time, there is considerable empirical evidence that consumer demand for insurance is sensitive to the financial position of the insurer; see e.g., Sommer (1996) and Grace, Klein, Kleindorfer & Murray (2003). Changes in demand due to changes in the financial standing of the insurer may appear both as a change in the premium an insurer can charge and as a change in the quantity that the insurer can sell. This provides incentives for insurance companies to maintain a sufficiently high level of surplus capital on their balance sheets.

In our analysis, we assume that both the insurer (or rather: its shareholders) and the policyholders behave rationally (that is, have preferences that comply with the expected utility hypothesis). We thus take a normative viewpoint. The main elements of the microeconomic model we develop to derive the optimal amount of surplus capital are briefly described below.

We consider an insurer that maximizes the expected utility from benefits accrued to the insurer's owners from its future operations. The concept of *charter value* is used. In our setting, the charter value is the present value of expected future dividends; it is to be maximized and acts as a reference point in the continuation decision that the insurer faces after a large loss has occurred.

Two standard types of capital market imperfections are considered in our model. First, we assume that holding equity capital involves a deadweight cost. This deadweight cost arises from various sources: (i) equity capital held by an insurer is taxed while it is on the balance sheet; (ii) agency concerns; and (iii) good insurers are indistinguishable from bad insurers so that the mere existence of bad insurers in the market gives rise to an additional "lemon" cost of capital. Second, we assume that raising equity capital involves a convex adjustment cost of capital in the spirit of Myers & Majluf (1984).

Specifically, the following trade-off will then be studied: increasing the amount of

[6] This was noticed also by Garven (1987, Section 3), asking for an explicit analysis of trade-offs.

equity capital means increasing the potential shareholder loss as well as increasing the deadweight costs involved in holding equity capital. The latter will have a direct negative effect on the dividends that the insurer can pay out and hence on the charter value. At the same time, increasing the amount of equity capital means increasing the expected payout to the policyholder in the case of default. This will have a positive effect on the premium that the policyholder is willing to pay, whence a positive effect on the dividend payouts and the charter value. Increasing the amount of equity capital, thus increasing the deadweight costs of holding equity capital, will also have a subtle indirect negative effect on the dividend payouts and charter value as it will increase the probability that the insurer stops activities after the occurrence of a large claim. This is because the costs of raising capital to set the surplus capital back to the original (target) level of surplus may become too large. The increase of the probability of ceasing activities (default) implies a decrease of the premium that the policyholder is willing to pay, whence a negative effect on the dividend payouts and the charter value.

This trade-off will be formalized in a structural model, which makes our approach distinct from previous work on the capital structure decision problem for insurance companies that we know of.[7,8] The model will be solved using dynamic optimization techniques.

We derive a.o. comparative statics with respect to the parameters of capital market imperfections, which are the main drivers of the capital structure decision, and with respect to the degree of skewness of losses, which is a key feature in insurance.

The rest of this paper is organized as follows. In Section 2 we introduce our base model. In Section 3 we solve this model and analyze its solution. Finally, Section 4 contains comparative statics of the base model.

2 The Base Model

We start with considering a market with one insurance company (the monopolist), one shareholder and one policyholder. Uncertainty in the insurance market is represented by

[7]The insurance literature has focused mainly on estimating measures of risk for calculating surplus capital or on mathematically evaluating properties (axioms) that such measures of risk satisfy. Closest to our work are Froot, Scharfstein & Stein (1993), Froot & Stein (1998) and Froot (2007) that build a framework for analyzing among other things the capital structure decision of financial institutions. However, in that strand of research the product market side (policyholders in this case) is modeled in reduced-form. Also, there exist important differences between the objective function used in the above-mentioned strand of research and the one that we use.

[8]To be strict we should note that also our approach, while structural in both supply and demand sides, is not entirely structural because the convex costs of external finance are modeled in reduced-form; we do so to restrict attention to the main determinants and our model can easily be generalized into an entirely structural model.

a set Ω of states of nature. One state is the true state. We consider the triplet (probability space) $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathcal{F}$ denoting the tribe (σ-algebra) of subsets of Ω that are events including the empty set $\emptyset$, and with $\mathbb{P}$ denoting a probability measure assigning to any event $A \in \mathcal{F}$ the probability $\mathbb{P}[A]$.

We consider the set of dates $\mathcal{T} = \{0, 1, 2, \ldots\}$. The flow of information is represented by the filtration $\mathbb{F}_T = \{\mathcal{F}_0, \ldots, \mathcal{F}_T\}$, $T \in \mathcal{T}$, which is a collection of subtribes of $\mathcal{F}$; as usual, we assume that $\mathcal{F}_s \subseteq \mathcal{F}_t$ whenever $s \leq t$ and that $\mathcal{F}_t \subseteq \mathcal{F}$ for each t, with $s, t \in \mathcal{T}$. A process $\{X_t\}_{t \geq 0}$ is said to be *adapted* if X_t is a random variable with respect to $(\Omega, \mathcal{F}_t)$, that is, if X_t is $\mathcal{F}_t$-measurable, $t \in \mathcal{T}$.

The insurance company is faced with the *capital structure decision.* This decision takes place at $t = 0$. We denote by k the target capital that the insurer decides to raise, and assume $k \geq 0$. The shareholder is considered to be risk-neutral and to discount time at rate ρ. The target capital is assumed to be entirely equity-financed and to earn the risk-free rate r per time period.

Holding equity capital involves a deadweight cost (due to taxes, possible agency problems between shareholders and management, and possible lemon costs). We assume it is constant and linear in the amount of capital, and denote it by γ per unit of capital per time period.

The policyholder has a strictly increasing and concave utility function U defined on the set L_+ of adapted non-negative consumption processes. We assume that the preferences are time additive, satisfying

$$U(c) := \mathbb{E}\left[\sum_{t \in \mathcal{T}} u_t(c_t) dt\right], \qquad c \in L_+, \tag{1}$$

for some $u_t : \mathbb{R}_+ \to \mathbb{R}$, $t \geq 0$, with $u_{t-1}(x) = \delta_t u_t(x)$, $t > 0$, for some number δ_t, $0 < \delta_t < 1$. The policyholder is endowed with a wage process w in L_+ We assume that $w_0 = 0$, and that w_t is degenerate at $w \geq 0$, $t \in \mathcal{T}$.

Let us temporarily restrict attention to the first time period. Within this time period, and similarly for all future time periods, the policyholder is exposed to a risk $X_1 \equiv X$ defined by

$$X := \mathbb{I}_{\text{Loss in time period 1}} Y,$$

where the random variable Y has a distribution function F_Y supported on $(0, +\infty)$ and represents the severity of the loss; and $\mathbb{I}_A$ denotes the indicator function of event A, which takes the value 1 if the event A takes place and 0 otherwise. We let $\mathbb{P}[\mathbb{I}_{\text{Loss in time period 1}} = 1] = p$ and assume that Y and $\mathbb{I}_{\text{Loss in time period 1}}$ are independent. We assume that losses are incurred by the policyholder and settled by the insurer at the *end* of each time period. Furthermore, we assume that both the policyholder and the insurer have complete information (on U, w, p, F_Y, k, γ, ρ, r and c (introduced below)).

We assume that the policyholder observes k and acts rationally (that is, according to its expected utility preferences). Then, the policyholder is willing to cede the risk X to the insurer at a premium π^* satisfying $\pi^* \leq \pi_X(k)$, where $\pi_X(k)$ is the equivalent utility premium.[9] Assuming that the insurance premium is paid at the end of each time period, and that there is full consumption and no savings on the policyholder behalf, the equivalent utility premium $\pi_X(k)$ solves:

$$\begin{aligned}\mathbb{E}[u_1(w - X)] = \mathbb{E}[u_1(w - \pi_X(k)) \mid \text{no insurer default}]\mathbb{P}[\text{no insurer default}] \\ + \mathbb{E}[u_1(w - \pi_X(k) - (1 - \Theta)X) \mid \text{insurer default}] \\ \times \mathbb{P}[\text{insurer default}]. \end{aligned} \tag{2}$$

Here, Θ denotes the (random) fraction of coverage in the case of default.

To specify in further detail the capital optimization problem that the insurer shareholders face, we first introduce some concepts. The *charter value*, denoted by V^{Charter}, is the expected sum of time-discounted benefits that accrues to the insurer's owners (shareholders) from its future operations. In times of financial distress, the charter value, which we determine endogenously, is at the core of the continuation decision that the insurer shareholders face.

Furthermore, we assume that there are *convex costs* to external financing represented by an exogenously given convex and increasing function $c : \mathbb{R} \to \mathbb{R}_0^+$, satisfying $c(x) = 0$, whenever $x \leq 0$. These convex costs of external financing are due to convex effort costs of e.g., screening or monitoring. The refinancing covers both the loss from the policyholder's claim and the deadweight cost involved in holding capital, which varies linearly with the amount of capital, less premium income.

We assume that the insurer anticipates rational behavior of the policyholders and sets $\pi^* = \pi_X(k)$. That is, we assume that the monopolistic insurer has all the bargaining power. Then, the random payoff at time $t = 1$ of the insurer's position is

$$f_1(k; \gamma, r, \rho, p, F_Y, U, w) := ke^{-\gamma+r} + \pi_X(k) - X.$$

At time $t = 1$, we distinguish between three cases:

[9] Assuming that $\{X_t, t \in \mathcal{T}\}$ is a series of i.i.d. random variables, the premium will be the same in case the policyholder would consider multiple periods due to his preferences being time additive of the form (1).

1. **Continuation without external financing:** $f_1(k) \geq k$: the $t = 1$ capital position of the insurer exceeds the target capital level. In this case $D = f_1(k) - k$ is paid out as dividend to the shareholders, the policyholder's claim (if any) is covered and the insurer continues its activities.

2. **Continuation with external financing:** $f_1(k) < k$ & $c(k - f_1(k)) \leq V^{\text{Charter}}$: the $t = 1$ capital position of the insurer is less than the target capital level but the convex costs of financing the shortage (target capital gap) is less than the charter value. In this case $k - f_1(k)$ is raised through external financing (that is, there is a negative dividend payment of $D = -c(k - f_1(k))$), the policyholder's claim is covered and the insurer continues its activities.

3. **Default:** $f_1(k) < k$ & $c(k - f_1(k)) > V^{\text{Charter}}$: the $t = 1$ capital position of the insurer is less than the target capital level and the convex costs of financing the shortage (target capital gap) exceeds the charter value. In this case the insurer stops its activities (default) and the policyholder's claim may or may not be fully covered; only the fraction $\Theta = \min\left\{1, \frac{ke^{-\gamma+r}+\pi_X(k)}{X}\right\}$ is covered and a final dividend payment of $D = ke^{-\gamma+r} + \pi_X(k) - \Theta X = \max\{0, ke^{-\gamma+r} + \pi - X\}$ is paid out.

The same structure applies to all future time periods.

Notice that default might take place when the surplus capital has not fully been depleted. This is the case when the convex costs of closing the shortage (target capital gap), i.e., the costs of setting the capital back to its target level, are too large. Notice furthermore that at this stage no conditions are imposed on the size of k, apart from being non-negative.

We then state the insurer's capital optimization problem:

$$\max_{k \geq 0} \left(V^{\text{Charter}}(k) - k\right) \tag{3}$$
$$\text{s.t.} \qquad V^{\text{Charter}}(k) \geq k;$$

with

$$V^{\text{Charter}}(k) := \mathbb{E}\left[\sum_{t=1}^{\tau} e^{-\rho t} D_t\right]. \tag{4}$$

Here, τ denotes the time of default, that is,

$$\tau := \inf\{t \in \mathcal{T} : f_t(k) < k \text{ \& } c(k - f_t(k)) > V^{\text{Charter}}\}. \tag{5}$$

Figure 2 summarizes the sequence of events. The base model captures the essence of our problem, while remaining tractable.

3 Solution to the Base Model

In this section, we solve our base model.

3.1 General Structure of the Solution

Notice first the stationarity (i.e., repetition) of the insurer's capital optimization problem in the base model: at the end of each time period, conditionally upon continuation, the capital is set back to its target ($t = 0$) level, which means that the insurer returns to its $t = 0$ state. It means that Bellman's value function is constant over time. Therefore, we often suppress indices $t = 1, 2, \ldots$ in the sequel.

One easily verifies that $F_X(x) := \mathbb{P}[X \leq x]$ satisfies

$$F_X(x) = \begin{cases} 0, & x < 0; \\ 1 - p, & x = 0; \\ 1 - p + pF_Y(x), & x > 0. \end{cases}$$

The insurer decides not to continue in case (cf. Case 3: Default)

$$c(k - f(k)) > V^{\text{Charter}}. \tag{6}$$

The event (6) is equivalent to

$$X > ke^{-\gamma+r} - k + \pi_X(k) + c^{-1}(V^{\text{Charter}}),$$

with c^{-1} the generalized inverse of c. Hence, the probability of default, denoted by q_1, is equal to

$$q_1 = \overline{F}_X \left(ke^{-\gamma+r} - k + \pi_X(k) + c^{-1}(V^{\text{Charter}})\right), \tag{7}$$

where, as usual, $\overline{F}_X(x) := 1 - F_X(x)$. Similarly, one can verify that the probability of a non-negative dividend payment, denoted by q_2, is given by

$$q_2 = F_X \left(ke^{-\gamma+r} - k + \pi_X(k)\right). \tag{8}$$

Notice that $q_2 = 0$ (and hence $V^{\text{Charter}} \leq 0$ and $q_1 = 1$) whenever $ke^{-\gamma+r} - k + \pi_X(k) < 0$. Finally, the probability of no default but negative dividend, denoted by q_3, is $q_3 = 1 - q_1 - q_2$.

Suppose $k > 0$ is known. Furthermore, suppose that we know the value of V^{Charter} used in the continuation decision made by the insurer at the end of each time period. Let us denote this value by $\widetilde{V}^{\text{Charter}}$ (and let us denote the corresponding probabilities by $\widetilde{q}_1, \widetilde{q}_2$ and $\widetilde{q}_3$). We will find the corresponding value of V^{Charter} in (4), which equals the expected sum of dividend payments to the shareholder up to and including random time τ of default, discounted by the time discount rate ρ. Then, by iteration, we will seek for $\widetilde{V}^{\text{Charter}}$ such that $\widetilde{V}^{\text{Charter}} = V^{\text{Charter}}$. Clearly, $\widetilde{V}^{\text{Charter}}$ should satisfy the feasibility condition that $\widetilde{V}^{\text{Charter}} > 0$.

Equation (2), determining the equivalent utility premium, can be rewritten as follows:

$$\begin{aligned}
&(1-p)u(w) + p\mathbb{E}[u(w-Y)] \\
=&F_X\left(ke^{-\gamma+r} - k + \widetilde{\pi} + c^{-1}(\widetilde{V}^{\text{Charter}})\right) u(w-\widetilde{\pi}) \\
&+ \overline{F}_X\left(ke^{-\gamma+r} - k + \widetilde{\pi} + c^{-1}(\widetilde{V}^{\text{Charter}})\right) \\
&\times \mathbb{E}\left[u(w-\widetilde{\pi} - X + \min\{ke^{-\gamma+r} + \widetilde{\pi}, X\}) | X > ke^{-\gamma+r} - k + \widetilde{\pi} + c^{-1}(\widetilde{V}^{\text{Charter}})\right]. \quad (9)
\end{aligned}$$

At the end of each period, either the insurer continues (with probability $\widetilde{q}_2 + \widetilde{q}_3$) or the insurer defaults (with probability $\widetilde{q}_1$). These probabilities do not depend on time. To verify this, recall the stationarity (i.e., repetition) of the problem. Therefore, the time of default has a Geometric distribution with probability of "success" (in this case, default) equal to $\widetilde{q}_1$. In short: $\widetilde{\tau} \sim \text{GEO}(\widetilde{q}_1)$. We note that the charter value in (4) can be expressed as

$$V^{\text{Charter}} = \mathbb{E}\left[\mathbb{E}\left[\sum_{t=1}^{\widetilde{\tau}} e^{-\rho t}\widetilde{D}_t | \widetilde{\tau}\right]\right],$$

where

$$\mathbb{E}[\widetilde{D}_t|A] = \begin{cases} 0, & A = \{t > \widetilde{\tau}\}; \\ \mathbb{E}[\max\{0, ke^{-\gamma+r} + \widetilde{\pi} - X\} | X > ke^{-\gamma+r} - k + \widetilde{\pi} + c^{-1}(\widetilde{V}^{\text{Charter}})], & A = \{t = \widetilde{\tau}\}; \\ \frac{\widetilde{q}_2}{1-\widetilde{q}_1}\left(ke^{-\gamma+r} - k + \widetilde{\pi} - \mathbb{E}[X | ke^{-\gamma+r} - k + \widetilde{\pi} - X \geq 0]\right) & \\ \quad + \frac{\widetilde{q}_3}{1-\widetilde{q}_1}\mathbb{E}[-c(k - ke^{-\gamma+r} - \widetilde{\pi} + X) | ke^{-\gamma+r} - k + \widetilde{\pi} - X < 0 & \\ \quad \& \ X \leq ke^{-\gamma+r} - k + \widetilde{\pi} + c^{-1}(\widetilde{V}^{\text{Charter}})], & A = \{t < \widetilde{\tau}\}. \end{cases}$$

Hence, we find that

$$\begin{aligned}
V^{\text{Charter}} =& \widetilde{q}_1 e^{-\rho}\mathbb{E}[\widetilde{D}_{\widetilde{\tau}}] \\
&+ \widetilde{q}_1(1-\widetilde{q}_1)\left(e^{-\rho}\mathbb{E}[\widetilde{D}_t | t < \widetilde{\tau}] + e^{-2\rho}\mathbb{E}[\widetilde{D}_{\widetilde{\tau}}]\right) \\
&+ \widetilde{q}_1(1-\widetilde{q}_1)^2\left(e^{-\rho}\mathbb{E}[\widetilde{D}_t | t < \widetilde{\tau}] + e^{-2\rho}\mathbb{E}[\widetilde{D}_t | t < \widetilde{\tau}] + e^{-3\rho}\mathbb{E}[\widetilde{D}_{\widetilde{\tau}}]\right) \\
&+ \dots.
\end{aligned}$$

By virtue of the geometric series, we thus obtain (notice that convergence is always satisfied provided $\rho > 0$)

$$V^{\text{Charter}} = \widetilde{q}_1 \left(\sum_{i=1}^{+\infty} (1-\widetilde{q}_1)^i \left(\sum_{j=1}^{i} \mathbb{E}[\widetilde{D}_t | t < \widetilde{\tau}] e^{-\rho j} + \mathbb{E}[\widetilde{D}_\tau] e^{-\rho(i+1)} \right) \right) + \widetilde{q}_1 \mathbb{E}[\widetilde{D}_\tau] e^{-\rho}$$

$$= \widetilde{q}_1 \left(\frac{\mathbb{E}[\widetilde{D}_t | t < \widetilde{\tau}] e^{-\rho}}{1-e^{-\rho}} \left(\frac{1-\widetilde{q}_1}{\widetilde{q}_1} - \frac{(1-\widetilde{q}_1)e^{-\rho}}{1-(1-\widetilde{q}_1)e^{-\rho}} \right) + \frac{\mathbb{E}[\widetilde{D}_\tau] e^{-\rho}}{1-(1-\widetilde{q}_1)e^{-\rho}} \right). \quad (10)$$

In sum, the optimization procedure is as follows: For a given $k > 0$ and a given $\widetilde{V}^{\text{Charter}}$, we derive the corresponding $\widetilde{\pi}$ using (9) and derive the corresponding V^{Charter} using (10). Then, by iteratively changing $\widetilde{V}^{\text{Charter}}$ we seek for the value of $\widetilde{V}^{\text{Charter}}$ that satisfies $\widetilde{V}^{\text{Charter}} = V^{\text{Charter}}$. Finally, we maximize V^{Charter} over all possible values of k. We thus solve our model backwards.

The mere role of surplus capital is to ensure that in the case of default the policyholders still receive most of their claims. Under the assumption of an optimal target level of capital that is (to be) reestablished after each period, the optimal amount of surplus capital is in essence determined by the risk aversion of the policyholder. The more risk-averse the policyholder, the more he is willing to pay for safety about receiving his claim and the more (costly) capital the insurer is willing to raise. A large amount of surplus capital does not necessarily imply a high probability that the insurer continues its activities; it does guarantee that a significant portion of the claim is always settled.

Adopting the widely applied Value-at-Risk to determine the level of surplus capital means that policyholders are ensured that with a pre-specified degree of certainty their claims will be settled. This does not mean that the insurer will continue its activities with the same degree of certainty nor does it guarantee a specified claim settlement in the case of default.

An exogenously imposed Value-at-Risk does not reflect the shareholder's preferences but rather represents the viewpoint of the regulatory authority that is assigned the legislative power with respect to the protection of policyholders, guaranteeing claim settlement with a certain degree of probability.

3.2 Preliminary Analysis: An Irrelevance Theorem

We first consider the simplified case in which there is no coverage in case of default, that is, we suppose $\Theta \equiv 0$ rather than $\Theta = \min\left\{1, \frac{ke^{-\gamma+r}+\pi_X(k)}{X}\right\}$.

Theorem 3.1 *Suppose $\Theta \equiv 0$, i.e., the coverage fraction in case of default is zero. Furthermore, suppose that holding capital is costless ($\gamma = r$) and that the rate at which shareholders discount time is zero ($\rho = 0$). Then the amount of capital is irrelevant.*

Proof One easily verifies that in this case $\widetilde{\pi}(k) = \widetilde{\pi}$, $\widetilde{q}_1(k) = \widetilde{q}_1$, $\widetilde{q}_2(k) = \widetilde{q}_2$, $\widetilde{q}_3(k) = \widetilde{q}_3$, $\mathbb{E}[\widetilde{D}_t](k) = \mathbb{E}[\widetilde{D}_t]$, for all t, and $V^{\text{Charter}}(k) = V^{\text{Charter}}$. This proves the stated result. □

The intuition behind this result is as follows. Suppose that holding capital is costless. In which of the three cases considered is holding a large amount of capital beneficial? When the company defaults, because in that case the coverage fraction in case of default will be large.

Corollary 3.1 *Suppose $\Theta \equiv 0$ and $\gamma = r$ but $\rho > 0$. Then the optimal amount of capital is 0.*

Proof In this case $\widetilde{\pi}(k) = \widetilde{\pi}$, $\widetilde{q}_1(k) = \widetilde{q}_1$, $\widetilde{q}_2(k) = \widetilde{q}_2$, $\widetilde{q}_3(k) = \widetilde{q}_3$ and $\mathbb{E}[\widetilde{D}_t|t < \widetilde{\tau}](k) = \mathbb{E}[\widetilde{D}_t|t < \widetilde{\tau}]$. Furthermore,

$$V^{\text{Charter}} - k = \mathbb{E}\left[\mathbb{E}\left[\sum_{t=1}^{\widetilde{\tau}-1} e^{-\rho t}\widetilde{D}_t|\widetilde{\tau}\right]\right] + \mathbb{E}[e^{-\rho\widetilde{\tau}}(k + \widetilde{\pi})] - k.$$

Under the stated assumptions, $\mathbb{E}[\mathbb{E}[\sum_{t=1}^{\widetilde{\tau}-1} e^{-\rho t}\widetilde{D}_t|\widetilde{\tau}]]$ and $\mathbb{E}[e^{-\rho\widetilde{\tau}}\widetilde{\pi}]$ do not depend on k. This means that, whenever $\rho > 0$, $V^{\text{Charter}} - k$ is decreasing in k. This proves the stated result. □

3.3 Degenerate Loss Severity

Suppose that Y is degenerate at $Y = L$, $L > 0$, so that

$$F_X(x) = \begin{cases} 0, & x < 0; \\ 1 - p, & 0 \le x < L; \\ 1, & x \ge L. \end{cases}$$

Furthermore, let $\rho > 0, \gamma > r$. Then it is optimal for the insurer (or rather: its shareholder) to either always continue its activities or always default upon the first occurrence of a positive loss, depending on the relevant economic parameters.

In the case that it is optimal to always default upon the first occurrence of a positive loss, one easily verifies that the equivalent utility premium is equal to 0, hence the charter value and the optimal amount of surplus capital equal 0 too. This means essentially that no insurance contract is written.

Next, consider the case that it is optimal to always continue activities. A loss can be financed either out of internal or out of external funds. In the presence of a positive deadweight cost of capital, internal financing is however only possible when the premium paid is larger than or equal to L, which is not the case in the expected utility framework as long as $p < 1$. This means that the loss will need to be settled by external finance. The equivalent utility premium is given by

$$\pi = w - u^{-1}((1-p)u(w) + pu(w-L)), \tag{11}$$

and does not depend on k in this case. The charter value is given by

$$V^{\text{Charter}} = \frac{e^{-\rho}}{1-e^{-\rho}} \left((1-p)(ke^{-\gamma+r} - k + \pi) - pc(-ke^{-\gamma+r} + k - \pi + L)\right). \tag{12}$$

We find that in this case the objective function $V^{\text{Charter}} - k$ is decreasing in k so that it is optimal to set $k = 0$. External financing the claim and continuing activities is feasible and optimal whenever

$$c(L-\pi) \le V^{\text{Charter}}.$$

Notice that whenever a feasible and optimal solution does not exist a feasible but suboptimal solution does not exist either.

The intuition behind this solution is as follows: recall that the mere role of capital is to ensure that in the case of default still most of the claim is settled. This implies that in case the insurer always continues its activities the optimal amount of capital is 0.

We formulate these results in the following theorem, characterizing the solution to the insurer's capital optimization problem under degenerate loss severity:

Theorem 3.2 *If*

$$c(L-\pi) \le V^{\text{Charter}}, \tag{13}$$

where π is given in (11) and V^{Charter} is given in (12) with $k = 0$, then it is optimal for the shareholders of the insurance company to always continue activities and the optimal amount of surplus capital is $k = 0$. If (13) does not hold, then it is optimal not to enter into an insurance contract.

It becomes apparent that when the agent facing the risk X is sufficiently risk averse (implying a large equivalent utility premium) and the shareholder's time discount rate as well as the convex costs of external financing are sufficiently small the optimal solution is to always continue activities and set $k = 0$.

3.4 Two-Point Loss Severity

We generalize the setup of Section 3.3, now considering a two-point loss severity distribution so that

$$F_X(x) = \begin{cases} 0, & x < 0; \\ 1 - p_1 - p_2, & 0 \leq x < l; \\ 1 - p_2, & l \leq x < L; \\ 1, & x \geq L; \end{cases}$$

where $0 \leq p_1, p_2, p \leq 1$ with $p := p_1 + p_2$ and $0 < l < L$. With $L >> l$ and $p_2 << p_1$ one may think of this loss severity distribution as representing a probabilistic mixture between a low-loss-high-frequency and a high-loss-low-frequency situation. Furthermore, let $\rho > 0$, $\gamma > r$.

Then one of the following three solutions is found for the insurance company:

1. Always continue activities;

2. Continue activities except when $X = L$;

3. Always default upon the first occurrence of a positive loss.

In the case that it is optimal to always default upon the first occurrence of a positive loss, one easily verifies (as in Section 3.3) that the equivalent utility premium is equal to 0, hence the charter value and the optimal amount of surplus capital are equal to 0 too. It essentially means that no insurance contract is written.

Next, consider the case that it is optimal to always continue activities. A loss can be financed either out of internal or out of external funds. In the presence of a positive deadweight cost of capital, internal financing is only possible when the premium paid is larger than or equal to Y. In the expected utility framework always $\pi < L$ as long as $p < 1$. This means that the loss will need to be settled by external finance whenever $X = L$. If $X = l$, internal financing is carried out when

$$ke^{-\gamma+r} - k + \pi - l \geq 0.$$

If however

$$ke^{-\gamma+r} - k + \pi - l < 0,$$

external finance is attracted with associated convex costs

$$c(-ke^{-\gamma+r} + k - \pi + l).$$

The equivalent utility premium is given by

$$\pi = w - u^{-1}((1-p)u(w) + p_1 u(w-l) + p_2 u(w-L)), \tag{14}$$

and does not depend on k in this case. The charter value is given by

$$\begin{aligned} V^{\text{Charter}} = \frac{e^{-\rho}}{1-e^{-\rho}} \big(& (1-p)(ke^{-\gamma+r} - k + \pi) + p_1 \big((ke^{-\gamma+r} - k + \pi - l) 1_{\{(ke^{-\gamma+r}-k+\pi-l)\geq 0\}} \\ & - c(-ke^{-\gamma+r} + k - \pi + l) 1_{\{(ke^{-\gamma+r}-k+\pi-l)<0\}} \big) \\ & - p_2 c(-ke^{-\gamma+r} + k - \pi + L) \big) . \end{aligned} \tag{15}$$

We find that in this case the objective function $V^{\text{Charter}} - k$ is decreasing in k so that it is optimal to set $k = 0$. Always continuing activities is feasible whenever

$$c(L - \pi) \leq V^{\text{Charter}}.$$

Finally, consider the case that it is optimal to continue activities except when $X = L$. A loss can be financed either out of internal or out of external funds. If $X = l$, internal financing is carried out when

$$ke^{-\gamma+r} - k + \pi - l \geq 0.$$

If however

$$ke^{-\gamma+r} - k + \pi - l < 0,$$

external finance is attracted with associated convex costs

$$c(-ke^{-\gamma+r} + k - \pi + l).$$

The equivalent utility premium solves in this case

$$\begin{aligned} (1-p)u(w) + & p_1 u(w-l) + p_2 u(w-L) \\ & = (1-p_2)u(w-\pi) + p_2 u(w - \pi - \max\{0, L - ke^{-\gamma+r} - \pi\}). \end{aligned} \tag{16}$$

It follows that as long as $k \leq (L-\pi)e^{\gamma-r}$, $\pi(k)$ is non-decreasing in k. Let

$$b_0 := (1-p)u(w) + p_1 u(w-l) + p_2 u(w-L).$$

The charter value is given by

$$
\begin{aligned}
V^{\text{Charter}} = p_2 \Bigg(\Bigg(& \frac{1-p}{1-p_2}(ke^{-\gamma+r} - k + \pi)\frac{e^{-\rho}}{1-e^{-\rho}} \\
& + \frac{p_1}{1-p_2}\left((ke^{-\gamma+r} - k + \pi - l)1_{\{(ke^{-\gamma+r}-k+\pi-l)\geq 0\}}\right. \\
& \left. -c(-ke^{-\gamma+r} + k - \pi + l)1_{\{(ke^{-\gamma+r}-k+\pi-l)<0\}}\right)\frac{e^{-\rho}}{1-e^{-\rho}}\Bigg) \\
& \times \left(\frac{1-p_2}{p_2} - \frac{(1-p_2)e^{-\rho}}{1-(1-p_2)e^{-\rho}}\right) \\
& + \max\{ke^{-\gamma+r} + \pi - L, 0\}\frac{e^{-\rho}}{1-(1-p_2)e^{-\rho}}\Bigg). \qquad (17)
\end{aligned}
$$

Continuing activities except when $X = L$ is feasible whenever

$$c(-ke^{-\gamma+r} + k - \pi + l) \leq V^{\text{Charter}}. \qquad (18)$$

It is feasible and optimal whenever the corresponding charter value satisfies in addition

$$V^{\text{Charter}} - k \geq V_0^{\text{Charter}},$$

where V_0^{Charter} is given in (15) with $k = 0$. To find the optimal amount of surplus capital we will solve $\frac{\partial V^{\text{Charter}}}{\partial k} - 1 = 0$ w.r.t. k. To this end, let us first derive $\frac{\partial \pi}{\partial k}$. By the implicit function theorem $\frac{\partial \pi}{\partial k} = -\frac{\partial \text{RHS}/\partial k}{\partial \text{RHS}/\partial \pi}$, where RHS denotes the right-hand side of (16). Assume $L - ke^{-\gamma+r} - \pi \geq 0$ (or equivalently $k \leq (L-\pi)e^{\gamma-r}$) holds. Holding more capital (than $(L-\pi)e^{\gamma-r}$) is suboptimal because it will not lead to an increase of the equivalent utility premium but does lead to an increase in the deadweight cost. Therefore we can make this assumption without loss of generality. We then find that

$$\frac{\partial \pi}{\partial k} = \frac{p_2 u'(w - L + ke^{-\gamma+r})e^{-\gamma+r}}{(1-p_2)u'(w-\pi)}.$$

Upon differentiation of (17) we find that

$$
\begin{aligned}
\frac{\partial V^{\text{Charter}}}{\partial k} = & p_2 \left(\frac{1-p}{1-p_2}\left(e^{-\gamma+r} - 1 + \frac{\partial \pi}{\partial k}\right)\frac{e^{-\rho}}{1-e^{-\rho}} \right. \\
& + \frac{p_1}{1-p_2}\left(\left(e^{-\gamma+r} - 1 + \frac{\partial \pi}{\partial k}\right)1_{\{(ke^{-\gamma+r}-k+\pi-l)\geq 0\}}\right. \\
& \left.\left. -c'(-ke^{-\gamma+r} + k - \pi + l)\left(-e^{-\gamma+r} + 1 - \frac{\partial \pi}{\partial k}\right)1_{\{(ke^{-\gamma+r}-k+\pi-l)<0\}}\right)\frac{e^{-\rho}}{1-e^{-\rho}}\right) \\
& \times \left(\frac{1-p_2}{p_2} - \frac{(1-p_2)e^{-\rho}}{1-(1-p_2)e^{-\rho}}\right).
\end{aligned}
$$

We distinguish between two cases: (i) $(ke^{-\gamma+r} - k + \pi - l) \geq 0$ and (ii) $(ke^{-\gamma+r} - k + \pi - l) < 0$. In the first case (18) is necessarily satisfied. Furthermore, in this case $\frac{\partial V^{\text{Charter}}}{\partial k}$ reduces to

$$\begin{aligned}
\frac{\partial V^{\text{Charter}}}{\partial k} =& p_2 \left(e^{-\gamma+r} - 1 + \frac{p_2 u'(w - L + ke^{-\gamma+r})e^{-\gamma+r}}{(1-p_2)u'(w-\pi)} \right) \\
& \times \frac{e^{-\rho}}{1-e^{-\rho}} \left(\frac{1-p_2}{p_2} - \frac{(1-p_2)e^{-\rho}}{1-(1-p_2)e^{-\rho}} \right) \\
=& d_1 + d_2 \frac{u'(w - L + ke^{-\gamma+r})}{u'(w-\pi)},
\end{aligned}$$

with

$$\pi = w - u^{-1} \left(\frac{b_0}{1-p_2} - \frac{p_2}{1-p_2} u(w - L + ke^{-\gamma+r}) \right), \tag{19}$$

and

$$\begin{aligned}
d_1 &= p_2(e^{-\gamma+r} - 1) \frac{e^{-\rho}}{1-e^{-\rho}} \left(\frac{1-p_2}{p_2} - \frac{(1-p_2)e^{-\rho}}{1-(1-p_2)e^{-\rho}} \right); \\
d_2 &= \frac{p_2^2 e^{-\gamma+r}}{1-p_2} \frac{e^{-\rho}}{1-e^{-\rho}} \left(\frac{1-p_2}{p_2} - \frac{(1-p_2)e^{-\rho}}{1-(1-p_2)e^{-\rho}} \right).
\end{aligned}$$

We then solve w.r.t. k the equation

$$\begin{aligned}
1 &= \frac{\partial V^{\text{Charter}}}{\partial k} \\
&= d_1 + d_2 \frac{u'(w - L + ke^{-\gamma+r})}{u'\left(u^{-1} \left(\frac{b_0}{1-p_2} - \frac{p_2}{1-p_2} u(w - L + ke^{-\gamma+r}) \right) \right)}.
\end{aligned} \tag{20}$$

Knowing the optimal value for k means that we can find π from (19) and V^{Charter} from (17) keeping in mind that $L - ke^{-\gamma+r} - \pi \geq 0$ and that $(ke^{-\gamma+r} - k + \pi - l) \geq 0$.

In the second case $\frac{\partial V^{\text{Charter}}}{\partial k}$ reduces to

$$\begin{aligned}
\frac{\partial V^{\text{Charter}}}{\partial k} =& p_2 \left(\frac{1-p}{1-p_2} \left(e^{-\gamma+r} - 1 + \frac{p_2 u'(w-L+ke^{-\gamma+r})e^{-\gamma+r}}{(1-p_2)u'(w-\pi)} \right) \frac{e^{-\rho}}{1-e^{-\rho}} \right. \\
&- \frac{p_1}{1-p_2} c'(-ke^{-\gamma+r} + k - \pi + l) \\
&\left. \times \left(-e^{-\gamma+r} + 1 - \frac{p_2 u'(w-L+ke^{-\gamma+r})e^{-\gamma+r}}{(1-p_2)u'(w-\pi)} \right) \frac{e^{-\rho}}{1-e^{-\rho}} \right) \\
&\times \left(\frac{1-p_2}{p_2} - \frac{(1-p_2)e^{-\rho}}{1-(1-p_2)e^{-\rho}} \right) \\
=& e_1 + e_2 \frac{u'(w-L+ke^{-\gamma+r})}{u'\left(w-\pi\right)} \\
&- e_3 c'(-ke^{-\gamma+r} + k - \pi + l) \\
&+ e_4 c'(-ke^{-\gamma+r} + k - \pi + l) \frac{u'(w-L+ke^{-\gamma+r})}{u'\left(w-\pi\right)},
\end{aligned}$$

with π given in (19) and

$$\begin{aligned}
e_1 &= p_2 \frac{1-p}{1-p_2}(e^{-\gamma+r} - 1) \frac{e^{-\rho}}{1-e^{-\rho}} \left(\frac{1-p_2}{p_2} - \frac{(1-p_2)e^{-\rho}}{1-(1-p_2)e^{-\rho}} \right); \\
e_2 &= \frac{p_2^2(1-p)e^{-\gamma+r}}{(1-p_2)^2} \frac{e^{-\rho}}{1-e^{-\rho}} \left(\frac{1-p_2}{p_2} - \frac{(1-p_2)e^{-\rho}}{1-(1-p_2)e^{-\rho}} \right); \\
e_3 &= \frac{p_1 p_2}{1-p_2}(-e^{-\gamma+r} + 1) \frac{e^{-\rho}}{1-e^{-\rho}} \left(\frac{1-p_2}{p_2} - \frac{(1-p_2)e^{-\rho}}{1-(1-p_2)e^{-\rho}} \right); \\
e_4 &= \frac{p_1 p_2^2 e^{-\gamma+r}}{(1-p_2)^2} \frac{e^{-\rho}}{1-e^{-\rho}} \left(\frac{1-p_2}{p_2} - \frac{(1-p_2)e^{-\rho}}{1-(1-p_2)e^{-\rho}} \right).
\end{aligned}$$

Notice that $e_1 < 0$ and $e_2, e_3, e_4 > 0$. We then solve w.r.t. k the equation

$$\begin{aligned}
1 =& \frac{\partial V^{\text{Charter}}}{\partial k} \\
=& e_1 + e_2 \frac{u'(w-L+ke^{-\gamma+r})}{u'\left(u^{-1}\left(\frac{b_0}{1-p_2} - \frac{p_2}{1-p_2} u(w-L+ke^{-\gamma+r}) \right) \right)} \\
&- e_3 c'\left(-ke^{-\gamma+r} + k - \left(w - u^{-1}\left(\frac{b_0}{1-p_2} - \frac{p_2}{1-p_2} u(w-L+ke^{-\gamma+r}) \right) \right) + l \right) \\
&+ e_4 c'\left(-ke^{-\gamma+r} + k - \left(w - u^{-1}\left(\frac{b_0}{1-p_2} - \frac{p_2}{1-p_2} u(w-L+ke^{-\gamma+r}) \right) \right) + l \right) \\
&\times \frac{u'(w-L+ke^{-\gamma+r})}{u'\left(u^{-1}\left(\frac{b_0}{1-p_2} - \frac{p_2}{1-p_2} u(w-L+ke^{-\gamma+r}) \right) \right)}.
\end{aligned} \tag{21}$$

Knowing the optimal value for k means that we can find π from (19) and V^{Charter} from (17) keeping in mind that $L - ke^{-\gamma+r} - \pi \geq 0$ and that now $(ke^{-\gamma+r} - k + \pi - l) < 0$.

We formulate these results in the following theorem, characterizing the solution to the insurer's capital optimization problem under a two-point loss severity:

Theorem 3.3 *Let k_1 and k_2 be the solution to (20) and (21), respectively, let π_1 and π_2 be the corresponding equivalent utility premium given in (19) and let V_1^{Charter} and V_2^{Charter} be the corresponding charter value given in (17), keeping in mind that it is presupposed that $L - ke^{-\gamma+r} - \pi \geq 0$ and that in the first case $k_1e^{-\gamma+r} - k_1 + \pi_1 - l \geq 0$ while in the second case $k_2e^{-\gamma+r} - k_2 + \pi_2 - l < 0$. Furthermore, let π_0 be given in (14) and let V_0^{Charter} be given in (15) with $k = 0$. Among the four possible solutions $(k; \pi, V^{\text{Charter}}) = \{(0; 0, 0), (0; \pi_0, V_0^{\text{Charter}}), (k_1; \pi_1, V_1^{\text{Charter}}), (k_2; \pi_2, V_2^{\text{Charter}})\}$ the optimal solution (triplet) is the one that satisfies the relevant feasibility conditions*

$$\begin{aligned}
c(L - \pi_0) &\leq V_0^{\text{Charter}}; \\
k_1e^{-\gamma+r} - k_1 + \pi_1 - l &\geq 0; \\
c(-k_2e^{-\gamma+r} + k_2 - \pi_2 + l) &\leq V_2^{\text{Charter}}; \\
k_2e^{-\gamma+r} - k_2 + \pi_2 - l &< 0; \\
L - k_ie^{-\gamma+r} - \pi_i &\geq 0, \qquad i = 1, 2;
\end{aligned}$$

and assumes the largest objective function (charter value minus initial equity) among the feasible solutions.

3.5 A Special Case: Exponential Loss Severity and Exponential Utility

In this subsection we derive some explicit results in the special case that the loss severity is exponential and the policyholder's utility function is exponential, too. So, let us assume that the policyholder has a utility function of the form

$$u(x) = 1 - e^{-\alpha x}, \qquad \alpha > 0.$$

As is well-known, an exponential utility function has constant absolute risk aversion (i.e., $-\frac{u''(x)}{u'(x)} = \alpha$). While constant absolute risk aversion is usually thought of as a less plausible description of risk aversion than e.g., constant relative risk aversion, this will not affect our analysis because at the end of each time period, conditionally upon continuation, the policyholder's wealth level returns to its initial level.

We assume that the policyholder's loss, conditionally upon loss occurrence, is exponentially distributed, i.e.,

$$F_Y(y) = 1 - e^{-\frac{y}{\lambda}}, \qquad \lambda > 0.$$

We write $Y \sim \mathrm{Exp}(\lambda)$. Exponential loss severity, with most probability mass near the origin and little probability mass far out in the tail, can be seen as a generalized mimic of a two-point loss severity distribution with a large probability of a small loss l and a small probability of a large loss L.

Below we state some straightforward or well-known results. One easily verifies that in this case

$$F_X(x) = \begin{cases} 0, & x < 0; \\ 1-p, & x = 0; \\ 1-p+p(1-e^{-\frac{x}{\lambda}}), & x > 0. \end{cases}$$

As is well-known,

$$\mathbb{E}[e^{\alpha Y}] = \frac{1}{1-\lambda\alpha}, \qquad \alpha < \frac{1}{\lambda}.$$

Next, for all $c > 0$,

$$\mathbb{E}[X|X \le c] = \frac{1-p}{F_X(c)}0 + \frac{p}{F_X(c)}\mathbb{E}[Y|Y \le c] = \frac{p}{F_X(c)}\mathbb{E}[Y|Y \le c].$$

Since

$$\mathbb{P}[Y \le x|Y \le c] = 1 - \frac{e^{-\frac{x}{\lambda}} - e^{-\frac{c}{\lambda}}}{1-e^{-\frac{c}{\lambda}}}, \qquad x \le c,$$

we find that

$$\mathbb{E}[Y|Y \le c] = \int_0^c \frac{e^{-\frac{x}{\lambda}} - e^{-\frac{c}{\lambda}}}{1-e^{-\frac{c}{\lambda}}}dx = \frac{\lambda - (\lambda+c)e^{-\frac{c}{\lambda}}}{1-e^{-\frac{c}{\lambda}}},$$

and hence

$$\mathbb{E}[X|X \le c] = \frac{p}{F_X(c)}\frac{\lambda - (\lambda+c)e^{-\frac{c}{\lambda}}}{1-e^{-\frac{c}{\lambda}}}.$$

Furthermore, for $0 \le c_1 < c_2$,

$$\mathbb{E}[X|c_1 < X \le c_2] = \mathbb{E}[Y|c_1 < Y \le c_2].$$

Since

$$\begin{aligned} \mathbb{P}[Y \le x|c_1 < Y \le c_2] &= \mathbb{P}[Y \le x - c_1|Y \le c_2 - c_1] \\ &= \frac{1-e^{-\frac{x-c_1}{\lambda}}}{1-e^{-\frac{c_2-c_1}{\lambda}}}, \qquad c_1 < x \le c_2, \end{aligned}$$

we calculate

$$\begin{aligned} \mathbb{E}[Y|c_1 < Y \le c_2] &= c_1 + \int_{c_1}^{c_2} \left(1 - \frac{1-e^{-\frac{x-c_1}{\lambda}}}{1-e^{-\frac{c_2-c_1}{\lambda}}}\right) dx \\ &= c_1 + (c_2 - c_1)\left(1 - \frac{1}{1-e^{-\frac{c_2-c_1}{\lambda}}}\right) + \lambda. \end{aligned}$$

Similarly, for $0 \le c_1 < c_2$,

$$\begin{aligned}
\mathbb{E}[Y^2|c_1 < Y \le c_2] &= \int_{c_1}^{c_2} x^2 d\left(\frac{1-e^{-\frac{x-c_1}{\lambda}}}{1-e^{-\frac{c_2-c_1}{\lambda}}}\right) \\
&= c_2^2 - 2\int_{c_1}^{c_2} x\frac{1-e^{-\frac{x-c_1}{\lambda}}}{1-e^{-\frac{c_2-c_1}{\lambda}}}dx \\
&= c_2^2 - \frac{2}{1-e^{-\frac{c_2-c_1}{\lambda}}}\left(\frac{1}{2}(c_2^2-c_1^2) - (-\lambda^2-\lambda c_2)e^{-\frac{c_2-c_1}{\lambda}} + (-\lambda^2-\lambda c_1)\right).
\end{aligned}$$

Finally, for all $c \ge 0$,

$$\begin{aligned}
\mathbb{E}[e^{\alpha X}|X > c] = \mathbb{E}[e^{\alpha Y}|Y > c] &= \int_c^{+\infty} e^{\alpha y} d\mathbb{P}[Y \le y|Y > c] \\
&= \int_c^{+\infty} e^{\alpha y} d\mathbb{P}[Y \le y - c] = e^{\alpha c}\int_0^{+\infty} e^{\alpha y} d\mathbb{P}[Y \le y] \\
&= e^{\alpha c}\frac{1}{1-\lambda\alpha}, \qquad \alpha < \frac{1}{\lambda}.
\end{aligned}$$

Hence, assuming $\Theta \equiv 0$, (2) becomes

$$\begin{aligned}
&(1-p)(1-e^{-\alpha w}) + p(1-e^{-\alpha w}\frac{1}{1-\lambda\alpha}) \\
&\quad = \left(1 - pe^{-(ke^{-\gamma+r}-k+\widetilde{\pi}+c^{-1}(\widetilde{V}^{\text{Charter}}))/\lambda}\right)(1-e^{-\alpha(w-\widetilde{\pi})}) \\
&\qquad + pe^{-(ke^{-\gamma+r}-k+\widetilde{\pi}+c^{-1}(\widetilde{V}^{\text{Charter}}))/\lambda}\left(1 - e^{-\alpha(w-\widetilde{\pi})}e^{\alpha(ke^{-\gamma+r}-k+\widetilde{\pi}+c^{-1}(\widetilde{V}^{\text{Charter}}))}\frac{1}{1-\lambda\alpha}\right).
\end{aligned} \tag{22}$$

This equation in $\widetilde{\pi}$ is of the form

$$C_1 e^{\alpha\widetilde{\pi}} - C_1C_2e^{\alpha\widetilde{\pi}-\widetilde{\pi}/\lambda} + C_1C_2C_3e^{2\alpha\widetilde{\pi}-\widetilde{\pi}/\lambda} = C_0, \tag{23}$$

with

$$\begin{aligned}
C_0 &= 1-(1-p)(1-e^{-\alpha w}) - p\left(1-e^{-\alpha w}\frac{1}{1-\lambda\alpha}\right); \\
C_1 &= e^{-\alpha w}; \\
C_2 &= pe^{-(ke^{-\gamma+r}-k+c^{-1}(\widetilde{V}^{\text{Charter}}))/\lambda}; \\
C_3 &= \frac{1}{1-\lambda\alpha}e^{\alpha(ke^{-\gamma+r}-k+c^{-1}(\widetilde{V}^{\text{Charter}}))}.
\end{aligned}$$

Setting $x = e^{\alpha\widetilde{\pi}}$, equation (23) reads

$$C_1x - C_1C_2x^{1-\frac{1}{\alpha\lambda}} + C_1C_2C_3x^{2-\frac{1}{\alpha\lambda}} = C_0,$$

which has in general no analytical solution. However, it can easily be solved numerically.

We state the following result:

Proposition 3.1 *Suppose that the policyholder has an exponential utility function and that the policyholder's loss, conditionally upon loss occurrence, is exponentially distributed. Furthermore, let* $\Theta \equiv 0$*. Then*

- $\frac{\partial \widetilde{\pi}}{\partial k} > 0$*, if* $\gamma < r$*;*
- $\frac{\partial \widetilde{\pi}}{\partial k} = 0$*, if* $\gamma = r$*;*
- $\frac{\partial \widetilde{\pi}}{\partial k} < 0$*, if* $\gamma > r$*.*

Proof By the implicit function theorem, $\frac{\partial \widetilde{\pi}}{\partial k} = -\frac{\partial \mathrm{RHS}/\partial k}{\partial \mathrm{RHS}/\partial \widetilde{\pi}}$, where RHS denotes the right-hand side of (22). Under the stated assumptions, one can verify that

$$\frac{\partial \mathrm{RHS}}{\partial k} \begin{cases} > 0, & \gamma < r; \\ = 0, & \gamma = r; \\ < 0, & \gamma > r; \end{cases} \tag{24}$$

and that

$$\frac{\partial \mathrm{RHS}}{\partial \widetilde{\pi}} < 0. \tag{25}$$

To verify (24), we need that

$$ke^{-\gamma+r} - k + \widetilde{\pi} + c^{-1}(\widetilde{V}^{\mathrm{Charter}}) > 0, \tag{26}$$

or equivalently

$$\widetilde{V}^{\mathrm{Charter}} > c(k - ke^{-\gamma+r} - \widetilde{\pi}),$$

which should hold by construction. Verifying (25) is more involved; details are provided in the Appendix. This proves the stated result. □

We state the following theorem:

Theorem 3.4 *Let the conditions of Proposition 3.1 be valid. Furthermore, let* $\rho = 0$*. Then*

- *if holding capital is costly* ($\gamma > r$)*, no capital is optimal;*
- *if holding capital is costless* ($\gamma = r$)*, capital is irrelevant;*
- *if holding capital generates money* ($\gamma < r$)*, infinite capital is optimal.*

Proof Follows from Proposition 3.1. □

Let us now return to the case in which $\Theta = \min\left\{1, \frac{ke^{-\gamma+r}+\pi_X(k)}{X}\right\}$. Notice that

$$(1-\Theta)X = \max\{0, X - (ke^{-\gamma+r} + \pi)\}.$$

Let $c_1 = ke^{-\gamma+r} + \pi$. Then

$$\max\{0, x - c_1\} = \begin{cases} x - c_1, & c_1 < x; \\ 0, & 0 < x \leq c_1. \end{cases}$$

For $c_2 \geq 0$, we calculate

$$\begin{aligned}
\mathbb{E}[e^{\alpha(1-\Theta)X} | X > c_2] =& \int_{\max\{c_1,c_2\}}^{+\infty} e^{\alpha(x-c_1)} d\mathbb{P}[Y \leq x | Y > c_2] \\
=& e^{\alpha(c_2-c_1)} \int_{\max\{c_1-c_2,0\}}^{+\infty} e^{\alpha x} d\mathbb{P}[Y \leq x] \\
=& e^{\alpha(c_2-c_1)} \int_{\max\{c_1-c_2,0\}}^{+\infty} e^{\alpha x} \frac{1}{\lambda} e^{-\frac{x}{\lambda}} dx \\
=& \frac{1}{1-\lambda\alpha} e^{\alpha(c_2-c_1)} e^{(\alpha-\frac{1}{\lambda})\max\{c_1-c_2,0\}}.
\end{aligned}$$

Hence, equation (2) (or (9)) determining the equivalent utility premium becomes

$$\begin{aligned}
&(1-p)(1-e^{-\alpha w}) + p(1-e^{-\alpha w}\frac{1}{1-\lambda\alpha}) \\
&= \left(1 - pe^{-(ke^{-\gamma+r}-k+\widetilde{\pi}+c^{-1}(\widetilde{V}^{\text{Charter}}))/\lambda}\right)(1-e^{-\alpha(w-\widetilde{\pi})}) \\
&\quad + pe^{-(ke^{-\gamma+r}-k+\widetilde{\pi}+c^{-1}(\widetilde{V}^{\text{Charter}}))/\lambda} \\
&\quad \times \left(1 - e^{-\alpha(w-\widetilde{\pi})}\frac{1}{1-\lambda\alpha} e^{\alpha(-k+c^{-1}(\widetilde{V}^{\text{Charter}}))} e^{(\alpha-\frac{1}{\lambda})\max\{k-c^{-1}(\widetilde{V}^{\text{Charter}}),0\}}\right).
\end{aligned}$$

This equation in $\widetilde{\pi}$ is of the form

$$C_1 e^{\alpha\widetilde{\pi}} - (C_1C_2 - C_1C_2C_4)e^{\alpha\widetilde{\pi}-\widetilde{\pi}/\lambda} = C_0, \tag{27}$$

with

$$\begin{aligned}
C_0 &= 1 - (1-p)(1-e^{-\alpha w}) - p\left(1 - e^{-\alpha w}\frac{1}{1-\lambda\alpha}\right); \\
C_1 &= e^{-\alpha w}; \\
C_2 &= pe^{-(ke^{-\gamma+r}-k+c^{-1}(\widetilde{V}^{\text{Charter}}))/\lambda}; \\
C_4 &= \frac{1}{1-\lambda\alpha} e^{\alpha(-k+c^{-1}(\widetilde{V}^{\text{Charter}}))} e^{(\alpha-\frac{1}{\lambda})\max\{k-c^{-1}(\widetilde{V}^{\text{Charter}}),0\}}.
\end{aligned}$$

Setting $x = e^{\alpha\widetilde{\pi}}$, equation (27) reads

$$C_1 x - (C_1C_2 - C_1C_2C_4)x^{1-\frac{1}{\alpha\lambda}} = C_0,$$

which has in general no analytical solution. However, it can easily be solved numerically.

ρ	0.05	α	0.2	k^*	4.51
γ	0.04	w	3	$V^{\text{Charter}*} - k^*$	5.14
r	0.03	p_1	0.25	π^*	1.53
$c(x)$	$x^2 1_{\{x>0\}}$	p_2	0.05	Continuation decision	Continue unless big loss; external finance
		l	2		
		L	10		

Table 1: Benchmark

When $\lambda\alpha = \frac{1}{2}$, we find that

$$\pi = \frac{1}{\alpha} \log \left(\frac{C_0 + \sqrt{C_0^2 - 4C_1^2 C_2 (C_4 - 1)}}{2C_1} \right)$$

In the case that the policyholder, when determining the premium that he is willing to pay for insurance, would ignore the possibility that the insurer default, we would obtain the traditional exponential premium: $\pi = \frac{1}{\alpha} \log \left(\frac{C_0}{C_1} \right) = \frac{1}{\alpha} \log \mathbb{E}[e^{\alpha X}]$.

4 Comparative Statics

In this section, we discuss comparative statics, assuming a two-point loss severity and an exponential utility function with coefficient of absolute risk aversion denoted by α. The benchmark we consider is summarized in Table 1.

4.1 Risk aversion policyholder

The more risk averse the policyholder, the higher the insurance premium he/she is willing to pay *ceteris paribus*, and the more profitable the insurance contract is to the shareholders (higher $V^{\text{Charter}} - k$). If the policyholder is almost risk neutral, no insurance contract will be signed. If the policyholder is moderately risk averse, the insurer will continue operations unless a big loss is experienced. The higher the degree of risk aversion, the earlier the insurer is able to continue operations without attracting external finance. Finally, if the policyholder is highly risk averse, the insurer will always continue operations.

Both in case no insurance contract is signed and in case the insurer always continues operations, the optimal amount of solvency capital is zero. Higher premium income allows reducing the amount of solvency capital. At the same time, higher risk aversion means valuing more a higher payout in case of big losses, hence an incentive to increase solvency capital. The optimal amount of solvency capital results from this trade-off. Figure 3 plots the net value ($V^{\text{Charter}*} - k^*$) as a function of the coefficient of absolute risk aversion.

4.2 Skewness of loss distribution

We vary *ceteris paribus* the parameters p_2 (probability of a big loss) and L (size of the big loss).

If p_2 is too low, then the premium is not sufficient to sign a contract. If p_2 is moderate, the insurer will continue operations unless a big loss is experienced. The higher p_2, the earlier the insurer is able to continue operations without attracting external finance. This does not mean, however, that the profitability of the insurance contract is monotonic in p_2: the profitability of the insurance contract is optimal for values of p_2 that are neither too big nor too small. This can be explained from the fact that when p_2 becomes relatively large the potential (expected) shareholder loss becomes large too, and is not sufficiently compensated by increased premiums. Figure 4 plots the net value ($V^{\text{Charter}*} - k^*$) as a function of the probability of a big loss.

The higher L, the higher the insurance premium he/she is willing to pay, and the more profitable the insurance contract is to the shareholders (higher $V^{\text{Charter}} - k$): risk averse policyholders pay more to reduce by the same amount bigger losses. If L is small (close to l), the premium is not sufficient and no insurance contract will be signed. If L is moderate, the insurer will continue operations unless a big loss is experienced. The higher L, the earlier the insurer is able to continue operations without attracting external finance. Finally, if L is very big, the insurer will always continue operations. Figure 5 plots the net value ($V^{\text{Charter}*} - k^*$) as a function of the size of the big loss.

4.3 Wage

By construction (property of the equivalent utility premium), the solution to the problem is not sensitive to the policyholder's level of wage.

4.4 Deadweight cost

The higher the deadweight cost involved in holding equity capital, the less profitable the insurance contract is to the shareholders. An increase in the deadweight cost leads to a decrease in the optimal amount of solvency capital, and further has a negative effect on the premium the policyholder is willing to pay. With a steep increase in the deadweight cost eventually no contract will be signed. Figure 6 plots the net value ($V^{\text{Charter}*} - k^*$) as a function of the deadweight cost.

4.5 Time discount rate

The higher the time discount rate, the less valuable the insurance contract is to the shareholders. An increase in the time discount rate leads to a decrease in the optimal amount of solvency capital, and further has a negative effect on the premium the policyholder is willing to pay. Figure 7 plots the net value ($V^{\text{Charter}*} - k^*$) as a function of the time discount rate.

5 Conclusions

We have developed a microeconomic financial model to analyze the trade-offs underlying the capital structure decision that insurance companies face. The analysis has made explicit that the role of surplus capital is merely to ensure that upon default, the policyholders' claims are still to a large extent settled. In particular, the settlement of large claims that do not lead to default, is effectively financed by external sources (in the model by raising new equity) and not by surplus capital. The model predicts that risk-averse policyholders are, in principle, willing to pay higher premiums to more solvent insurance companies, which trades off the high costs associated with holding surplus capital, and that this willingness increases with the degree of policyholder risk aversion.

However, a large capital buffer only guarantees that a significant portion of the policyholders' claim is settled and does not automatically imply a high probability that the insurer continues its activities, that is, does not default. In particular, the commitment of shareholders decreases with the target level of the buffer. This is because with higher and more costly target capital, upon the occurrence of large losses, the amount to be refinanced ('target capital gap') is higher. This has an oppressive effect on the premiums the policyholders are willing to pay.

The optimal capital structure implied by our model differs from a Value-at-Risk based capital structure. With a Value-at-Risk based level of surplus capital, policyholders are ensured with a pre-specified degree of certainty that their claims will be settled. Important to note is that, as can be seen from our model, such a Value-at-Risk based capital structure does not imply that the insurer will continue its activities with the same degree of certainty nor does it guarantee that upon default a pre-specified portion of the policyholders' claims is always settled.

Let us finally discuss some limitations and possible extensions of this work. In our model, the premium (price) system fully accounts for the quality of insurance protection offered. Under this price system, the expected utility of the policyholder therefore does not depend on the level of surplus capital. It means that as long as the price accounts for the quality of the insurance protection, there is no need for a regulatory authority to protect the policyholder. In particular, a regulator's social welfare optimization problem, weighing the benefits accrued to the shareholders on the one hand and the expected utility of the policyholder on the other, would have the same solution as the insurer's capital optimization problem in our model. In reality, however, the policyholder may not precisely know the quality of insurance protection and the insurance premium may not fully reflect the quality of insurance protection offered, which creates room for a regulatory authority. This setting will be investigated in future research. In a setting with a regulator, higher capital buffers may prevent from falling below the required capital buffer upon the occurrence of large losses, and thus reduce the immediate need for external financing.

In our model, to reduce the complex problem to its essence, we have assumed that the target capital level is constant over time and that at the end of each time period, upon continuation, the actual capital buffer is restored to its target value. Of course, in reality (temporary) deviations from the target capital level can and will be observed. The impact of the possibility to (temporarily) deviate from the target capital level on the (in this case *dynamic*) optimal capital structure will be investigated in future research.

We have also assumed that the convex cost associated to raising new equity does not depend on the specific target capital level. However, one can imagine that the convex cost of new equity may fall with the target stock. This would provide stronger commitment of shareholders when capital buffers are large, and may be another issue for further investigation.

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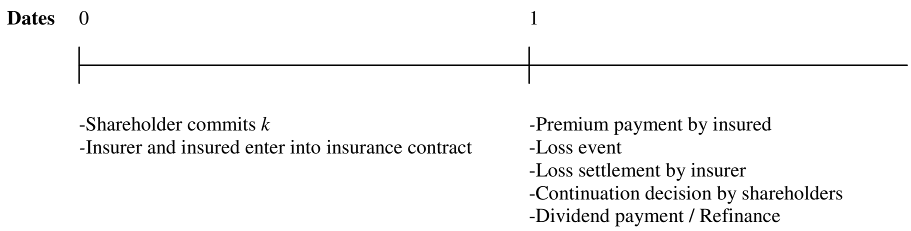
A Sequence of Events

Figure 2: Sequence of Events



B Figures Comparative Statics

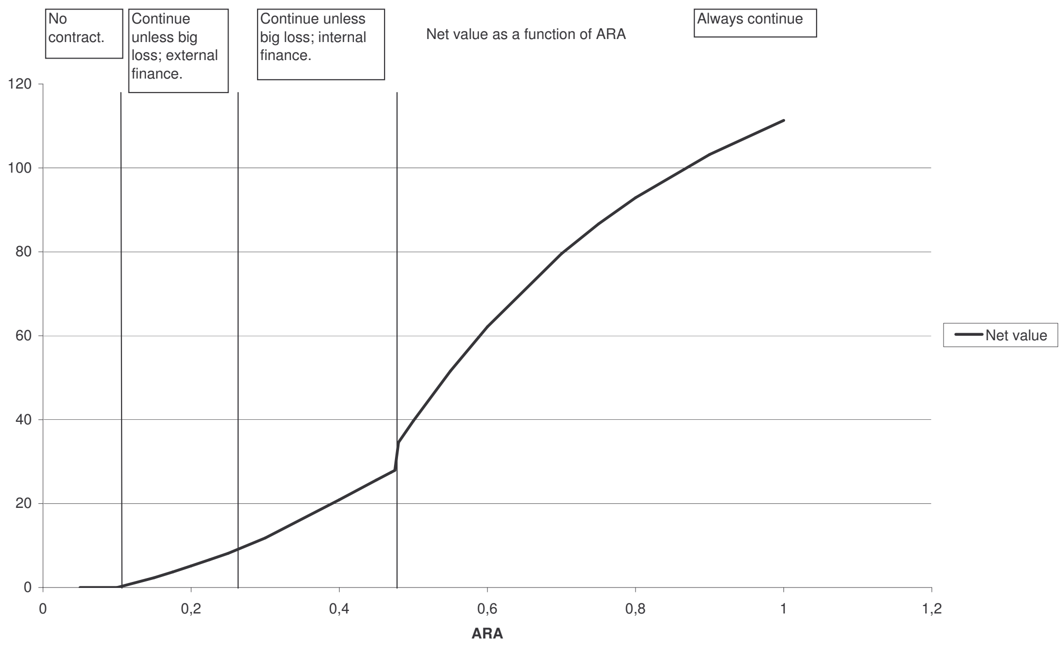


Figure 3: Comparative Statics ARA

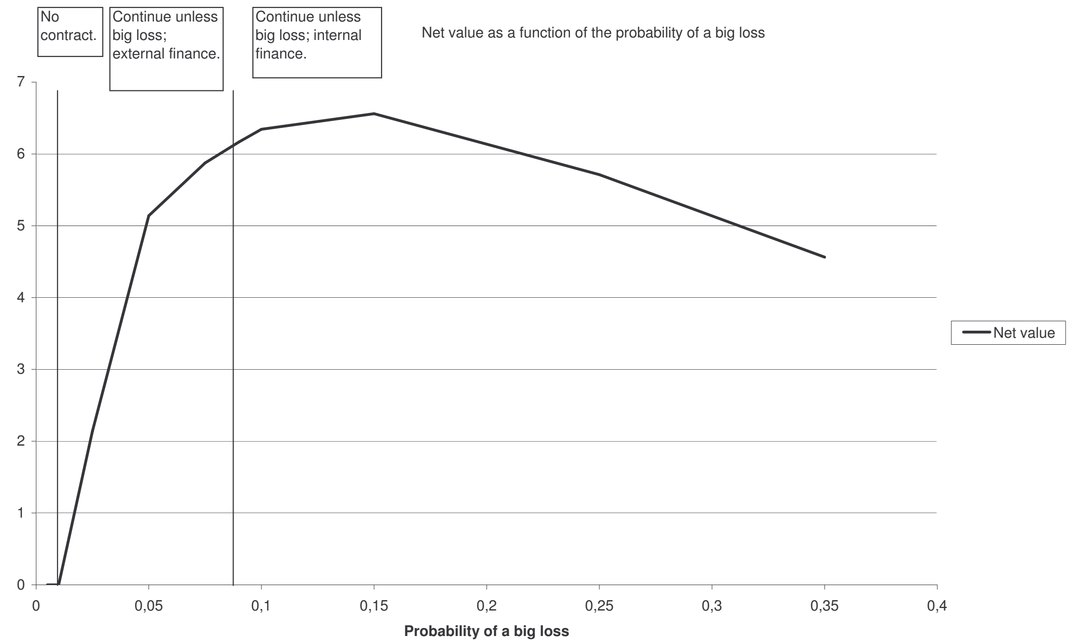


Figure 4: Comparative Statics p_2

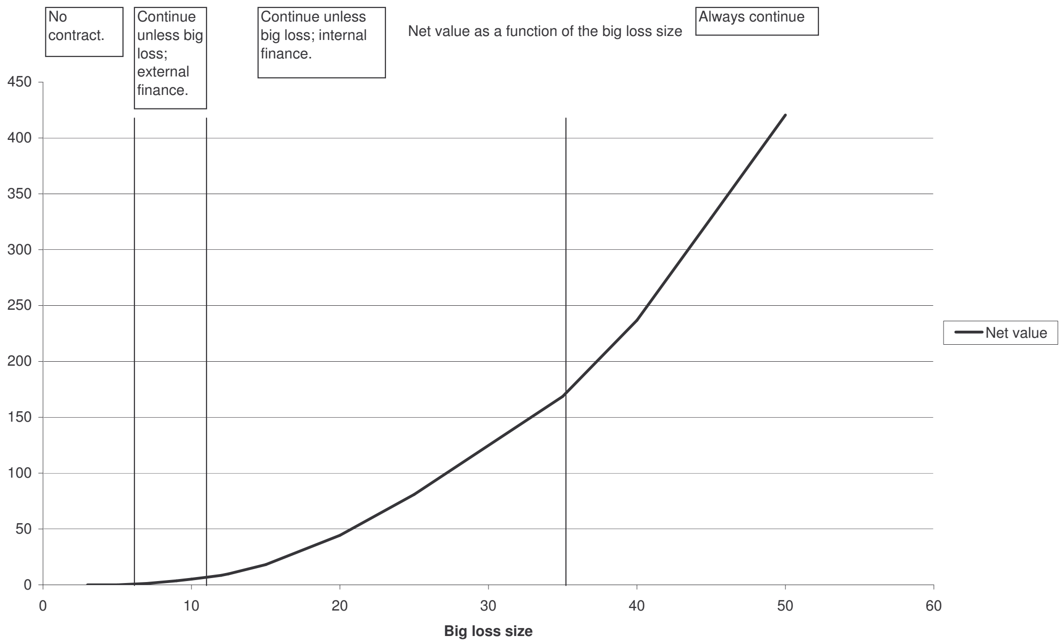


Figure 5: Comparative Statics L

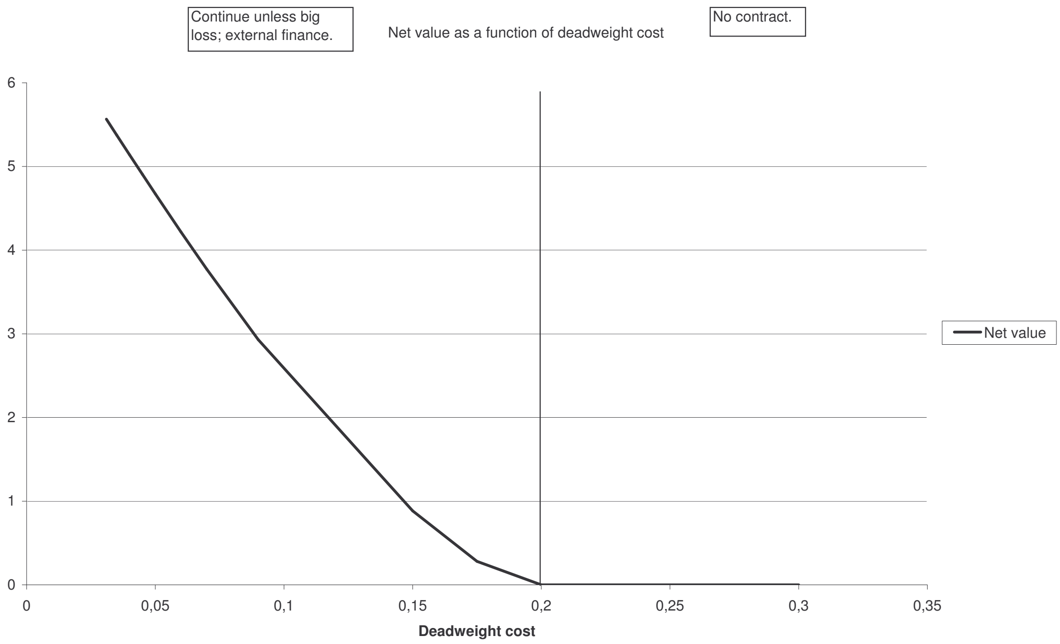


Figure 6: Comparative Statics γ

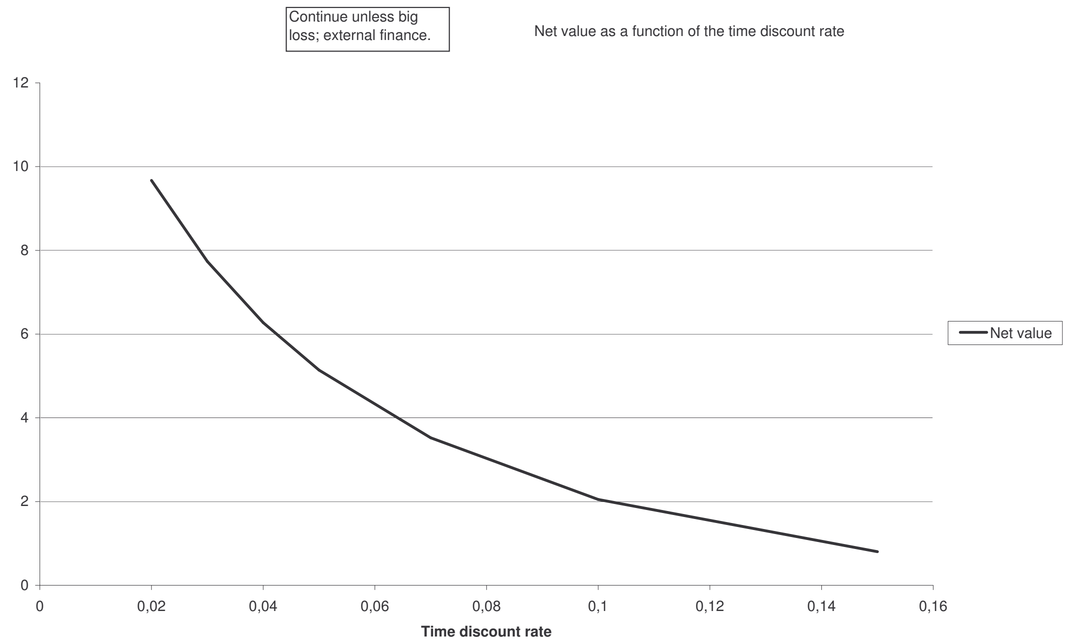


Figure 7: Comparative Statics ρ

C Additional Proofs

C.1 Additional proof of Proposition 3.1

From (22), we derive

$$\frac{\partial \mathrm{RHS}}{\partial \widetilde{\pi}} = -\alpha e^{-\alpha(w-\widetilde{\pi})}\left(1+\frac{1-\lambda\alpha}{\lambda\alpha}pe^{-(ke^{-\gamma+r}-k+\widetilde{\pi}+c^{-1}(\widetilde{V}^{\mathrm{Charter}}))/\lambda}\right.$$
$$\left.\times\left(1-\frac{1-2\lambda\alpha}{(1-\lambda\alpha)^2}e^{\alpha(ke^{-\gamma+r}-k+\widetilde{\pi}+c^{-1}(\widetilde{V}^{\mathrm{Charter}}))}\right)\right).$$

Let $c = ke^{-\gamma+r}-k+\widetilde{\pi}+c^{-1}(\widetilde{V}^{\mathrm{Charter}})$. We know that $c > 0$; see (26). To prove $\frac{\partial \mathrm{RHS}}{\partial \widetilde{\pi}} < 0$, it suffices to prove that

$$\lambda\alpha + (1-\lambda\alpha)e^{-\frac{c}{\lambda}}\left(1-\frac{1-2\lambda\alpha}{(1-\lambda\alpha)^2}e^{\alpha c}\right) > 0. \tag{28}$$

We distinguish between two cases: i) $1-2\lambda\alpha \leq 0$. In this case inequality (28) is trivially satisfied. ii) $1-2\lambda\alpha > 0$. In this case, we replace $\frac{1-2\lambda\alpha}{(1-\lambda\alpha)^2}$ by 1 and minimize $e^{-\frac{c}{\lambda}} - e^{(\alpha-\frac{1}{\lambda})c}$. The (global!) minimum is obtained when $c = \frac{1}{\alpha}\log\left(\frac{1}{1-\lambda\alpha}\right)$. Substituting this expression for c in

$$\lambda\alpha + (1-\lambda\alpha)(e^{-\frac{c}{\lambda}} - e^{(\alpha-\frac{1}{\lambda})c}),$$

we obtain

$$\begin{aligned}
&\lambda\alpha + (1-\lambda\alpha)^{1+\frac{1}{\lambda\alpha}} - (1-\lambda\alpha)^{\frac{1}{\lambda\alpha}}\\
&= \lambda\alpha + (1-\lambda\alpha)^{\frac{1}{\lambda\alpha}}(1-\lambda\alpha-1)\\
&= \lambda\alpha(1-(1-\lambda\alpha)^{\frac{1}{\lambda\alpha}})\\
&> 0.
\end{aligned}$$

This proves that $\frac{\partial \mathrm{RHS}}{\partial \widetilde{\pi}} < 0$.

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