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Measuring the Cost of Regulatory Funding Ratio Constraints for Defined Benefit Pension Plans

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Measuring the Cost of Regulatory Funding Ratio Constraints for Defined Benefit Pension Plans

by
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Abstract

As a risk management tool, pension regulators require pension funds to maintain their short term and nominal funding ratios higher than a certain level with a high probability, while pension funds typically have long term and real ambitions. This thesis aims at measuring the effects of such regulatory constraints on pension funds' investment decisions. We address the problem in a complete market where equity risk, stochastic interest rate and stochastic inflation are present.

We find that regulatory constraints significantly limit pension funds' portfolio choice. When the funding ratio constraints are binding, pension funds are forced to hold more nominal assets and less equity than without constraints, thus leading to utility losses. The utility losses are more significant when constraints are short term as opposed to long term and when investors are less risk averse. When stochastic interest rate and stochastic inflation is present, the constrained investors suffer from much less utility losses compared to the myopic portfolio case because of the demand to hedge the liabilities. Another important finding of our paper is that when the current funding ratio is close to the regulatory "floor", investors with different risk aversion hold similar portfolios since they are obliged to meet the constraints and have to put their own risk appetite aside in such situations.

Chapter 1

Introduction

The European pension system consists of three pillars: the statutory old-age benefits provided by the government, the occupational pensions and the supplementary individual pension plans. We focus on the second pillar pension schemes which involve the agreements between the sponsor, the fund and the pensioner. In defined contribution (DC) pension plans, pensioners accumulate certain amounts of contribution and the benefits received are based on these contributions. Another type of pension plans, the defined benefit (DB) plans require pension funds (or employers) to provide the promised benefits regardless of their investment outcome. There are variations of DB and DC plans with different risk sharing mechanisms, for example collective DC, combinations of DB and DC, and conditional indexation in DB plans.

Despite the pure DC plans where investment risks are entirely borne by pensioners, almost all pension plans involve risk of underfunding where pension funds are unable to fulfill their obligations to pensioners. Regulators thus impose explicit risk management rules on pension funds, from accounting standards for calculating pension liabilities to prudential regulations on funding levels. We focus on the latter type of regulations.

The Dutch regulations for pensions, the Financial Assessment Framework (FTK) requires that "there is at least a 97.5% chance that a fund will have a (nominal) funding ratio of 105% one year later". If pension funds fail to meet such requirements, they are obliged to draw up recovery plans and stop taking more investment risks within a recovery period. In practice such regulatory constraints are not strictly enforced, but it is interesting to investigate the effects of these constraints on pension wealth and investment strategy if they were explicitly taken into account.

With modern asset and liability management (ALM) techniques, the pension manager can address the regulations directly by adapting their investment strategies. However, the regulatory constraints as stated in the FTK, with a short horizon and in nominal terms, are naturally in conflict with pension managers' long term investment goal and real ambition. So our research question is, in a continuous-time dynamic asset allocation framework, how can we reasonably capture this dynamics and what are the impact of such regulatory constraints on pension investors' asset allocation decisions?

This thesis is developed on the basis of Martellini and Milhau (2012) and Shi and Werker

(2012). Both papers considered institutional investors with CRRA utilities who dynamically invest in a complete financial market with risky stocks and stochastic interest rates. Martellini and Milhau (2012) compares the dynamic allocation strategies of a pension fund facing fully indexed liabilities and seeking to maximize expected utility over terminal real funding ratio, under either no constraints or minimum real funding ratio constraints. They show that the dynamic strategies with constraints are reminiscent of portfolio insurance strategies. Shi and Werker (2012) also deals with this type of dynamic problems using the Martingale approach by Cox and Huang (1989) and dynamic programming. Instead of the portfolio insurance type constraints, they investigate the different strategies under no constraints, long term VaR-type constraints or short term VaR-type constraints. They show that the short term constraints, while preventing from large losses, also limits the institutional investor's ability to benefit from upside volatility and thus entails utility losses.

This thesis is an extension to Martellini and Milhau (2012) in the sense that it distinguishes between regulatory constraints over nominal funding ratio and the investor's utility over real funding ratio. We also adopt the VaR-type constraint instead of portfolio insurance-type constraints which is more precise in describing the FTK regulations. We extend Shi and Werker (2012) by introducing inflation risk and stochastic liabilities.

Besides these two important references, this thesis is related to several strands of literature.

First, we consider the dynamic asset allocation problem in a complete market setting. Since Merton (1969, 1971) developed the continuous time portfolio choice problem using dynamic programming, the dynamic asset allocation literature has evolved based on Merton's findings. The dynamic programming approach to asset allocation is general enough to incorporate various preferences and investment opportunity sets, but a closed-form solution is not always available. This approach is well explained in textbooks such as Duffie (2001), and widely adopted in the dynamic asset allocation literature, for example Munk et al. (2004). On the other hand, inspired by Karatzkas et al. (1987) and Cox and Huang (1989), there are many papers that adopt their "martingale approach" to analyze the portfolio choice problem. In this paper we assume stochastic stock prices, stochastic interest rates, and take into account inflation risk. The investment risk is spanned by a stock index, a nominal zero-coupon bond and an inflation-indexed zero-coupon bond so that we can assume a unique state price density and apply the martingale method. Explicit solutions for the optimal asset allocation strategies to this kind of problems have been found using the martingale approach. Wachter (2002) derives an exact solution for an investor with power utility over consumption and , constant interest rate and mean-reverting stock returns.

The second strand of literature are those related to asset liability management of pension funds. Since the regulation as well as investment goals are over funding ratio rather than wealth, we need to adequately value and index the pension liabilities. Papers such as Hoevenaars et al. (2008) model the liability process as a real bond. Martellini and Milhau (2012) considers continuous or discrete pension payments in the generic case, and one fully-indexed lump-sum payment in the zero coupon case. It is also possible to incorporate conditional indexation as proposed in De Jong (2006) and Kleinow (2010), where full indexation is only possible when funding ratio of the fund is above certain threshold. For simplicity, we adopt the setup in Martellini and Milhau (2012), and consider

To measure the cost and benefit of regulatory constraints, we write the optimization problem taking into account explicit funding ratio requirements. Thus this paper is related to the literature about optimal asset allocation under constraints, especially VaR-type ones. The portfolio insurance literature is a special case of the VaR-type constraints where probability of underfunding is zero. Basak and Shapiro (2001) is one of the first papers that deal with this type of constraints in dynamic allocation problems. Their findings show that the VaR constraint keep the portfolio value above or at the threshold in favorable states but incurs sizable loss in unfavorable states. Cuoco et al. (2008) consider the problem where the VaR constraints are short term and based on conditional information. They show that when dynamically reevaluated, the losses in unfavorable states incurred by long term VaR constraints are no longer present. Van Binsbergen and Brandt (2009) distinguish between ex-ante and ex-post risk constraints. Kraft and Steffensen (2012) solves the constrained program by dynamic programming. Shi and Werker (2012) takes a step further to compare the long term and short term regulatory horizon. In the long term case, regulatory horizon is the same as the investment horizon; in the short term case, the program is embedded with subsequent and non-overlapping VaR constraints.

Van Binsbergen and Brandt (2006) also address the effects of regulatory constraints on the investment plans of DB pension plans. They find that ex ante risk constraints like the VaR constraint in our problem tend to decrease the gain to dynamic investment while ex post risk constraint lead to large utility gains.

The contribution of this thesis lies in three aspects. First, we extend the literature on VaR-based asset allocation to incorporate stochastic liability. The results of Basak and Shapiro (2001) are applied to the pension fund ALM setting where regulatory constraints are a great concern of pension funds. Second, we distinguish between nominal and real funding ratio constraints to reveal the conflict between regulatory constraints and pension fund real objectives, while existing literature have only focused on real funding ratio constraints. Lastly, we use the dynamic programming approach in the presence of short term VaR-type constraints while using funding ratio and inflation as state variables.

The rest of this thesis proceeds as follows. Chapter 2 introduces the financial market settings, assumptions on the pension fund, and regulatory constraints. Chapter 3 describes the theoretical background and numerical procedures to solve dynamic asset allocation problems, namely the Martingale Approach and Dynamic Programming. Chapter 4 solves the problem in a "Merton" setting with constant interest rate and no inflation. Chapter 5 take a step further to add stochastic inflation and stochastic interest rate. We distinguish between a fixed weight strategy, which is more realistic in pension fund practice, and a dynamic strategy. Chapter 6 looks at how our results change with different assumptions on asset returns and volatilities. Chapter 7 concludes.

Chapter 2

Financial Market and Pension Fund Settings

In this chapter we introduce the basic assumptions on the financial market, the pension fund, and regulatory constraints relevant to our research question.

2.1 Financial Market

We consider a continuous-time stochastic economy on a finite horizon $[0, T_0]$. There are three sources of risk that are most critical to portfolio choice problems of defined benefit pension plans, namely equity risk, interest rate risk and inflation risk. The market is dynamically complete in the sense that all risks in the economy are spanned by existing assets, so that any possible state-of-the-world can be replicated by dynamically trading these assets. The complete market assumption enables us to solve for closed-form solutions using the Martingale Approach, which we will explain in further detail in Chapter 3.

More specifically, the asset menu on the financial market includes a short term risk-free asset (cash), a stock index, a default-free nominal bond and a default-free real bond. The nominal short term interest rate r_t follows the mean-reverting Vasicek process; this is the continuously compounded logarithmic return on the cash account. The price S_t of the stock indexn and the Consumer Price Index Φ_t follows a Geometric Brownian Motion. The CPI is the nominal price of one unit of real consumption good, so the real price of any asset is its nominal price deflated by Φ_t.

In the objective probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

$$\begin{aligned} dr_t &= a(b + \frac{\sigma_r \lambda_r}{a} - r_t)dt + \sigma_r dz_{r,t} \\ dS_t &= S_t((r_t + \sigma_s \lambda_s)dt + \sigma_s dz_{s,t}) \\ d\Phi_t &= \Phi_t((\varphi + \sigma_\Phi \lambda_\Phi)dt + \sigma_\Phi dz_{\Phi,t}) \end{aligned}$$

where the risks in the economy are represented by $\{z_{r,t}, z_{s,t}, z_{\Phi,t}\}$, standard Brownian Motions with pairwise correlations $\rho_{rs}, \rho_{r\Phi}, \rho_{s\Phi}$, and $(\lambda_r, \lambda_s, \lambda_\Phi)$ are corresponding prices of risk. Note that since

CPI is not directly traded on the market, the "price of inflation risk" is not directly observable. For simplicity, the prices of risk $\lambda_r, \lambda_s, \lambda_\Phi$, the volatilities $\sigma_r, \sigma_s, \sigma_\Phi$ and the instantaneous expected inflation φ are assumed to be constant over time.

To write the model in a more compact way, denote with $\{\boldsymbol{z}_t\}$ a three-dimensional standard Brownian Motion with mutually independent entries and rewrite the volatilities and prices of risk in a vector form. $\{\boldsymbol{z}_t\}$ can be seen as an normalized version of the risks $\{z_{r,t}, z_{s,t}, z_{\Phi,t}\}$, and the dependencies between $\{z_{r,t}, z_{s,t}, z_{\Phi,t}\}$ are captured through the volatility matrix $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_r, \boldsymbol{\sigma}_s, \boldsymbol{\sigma}_\Phi)$. There are infinitely many ways of normalizing the risks, and the complete market assumption implies existence of a unique price of risk vector $\boldsymbol{\lambda}$ and a unique pricing kernel M_t in each way of normalization. See B for one way of performing such normalization.

$$
\begin{aligned}
dr_t &= a(b + \frac{\boldsymbol{\sigma}_r'\boldsymbol{\lambda}}{a} - r_t)dt + \boldsymbol{\sigma}_r' d\boldsymbol{z}_t \\
dS_t &= S_t((r_t + \boldsymbol{\sigma}_s'\boldsymbol{\lambda})dt + \boldsymbol{\sigma}_s' d\boldsymbol{z}_t) \\
d\Phi_t &= \Phi_t((\varphi + \boldsymbol{\sigma}_\Phi'\boldsymbol{\lambda})dt + \boldsymbol{\sigma}_\Phi' d\boldsymbol{z}_t)
\end{aligned}
$$

According to Girsanov's Theorem, the dynamics of the financial market can be expressed with an equivalent risk neutral probability measure $\mathbb{Q}$, under which $\{\tilde{\boldsymbol{z}}_t\}$ is a three-dimensional standard Brownian Motion:

$$
\begin{aligned}
d\tilde{\boldsymbol{z}}_t &= \boldsymbol{\lambda} dt + d\boldsymbol{z}_t \\
\frac{d\mathbb{Q}}{d\mathbb{P}} &= \exp\left[-\int_0^{T_0} \boldsymbol{\lambda}' d\boldsymbol{z}_t - \frac{1}{2}\int_0^{T_0} ||\boldsymbol{\lambda}||^2 dt\right] \\
dr_t &= a(b - r_t)dt + \boldsymbol{\sigma}_r' d\tilde{\boldsymbol{z}}_t \\
dS_t &= S_t(r_t dt + \boldsymbol{\sigma}_s' d\tilde{\boldsymbol{z}}_t) \\
d\Phi_t &= \Phi_t(\varphi dt + \boldsymbol{\sigma}_\Phi' d\tilde{\boldsymbol{z}}_t).
\end{aligned}
$$

The unique nominal pricing kernel is thus defined as

$$
\begin{aligned}
M_t =& M_0 \exp\left(-\int_0^t r_s ds\right) \mathbb{E}_t\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] \\
=& M_0 \exp\left(-\int_0^t r_u du - \boldsymbol{\lambda}'(\boldsymbol{z}_t - \boldsymbol{z}_0) - \frac{1}{2}||\boldsymbol{\lambda}||^2(t-0)\right), M_0 = 1
\end{aligned}
$$

or equivalently,

$$
\frac{dM_t}{M_t} = -r_t dt - \boldsymbol{\lambda}' d\boldsymbol{z}_t.
$$

With the pricing kernel it is possible to calculate the price of any existing asset as the expectation (under objective measure $\mathbb{P}$) of the product of the pricing kernel and the asset payoff. See appendix C for the verification of this pricing kernel.

The nominal bond is a zero-coupon bond that pays 1 unit of cash at maturity τ_1 and the real bond is a zero-coupon bond that pays Φ_{τ_2} amount of cash at maturity τ_2. Their (nominal) prices at time t are denoted $B(t,\tau_1)$ and $I(t,\tau_2)$ respectively,

$$B(t,\tau_1) = \exp(-\alpha(\tau_1 - t)r_t + \beta_1(\tau_1 - t))$$
$$I(t,\tau_2) = \Phi_t \exp(-\alpha(\tau_2 - t)r_t + \beta_2(\tau_2 - t))$$

where

$$\alpha(s) = \frac{1 - e^{-as}}{a}$$
$$\beta_1(s) = (\alpha(s) - s)R_\infty - \frac{||\boldsymbol{\sigma_r}||^2}{4a}\alpha(s)^2, R_\infty = b - \frac{||\boldsymbol{\sigma_r}||^2}{2a^2}$$
$$\beta_2(s) = b\alpha(s) + (\varphi - b - \frac{||\boldsymbol{\sigma}_\Phi||^2}{2})s + \frac{1}{2}\int_0^s ||\alpha(u)\boldsymbol{\sigma}_r - \boldsymbol{\sigma}_\Phi||^2 du$$

or equivalently, the dynamics of the bond prices under $\mathbb{P}$ is

$$\frac{dB(t,\tau_1)}{B(t,\tau_1)} = (r_t - \alpha(\tau_1 - t)\boldsymbol{\sigma_r'\lambda})dt - \alpha(\tau_1 - t)\boldsymbol{\sigma_r' dz_t}$$
$$\frac{dI(t,\tau_2)}{I(t,\tau_2)} = (r_t + (\boldsymbol{\sigma_\Phi} - \alpha(\tau_1 - t)\boldsymbol{\sigma_r})'\boldsymbol{\lambda})dt + (\boldsymbol{\sigma_\Phi} - \alpha(\tau_1 - t)\boldsymbol{\sigma_r})'\boldsymbol{dz_t}$$

From the dynamics above we can see that the volatility vectors of the bonds are $\boldsymbol{\sigma}_B = -\alpha(\tau_1 - t)\boldsymbol{\sigma_r}$ and $\boldsymbol{\sigma}_I = \boldsymbol{\sigma_\Phi} - \alpha(\tau_1 - t)\boldsymbol{\sigma_r}$, respectively. Note that by definition, $\alpha(\tau_1 - t)$ is the duration of the nominal bond,

$$-\frac{dB(t,\tau_1)}{dr_t}\frac{1}{B(t,\tau_1)} = -\exp(-\alpha(\tau_1 - t)r_t + \beta_1(\tau_1 - t))(-\alpha(\tau_1 - t)r_t)\frac{1}{B(t,\tau_1)} = \alpha(\tau_1 - t)$$

and similarly, $\alpha(\tau_2 - t)$ is the duration of the real bond.

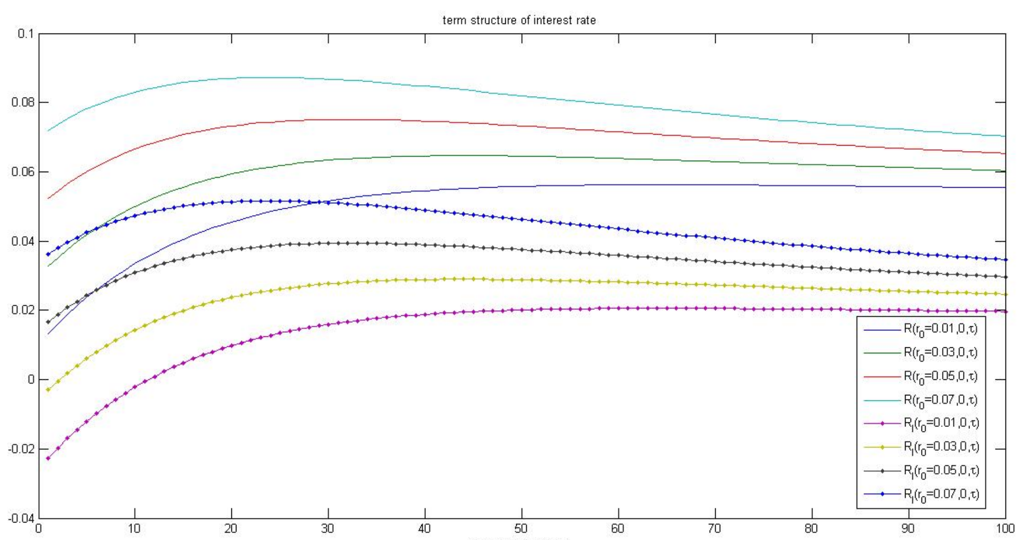


Figure 2.1: Term structure of nominal and real interest rates

The derivation of the bond price, dynamics and term structure for the nominal and real bond are shown in Appendix D and E.

The calibrated parameters are the same as Munk et al. (2004), which has the same stock and nominal short rate processes as ours. See Appendix A for the base case parameters used in this thesis.

2.2 Pension Fund

We assume a closed pension fund, which means the fund ceases to exist and liquidates its assets after a fixed date in the future. The fund only invests on behalf of its current members, and no new entrants nor contribution payment is allowed. In reality pension funds operate on a "going concern" basis, assuming a fixed length of investment horizon as time proceeds, and allowing new entrants and/or new contribution payments. In such cases the allocation problem of the pension fund reduces to a static one. The closed fund assumption however, is more aligned with prudential regulations and pension managers' ALM practice. It is also partially justified by the trend of transitioning from DB to DC pension plans: the fund may be motivated to choose a future date after which the DB plan is closed and switch to DC.

The wealth of the pension fund is a portfolio with initial endowment A_0 that can be continuously traded in the aforementioned assets,

$$dA_t = A_t(r_t + \boldsymbol{\omega}_t'\boldsymbol{\sigma}_t'\boldsymbol{\lambda})dt + A_t\boldsymbol{\omega}_t'\boldsymbol{\sigma}_t'd\boldsymbol{z}_t.$$

In line with pension fund practice in reality, we impose short sale constraints that $0 \leq \boldsymbol{\omega}_t \leq 1, \boldsymbol{\omega}_t'\boldsymbol{\iota} \leq 1$ ($\boldsymbol{\iota}$ is a vector of ones).

On the liability side, we model the pension liability as a lump-sum payment at time T_0,

$$L_t = L_0 \exp[\int_0^t (r_u + \boldsymbol{\sigma}_{L,u}'\boldsymbol{\lambda} - \frac{1}{2}||\boldsymbol{\sigma}_{L,u}||^2)du + \int_0^t \boldsymbol{\sigma}_{L,u}'d\boldsymbol{z}_u]$$

Although pension liabilities in the real world are rarely a single lump-sum payment, we can value the stream of payments and calculate the duration for an equivalent lump-sum payment.

Pension funds mainly aim at providing retirement protection for its members, therefore its investment objective is long term and indexed to wage or consumer price. However, they are only required by regulators to fulfill their nominal obligations to pensioners and only need to provide indexation if funding ratio is sufficient. We assume away the complication of conditional indexation and consider real liabilities fully indexed to CPI, thus the same as a real bond: $L_t = I(t, T_0), \boldsymbol{\sigma}_{L,t} = \boldsymbol{\sigma}_I = \boldsymbol{\sigma}_\Phi - \alpha(T_0 - t)\boldsymbol{\sigma}_r$. The real funding ratio is calculated as the ratio of pension asset versus the real liability $F_t = A_t/L_t$.

In contrast, the nominal funding ratio is calculated with nominal liabilities, or a nominal bond $L_{nom,t} = \boldsymbol{\sigma}_B = B(t, T_0), \boldsymbol{\sigma}_{Lnom,t} = -\alpha(T_0 - t)\boldsymbol{\sigma}_r, F_{nom,t} = A_t/L_{nom,t}$. The nominal funding ratio appears in regulatory funding ratio constraints, as will be discussed in the next section.

The pension fund has constant relative risk aversion(CRRA) utility over the surplus of terminal real funding ratio above a subsistence level and investment horizon $[0, T]$.

$$U(F_T) = \begin{cases} \frac{(\frac{A_T}{L_T} - h)^{1-\gamma}}{1-\gamma} & \text{if} F_T > h \\ -\infty & \text{if} F_T \leq h \end{cases}$$

Due to its property of leading to the same asset allocation regardless of initial wealth, the CRRA utility is one of the most commonly used utility function in pension fund ALM practice. However this property is lost with the introduction of the subsistence level. The subsistence level is the lowest real funding level that the pension fund imposes on itself. Because of the ambition to provide inflation indexed pension payments the utility is derived from real funding ratio instead of nominal ones. Intuitively, the subsistence level is the amount of real funding ratio that in case of a closing fund, the fund aims to maintain "at all cost". The subsistence level also reflects the fund's relative risk aversion since

$$RRA = -\frac{U''F}{U'} = \frac{\gamma}{1 - \frac{h}{F_T}}.$$

This parameter is hard to calibrate since pension funds in reality are rarely going bankrupt; we calibrate this level h as the percentage allocation of pension wealth to index linked bonds, around 8%.

2.3 Regulatory Constraints

To ensure pension funds fulfill their obligations to pensioners, regulators impose a number of accounting and prudential constraints on pensions. The Dutch pension regulations "Financial Assessment Framework" (FTK) requires pension funds to draw up recovery plans in the case of underfunding to recover from the situation within a period of three years. Furthermore, pension plans face VaR-type constraints imposed on nominal funding ratios, that the probability of having a nominal funding ratio τ years later higher than k is at least $1 - p$,

$$\Pr_t\left(\frac{A_{t+\tau}}{L_{t+\tau}^{nom}} < k\right) \leq p.$$

The so-called long term constraints are those imposed at the beginning of the investment horizon over terminal funding ratio; short term constraints are those when investment horizon is divided into several non-overlapping periods (usually in one-year periods), imposed at the beginning of each period over the end-period funding ratio. The current FTK regulations are similar to the short term VaR constraints where $\tau = 1$. We make the distinction between long term and short term constraints and nominal and real constraints in order to analyze the impact of each type of constraints on the fund's asset allocation decisions.

In special cases, for example $p = 1$, the VaR constraint becomes non-binding; when $p = 0$, the VaR-constrained problem becomes a portfolio insurance one. We take the FTK setting and set $k = 1.05, p = 25\%$.

Furthermore, to unveil the possible differences between regulatory constraints imposed on nominal and real funding ratios, we note that the ratio of nominal funding to real funding level is on average quite stable. Dividing the nominal regulatory floor k by this ratio at time 0, we get an “equivalent” regulatory floor on real funding ratios, thus making the real and nominal regulations comparable. We explain this in details in Chapter 5.

Chapter 3

Approach

We address the dynamic allocation problem by solving for the optimization program where we maximize expected utility over terminal (real) funding ratio, both in the unconstrained case and under VaR-type constraints.

Expressions for optimal wealth and weights can be found in the appendices. When closed form solutions are not available, numerical results are obtained by dynamic programming methods, which we explain in section 3.3.

3.1 The Optimization Problems

Unconstrained case

$$\max_{\omega_t} \ E\left[\frac{(A_T/L_T - h)^{1-\gamma}}{1-\gamma}\right]$$
$$s.t. A_T = A_0 \exp\left[\int_0^T (r_u + \boldsymbol{\omega}_u'\boldsymbol{\sigma}_u'\boldsymbol{\lambda} - \frac{1}{2}||\boldsymbol{\omega}_u'\boldsymbol{\sigma}_u'||^2)du + \boldsymbol{\omega}_u'\boldsymbol{\sigma}_u' d\boldsymbol{z}_u\right]$$

Long term VaR-type constraints

$$\max_{\omega_t} \ E\left[\frac{(A_T/L_T - h)^{1-\gamma}}{1-\gamma}\right]$$
$$s.t. A_T = A_0 \exp\left[\int_0^T (r_u + \boldsymbol{\omega}_u'\boldsymbol{\sigma}_u'\boldsymbol{\lambda} - \frac{1}{2}||\boldsymbol{\omega}_u'\boldsymbol{\sigma}_u'||^2)du + \boldsymbol{\omega}_u'\boldsymbol{\sigma}_u' d\boldsymbol{z}_u\right]$$
$$\text{Pr}[\frac{A_T}{L_{Nom,T}} < k] \leq p$$

Basak and Shapiro (2001) shows a closed form solution for this problem. Our assumption of lump-sum payment and full indexation enables us to write the value of liabilities exogenously, so optimal portfolio weights can also be obtained by applying Ito's lemma and the martingale representation theorem. Otherwise only numerical solution is available.

Short term VaR-type constraints

$$\max_{\omega_t} \; E[\frac{(A_T/L_T - h)^{1-\gamma}}{1-\gamma}]$$
$$s.t. A_T = A_0 \exp\left[\int_0^T (r_u + \boldsymbol{\omega}_u' \boldsymbol{\sigma}_u' \boldsymbol{\lambda} - \frac{1}{2}||\boldsymbol{\omega}_u' \boldsymbol{\sigma}_u'||^2)du + \boldsymbol{\omega}_u' \boldsymbol{\sigma}_u' d\boldsymbol{z}_u\right]$$
$$\Pr_t[\frac{A_{t+\tau}}{L_{Nom,t+\tau}} < k] \leq p, \forall t \in [0, T).$$

3.2 The Martingale Approach

Karatzas and Shreve (1987), Cox and Huang (1989) developed the martingale approach to portfolio choice in continuous time. Assuming a complete financial market, there exists a unique pricing kernel $\{M_t\}$ and all outcomes of the world are attainable by dynamically trading the available assets. As a result, the budget constraint that A_T is attained by holding $\{\boldsymbol{\omega}_t\}$ assets with initial wealth A_0 can be replaced by simply writing $A_0 = E[M_T A_T]$. In such a setting the dynamic asset allocation problem is reduced to a static one since we can first solve for the optimal terminal wealth:

$$\max_{A_T} E\left[\frac{(A_T/L_T - h)^{1-\gamma}}{1-\gamma}\right]$$
$$s.t. A_0 = E[M_T A_T]$$

and then solve for all previous optimal wealth using the martingale property of $\{M_t A_t\}$,

$$A_t = \frac{1}{M_t} E[M_T A_T]$$

Then the exposure of the obtained optimal intermediate wealth to the risk terms is the same as those obtained by the trading strategy $\boldsymbol{\omega}_t$ that replicates the optimal wealth,

$$dA_t = (...)dt + A_t \boldsymbol{\omega}_t' \boldsymbol{\sigma}_t' d\boldsymbol{z}_t.$$

See Appendix G for details of the derivation without funding ratio constraints. When long term funding ratio constraints are present, we adapt the static optimization problem in the first step to ensure that terminal wealth satisfies the constraints in some states of the world. The derivation of optimal terminal wealth is relegated to Appendix H. In the Merton case, the optimal asset weights can also be solved, which we show in Appendix I.

When short term constraints are imposed, the martingale approach becomes infeasible. Thus we turn to numerical method, explained in the next section.

3.3 Dynamic Programming

To solve for the dynamic asset allocation problem, the investor has to take into account all possible future investment opportunities and optimal allocations. Only under certain circumstances can closed-form solutions be found for such problems. Most of the times we have to rely on numerical techniques like numerical dynamic programming.

The idea of using dynamic programming to solve dynamic optimization problems is a backward recursive procedure. Bellman's optimality condition states that,

"Principle of Optimality: An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

In other words, the investor starts with the last period in time, where the closed form of value function (the maximized expected terminal utility conditional on information available at each time period) is known. Working backwards in time, the investor is able to solve the dynamic allocation problem in all periods.

Now we explain the dynamic programming procedure used in this thesis.

Step 0: discretize state space into periods of length dt and simulate a large number N of sample paths for state variables, asset returns (short rate, stock excess returns and the two bond returns) and liability over the horizon T.

Step 1: Since the portfolio return is endogenous in current funding ratio, make a grid for (real) funding ratio F ; for each value of F generate the funding ratio return given a portfolio weight for all sample paths, so we have N paths for next period's funding ratio.

Step 2: The value function at terminal horizon is just the utility function over F, conditional on the state variable(s). We start in the final period, which is one period ahead of the terminal horizon.

$$V_{T-1}(F_{T-1}, \boldsymbol{x}_{T-1}) \equiv \max \mathrm{E}_{T-1}\left[\frac{(A_T/L_T - h)^{1-\gamma}}{1-\gamma}\right]$$

$$s.t. F_T = F_{T-1}\exp\left[\int_{T-1}^{T}(\boldsymbol{\omega}_u'\boldsymbol{\sigma}_u'\boldsymbol{\lambda} - \boldsymbol{\sigma}_{L,u}'\boldsymbol{\lambda} - \frac{1}{2}||\boldsymbol{\omega}_u'\boldsymbol{\sigma}_u'||^2 + \frac{1}{2}||\boldsymbol{\sigma}_{L,u}||^2)du + \int_{T-1}^{T}(\boldsymbol{\omega}_u'\boldsymbol{\sigma}_u' - \boldsymbol{\sigma}_{L,u}')d\boldsymbol{z}_u\right],$$

$$\boldsymbol{\omega}_{T-1} \geq 0, \boldsymbol{\iota}'\boldsymbol{\omega}_{T-1} \leq 1.$$

$\boldsymbol{x}_t$ is a vector of state variables and F_t is the real funding ratio. The value function one period ahead is the conditional expectation of next period's value function, choosing the portfolio weight that maximizes this expectation. Since we work with funding ratios, the effect of short interest rate is canceled out in the asset and liability side, so we only have one state variable Φ_t in the nominal constrained case and no state variables in the real constrained case. We use a fourth-order polynominal expansion of the state variable.

Step 3: Recursing backwards until time 0, we have an optimal weight and a value function for each grid point of F and each simulation path. Now given an initial funding ratio and a sample path we can determine the optimal allocation.

Step 4a: To add short term probability constraints, note that the conditional distribution of next period's funding ratio is lognormal and only depends on current funding ratio. In the Merton case and when the constraints are on real funding ratio, there is no state variables except for F_t,

$$
\begin{aligned}
F_{t+\tau} &= F_t \frac{A_T/A_{t+\tau}}{L_{t+\tau}/L_t} \\
&= F_t \frac{\exp[\int_t^{t+\tau}(r_u + \boldsymbol{\omega}_u'\boldsymbol{\sigma}_u'\boldsymbol{\lambda} - \frac{1}{2}||\boldsymbol{\omega}_u'\boldsymbol{\sigma}_u'||^2)du + \int_t^{t+\tau}\boldsymbol{\omega}_u'\boldsymbol{\sigma}_u' d\boldsymbol{z}_u]}{\exp[\int_t^{t+\tau}(r_u + \boldsymbol{\sigma}_{L,u}'\boldsymbol{\lambda} - \frac{1}{2}||\boldsymbol{\sigma}_{L,u}||^2)du + \int_t^{t+\tau}\boldsymbol{\sigma}_{L,u}' d\boldsymbol{z}_u]} \\
&= F_t \exp[\int_t^{t+\tau}(\boldsymbol{\omega}_u'\boldsymbol{\sigma}_u'\boldsymbol{\lambda} - \boldsymbol{\sigma}_{L,u}'\boldsymbol{\lambda} - \frac{1}{2}||\boldsymbol{\omega}_u'\boldsymbol{\sigma}_u'||^2 + \frac{1}{2}||\boldsymbol{\sigma}_{L,u}||^2)du + \int_t^{t+\tau}(\boldsymbol{\omega}_u'\boldsymbol{\sigma}_u' - \boldsymbol{\sigma}_{L,u}')d\boldsymbol{z}_u]
\end{aligned}
$$

Therefore for each funding ratio grid point it can be calculated beforehand which weights satisfy the probability constraint and which do not. Excluding the infeasible points in the grid search over all possible weights, we can repeat Step 2 and get the optimal strategy under probability constraints.

Step 4b: When the regulatory constraints are on nominal funding ratio, Φ_t acts as another state variable. For each feasible weight and each funding grid point, perform a cross-sectional regression of realized values of $V_{t+1}(F_{t+1})$ on a fourth order expansion of the state variable Φ_t. With the estimated parameters, the fitted value for each sample path is the conditional expectation of next period value function, and the optimal weight for each sample path can be found.

3.4 Cost and Benefit Analysis

Once the optimal portfolio wealth (and portfolio weight) is obtained for each type of strategy, the "cost of regulatory constraints" can be measured by the certainty equivalent loss in initial funding ratios,

$$V_0^u(F_0 \cdot \exp^{-r_{ce}T}) = V_0^{constr}(F_0).$$

This certainty equivalent loss rate r_{ce} is the annualized rate of initial funding ratio loss under the unconstrained strategy if it were to have the same expected utility as the constrained strategy.

The benefit of said constraints can be measured by inspecting e. g. probability of shortfall,

$$PSF = \Pr[F_T < k],$$

and expected shortfall in funding ratios,

$$ES = E[\max(k - F_T, 0)].$$

Chapter 4

Myopic Portfolio Choice, Constant Investment Opportunities

In this chapter we consider the simplified case with constant interest rate and no inflation. Investor chooses between the risky stock and riskless cash. Liabilities behave like a zero-coupon bond with the same interest rate. In this case the investment opportunity set is constant through time and there is no demand for hedging liabilities. This is the so-called myopic portfolio choice.

The financial market is as follows.

$$
\begin{aligned}
dS_t &= S_t((r_t + \sigma_s' \lambda_s)dt + \sigma_s' dz_t), r_t \equiv r_0 \\
L_t &= \exp(-r_0(T_0 - t)) \\
M_T &= M_t \exp\left(-r_0(T-t) - \frac{1}{2}\lambda_s^2(T-t) - {\lambda_s}'(z_T - z_t)\right), M_0 = 1
\end{aligned}
$$

4.1 Unconstrained

We confirm the classic Merton formula that the optimal portfolio weight in the unconstrained problem is

$$
\boldsymbol{\omega}_t = \frac{1}{\gamma}{\sigma_s}^{-1}\lambda_s(1 - \frac{h}{F_t}).
$$

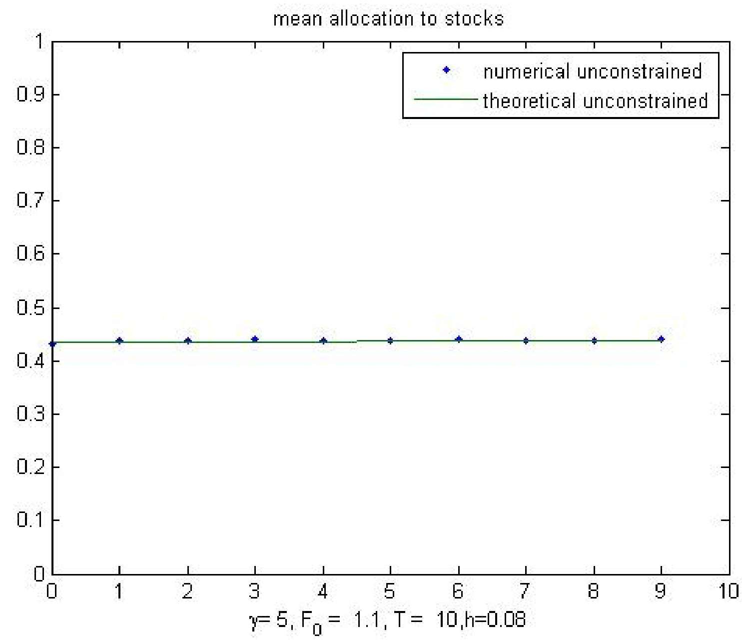


Figure 4.1: Numerical and theoretical weights in the unconstrained case.

Figure 4.1 shows the average allocation to stocks for an unconstrained investor. The line is the theoretical optimal weight in the above formula, and the dots are results from numerical dynamic programming with a rebalancing period of one year. The numerical results confirm the theoretical optimal weights.

One interpretation for the optimal weight is that the investor sets aside at time 0 an amount of cash that pays hL_T at terminal horizon to make sure that funding ratio is always above the subsistence level, then invests the fraction of the rest in the classic “speculative portfolio”. When $h = 0$, the optimal strategy is irrelevant of inital funding ratio and constant through time. As h gets larger, the investor invests less in stocks; the subsistence level acts as a measure for risk aversion, or more precisely the aversion to being unable to fulfil one’s liability obligations. We illustrate this with a higher subsistence level $h = 0.5$,

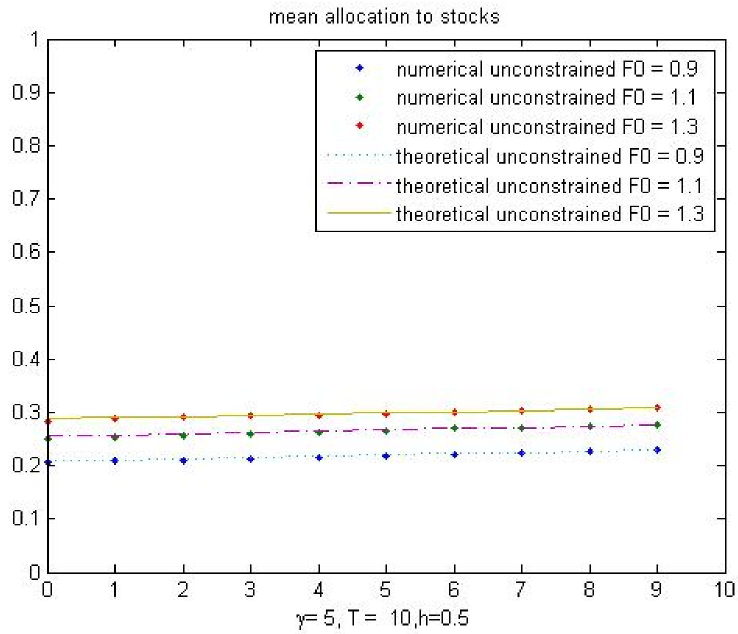


Figure 4.2: Numerical and theoretical weights with different γ's in the unconstrained case, $h = 0.5$.

From Fig. 4.2 we can see the effects of the subsistence level. The larger the current funding ratio is, the more risk the investor is able to take. The funding ratio has on average a positive drift, so as time goes on the allocation to stocks is larger. If funding ratio is very close to (but still higher than) h, the investor takes no stock exposure.

4.2 Long term constraints

The optimal wealth under long term VaR-constraints is (see Appendix H for derivation)

$$A_T^{VaR} = \begin{cases} (yM_T)^{-\frac{1}{\gamma}} L_T^{1-\frac{1}{\gamma}} + hL_T & \text{if } M_T < \min\{\underline{M}, \overline{M}\}, \\ kL_T & \text{if } \min\{\underline{M}, \overline{M}\} \leq M_T < \overline{M}, \\ (yM_T)^{-\frac{1}{\gamma}} L_T^{1-\frac{1}{\gamma}} + hL_T & \text{if } M_T \geq \overline{M} \end{cases}$$

where $\underline{M}$ is such that $(yM_T)^{-\frac{1}{\gamma}} L_T^{1-\frac{1}{\gamma}} + hL_T = kL_T$; $\overline{M}$ is such that $\Pr[M_T > \overline{M}] = p$. In the constant investment opportunity case in this chapter, $\underline{M} = \frac{1}{y} e^{r(T_0 - T)}(k-h)^{-\gamma}$,
$\overline{M} = \exp(\lambda_s N^{-1}(1-p)\sqrt{T} - (r_0 + \lambda_s^2/2)T)$.

The following graph shows the dependence of terminal funding ratio on the state of the world (the pricing kernel). The upper barrier $\overline{M}$ is the level such that the time 0 probability of a worse final state M_T is exactly p. The lower barrier $\underline{M}$ is such that the unconstrained terminal wealth and the required “floor” coincides. It is shown on the graph that when the economy is so good (pricing kernel is small enough) that an unconstrained strategy does not lead to breaching the VaR constraint, the investor behaves like an unconstrained investor. In intermediate states of the world, the VaR-constrained investor has to hedge against the constraint to keep the funding ratio above the required level. When the economy is very bad, it is very expensive to hold hedging assets so the investor again behaves like an unconstrained investor. As long as the probability of having a 'bad' economy is kept to the required level, the magnitude of the loss is ignored. This is a commonly criticized shortcoming of VaR constraints and has been analyzed in Basak and Shapiro (2001) and Shi and Werker (2012).

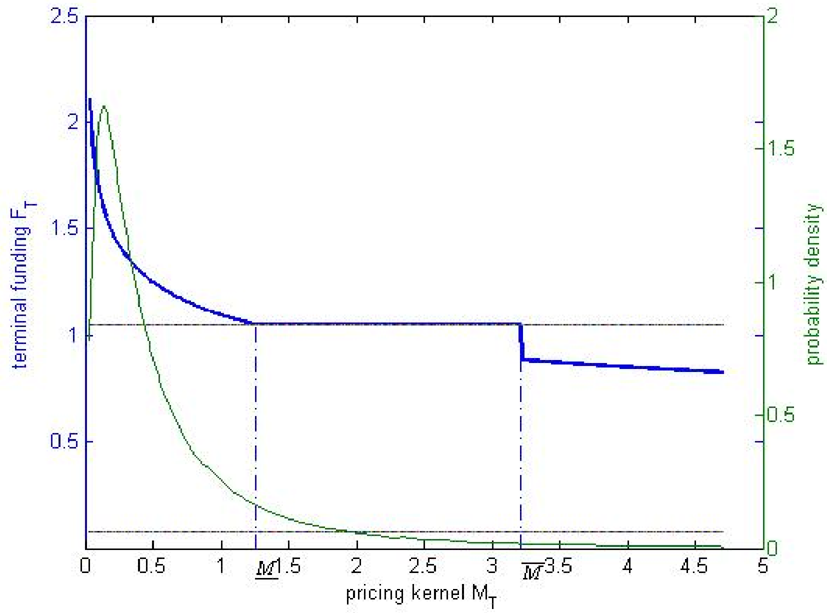


Figure 4.3: Terminal funding ratio for the long term VaR constrained problem in different states of the world. y-axis on the right is the PDF of pricing kernel.

Using the martingale property of $\{M_tA_t\}$ and the Itô's rule, the optimal asset weights for the long term constrained problem can be solved as

$$\begin{aligned}\omega_t &= \frac{1}{\gamma}\sigma^{-1}\lambda(1 - \frac{h}{F_t} - \frac{(k-h)[N(d_1(\overline{M})) - N(d_1(\min\{\underline{M},\overline{M}\}))]}{F_t}) \\ &+ \frac{(k-h)\sigma^{-1}[\phi(d_1(\overline{M})) - \phi(d_1(\min\{\underline{M},\overline{M}\}))]}{\sqrt{T-t}F_t} \\ &- y^{-\frac{1}{\gamma}}\Gamma_t M_t^{-\frac{1}{\gamma}}[\phi(d_2(\overline{M})) - \phi(d_2(\min\{\underline{M},\overline{M}\}))]\frac{1}{\sqrt{T-t}F_t},\end{aligned}$$

$$\begin{aligned}\text{where } \Gamma_t &= \exp(-(1-\frac{1}{\gamma})r(T_0 - t) + \frac{1-\gamma}{2\gamma^2}\lambda^2(T-t)) \\ d_1(x) &= \frac{\log(\frac{x}{M_t}) - (-r + \lambda^2/2)(T-t)}{\lambda\sqrt{T-t}} \\ d_2(x) &= \frac{\log(\frac{x}{M_t}) - (-r - \lambda^2/2 + (1-\frac{1}{\gamma})\lambda^2)(T-t)}{\lambda\sqrt{T-t}}.\end{aligned}$$

Note that this optimal weight requires dynamic rebalancing so our discretization might lead to inconsistencies. To illustrate how well the discrete version of the long term constrained strategies perform, we show in Figure 4.4 the terminal funding ratio distribution for an investor who adopts the optimal allocation given above and rebalances at annual, semiannual, monthly, daily, and continuous frequencies. As is shown in the graph, the discrete version of the strategy achieves a terminal wealth closer to the theoretical optimal one as rebalancing approaches continuous trading. Considering the tradeoff between accuracy and efficiency, we adopt the monthly rebalancing frequency here and also for the comparison of strategies in the next section.

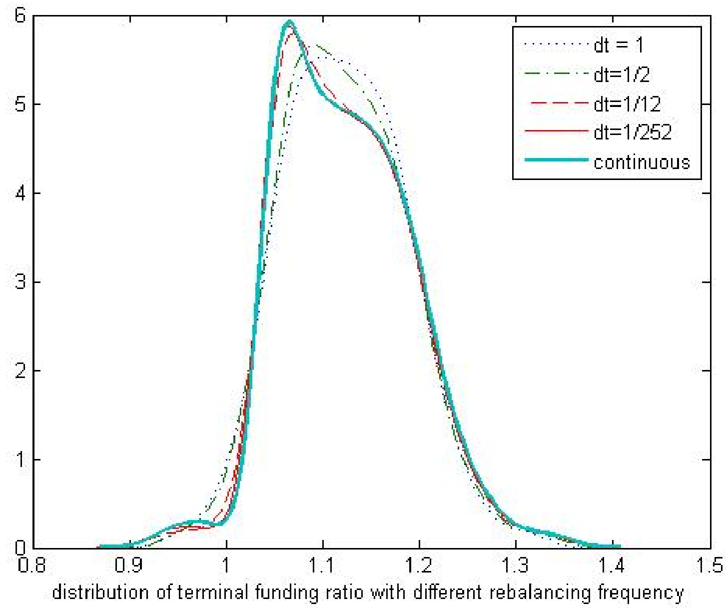
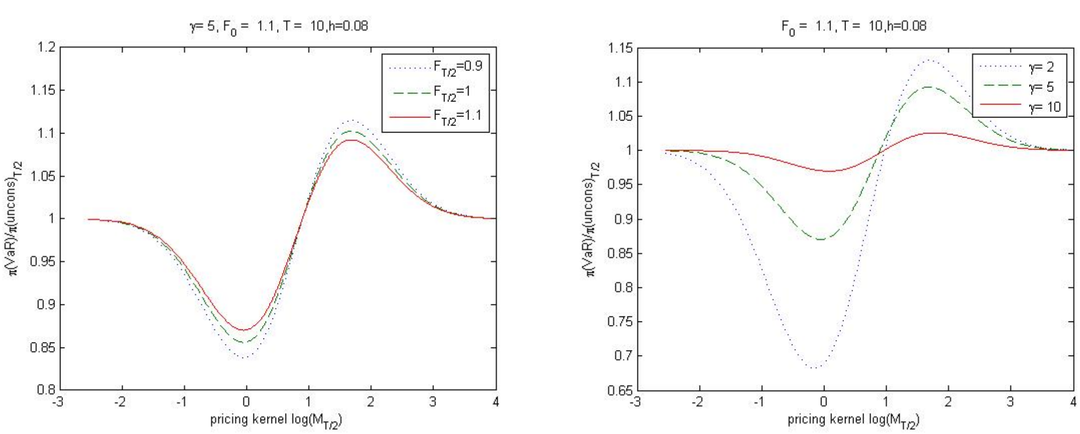


Figure 4.4: Terminal funding ratio distribution for the long term VaR constrained problem with different rebalancing frequencies.



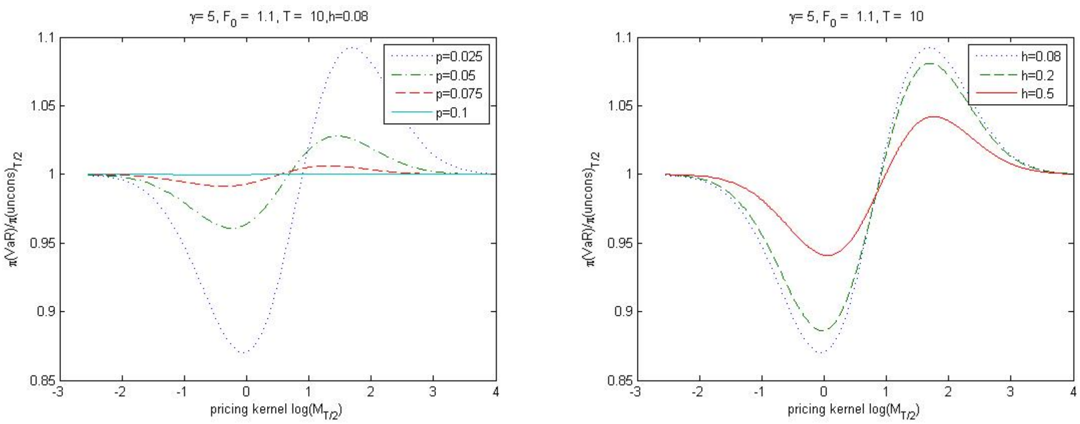


Figure 4.5: Relative allocation to stocks at time T/2.

Figure 4.5 shows the demand for stocks with long term VaR constraints relative to the unconstrained case under varying assumptions. In general, when state of the world gets worse, the VaR constrained investor first decreases his stock portfolio to satisfy the constraint. As the economy further deteriorates, the stock return is smaller but there is still chance to meet the constraint, so the investor puts more allocation to stocks as a "gambling" behavior. When state of the world is so bad that no strategy is available to meet the constraint, the constrained investor behaves like an unconstrained one and let the loss occur.

The top left panel shows the impact of current funding ratio on the allocation decision. A constrained investor facing lower current funding level is more sensitive to the economy since the constraint is more binding. The top right panel shows the decisions of investors with different risk aversion. More risk averse investors also invest more conservatively facing constraints. As seen from the bottom left panel, stricter regulations can lead to more extreme portfolio weights. Relaxing the allowed probability also shifts the start of the gambling behavior to better states of the world. The bottom right panel shows that the investor with higher internal subsistence level also invests more conservatively when facing constraints.

4.3 Short term constraints

With the short term constraints, the investor seeks to maximize expected terminal utility while keeping the probability of funding ratio one year later falling under k lower than p, taking into account possible future allocation decisions. We use dynamic programming to solve for the short term constrained strategies. The probability constraints are calculated assuming that asset weights are kept constant during the coming one year period. In addition, we impose short sales and borrowing constraints. All infeasible weights are excluded from the investor's choice set when performing dynamic programming.

In the following graphs we compare optimal strategies under different types of constraints.

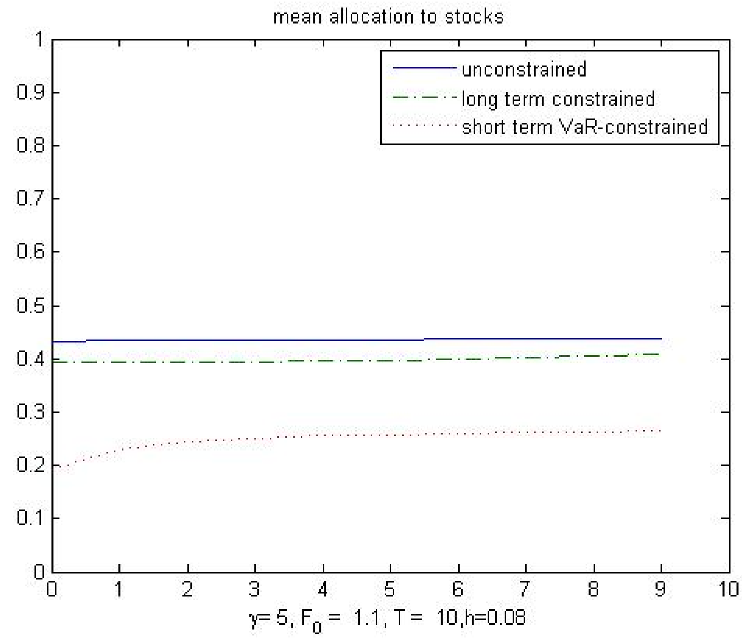


Figure 4.6: Mean allocation to stocks in the unconstrained, long term and short term constrained case

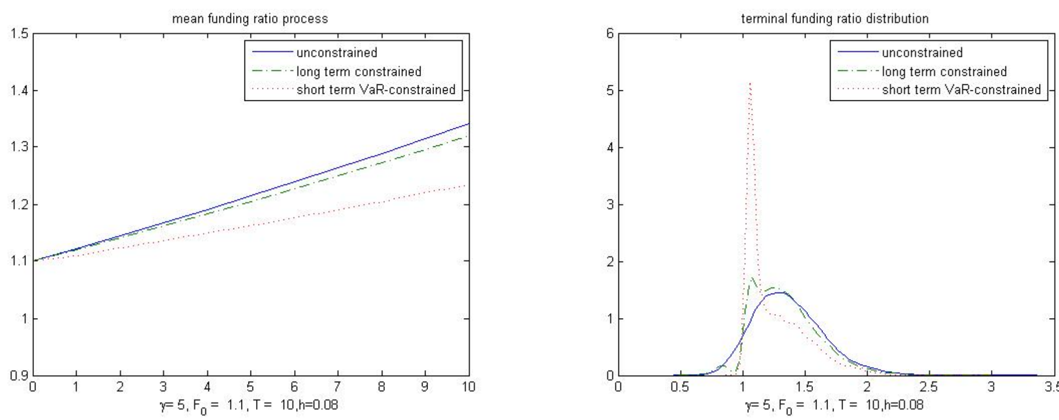


Figure 4.7: Left panel: mean funding ratio process in the unconstrained, long term and short term constrained case. Right panel: distribution of terminal funding ratio

Figure 4.6 shows the average allocation to stocks under the unconstrained, long term constrained and short term constrained cases. On average the unconstrained portfolio has the largest stock exposure compared with constrained ones and the short term constrained investor allocates much less to stocks than other two types of investors. Figure 4.7(a) shows the average funding ratio through time. As a result of the low stock exposure, the short term constrained investor has a much lower mean terminal funding ratio.

Figure 4.7(b) shows the probability density distribution of terminal funding ratios in the three cases. As is shown in the graph, the short term constraint guarantees the terminal funding ratio above the required level with a high probability, but loses probability mass in higher funding ratios. The long term constrained portfolio also results in a terminal funding ratio distribution that is more concentrated than the unconstrained case and loses probability of having a higher funding level. Furthermore, the long term constraints would incur large losses with a small probability.

To demonstrate how the different constraints impact the investment behavior, we show in Figure 4.8 the allocation strategies when the stock return path is set at its $25\%, 50\%, 75\%$ quantile at each time step respectively. When state of the world is bad, the long term constrained investor gambles in order to fulfil the constraint and the short term constrained investor is allowed to hold only a very small stock portfolio. In the median case the funding ratio tends to increase over time and the short term constraints become less binding and the short constrained strategy converges to the unconstrained one. In the last panel, the short constrained strategy quickly converges to the unconstrained case while the long term constrained investor takes less stock exposure because the terminal funding ratio is guaranteed to exceed the required "floor".

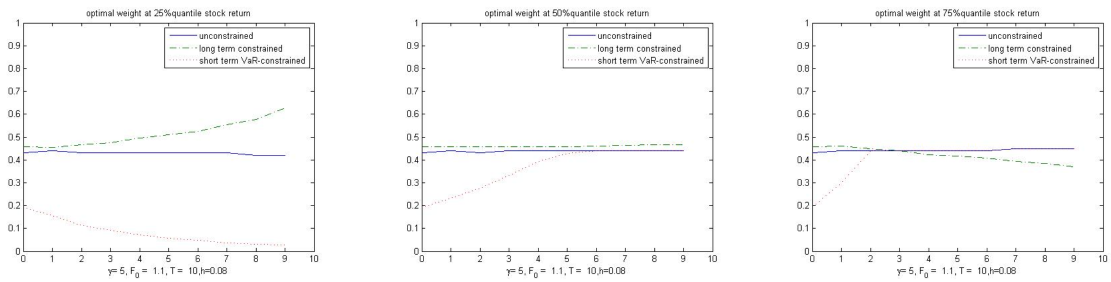


Figure 4.8: stock exposure when returns are at $25\%, 50\%, 75\%$ quantile

Finally, the certainty equivalent loss rate of the two types of constraints as opposed to unconstrained case is shown in Figure 4.9. This rate can be interpreted as the percentage of initial funding ratio reduction per year on the unconstrained strategy if it were to have the same expected utility as the constrained ones. For example, with our base parameter setting and a certainty equivalent loss rate of 10 basis points, a constrained investor starting with $F_0 = 1.1$ is equivalent to an unconstrained investor starting with approximately $F_0 = 1.09$. As initial funding ratio gets higher, the constraints are less binding, and certainty equivalent loss is smaller. The short term constraints lead to significantly higher certainty equivalent loss, around 40 basis points when $F_0 = 1.1$ and $\gamma = 5$, as opposed to long term constraints with 5 basis points.

The losses from long term or short term funding ratio constraints are higher when investor is less risk averse. Apparently when the investor is more risk-seeking, he would prefer investing in a larger equity portfolio, which is unacceptable because of the constraint.

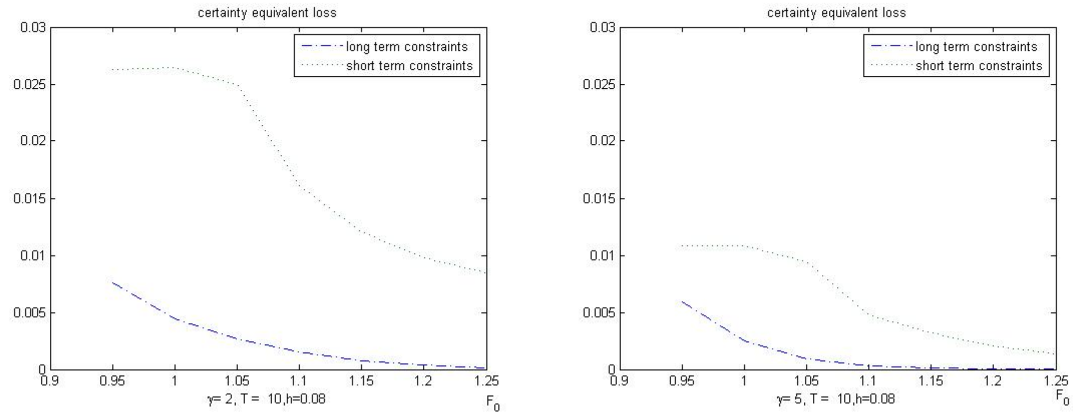


Figure 4.9: Certainty equivalent loss rate for short term and long term constrained investors. Left panel: $\gamma = 2$. Right panel: $\gamma = 5$.

In conclusion of this chapter, the VaR-type regulations on funding ratios (especially the short term ones) have a significant impact on investors allocation decisions. Although effectively reducing the probability of ending up with low funding levels, the constraints also confine the funding level close to the regulatory floor and inhibit the investor from harvesting higher potentials of stock return. The long term regulatory constraints do not have such a strong impact on the average stock exposure, but might lead to gambling behavior and the small probability of incurring large losses is unhedged. Both types of constraints lead to large utility losses when initial funding ratio is close to the regulatory floor.

Chapter 5

Dynamic Portfolio Choice, Stochastic Inflation and Stochastic Interest Rate

In the last chapter we showed the gains and losses of long term and short term funding ratio contraints when there is only one risky asset and interest rate is constant. In the real world this is rarely the case; pension fund managers have to take into account inflation risk and interest rate risk, among many others. In this chapter we add these complications and investigate the demand for nominal and real bonds under different types of constraints. We first look at the fixed weight strategy which is more realistic to pension fund: investors make decisions on strategic asset allocations over a relatively long horizon and rebalance to the same weights every period during the horizon. This strategy is dynamic in nature and is cost-efficient to implement, but often leads to "buying high and selling low". In a next section we turn to dynamic strategies where investors take into account possible future scenarios and allocation decisions when making decisions for each period. Without transaction costs and liquidity concerns this is superior to other strategies.

5.1 Fixed Weight Strategy

When $h > 0$, there is always a positive possibility that real funding ratio falls below the subsistence level no matter what fixed weight the investor chooses. In such case utility as well as expected utility goes to minus infinity, which makes the optimization problem infeasible. One way to circumvent this problem is to set a floor ε to the utility function: $U^*(A_T/L_T) = \max\{\frac{(F_T-h)^{1-\gamma}}{1-\gamma}, \varepsilon\}$. However, the truncation of the utility function introduces a bias into our solution. An alternative way is to invest in a real hedging portfolio with payoff hL_T and the rest in a fixed weight portfolio. As long as the investor rebalances the fixed weight portfolio continuously, this part of wealth is always positive, so the total real funding ratio is always above h. In other words, $A_t = hL_t + A_t^*, A_0 = hL_0 + A_0^*, A_T^* = A_0^* \exp[\int_0^T (r_u + \boldsymbol{\omega'\sigma'_u\lambda} - \frac{1}{2}||\boldsymbol{\omega'\sigma'_u}||^2)du + \boldsymbol{\omega'\sigma'}_u d\boldsymbol{z}_u]$. This is equivalent to $F_T^* = \frac{A_T^*}{L_T}, F_0^* = F_0 - h, \max_\omega E[\frac{(F_T^*)^{1-\gamma}}{1-\gamma}]$. As a result,we can focus on the "fixed weight" portfolio where effectively $h = 0$. Then the optimal fixed weight strategy consists of the famous speculative portfolio and liability hedging portfolio:

$$\omega = \left(\int_0^T \boldsymbol{\sigma}_u' \boldsymbol{\sigma}_u du\right)^{-1} \left(\frac{1}{\gamma}\int_0^T \boldsymbol{\sigma}_u' \boldsymbol{\lambda} du + (1-\frac{1}{\gamma})\int_0^T \boldsymbol{\sigma}_u' \boldsymbol{\sigma}_{L,u} du\right)$$

Next, we incorporate probabilistic constraints on terminal funding ratio. In the numerical setting we also impose short sale constraints that the weights on stocks, nominal bonds and real bonds are all between 0 and 1 and sum up to equal or smaller than one. Note that the choice set of feasible weights depends on initial funding ratio, so there does not exist such weights that satisfy all intermediate funding ratio constraints.

The nominal funding ratio is always larger than the real funding ratio because of the indexation. The magnitude of the ratio of nominal to real funding is

$$\begin{aligned} F_{nom,t}/F_{real,t} &= \frac{A_t/L_{nom,t}}{A_t/L_{real,t}} = \frac{L_t}{L_{nom,t}} = \frac{I(t,T_0)}{B(t,T_0)} \\ &= \Phi_t \exp[\varphi(T_0-t) + \frac{1}{a}\boldsymbol{\sigma}_r' \boldsymbol{\sigma}_\Phi(\alpha(T_0-t)-(T_0-t))] \end{aligned}$$

This ratio is on average quite stable through time, in our parameter setting approximately 1.708 with several basis points variation in a 10 year horizon. We fix this ratio at $F_{nom2real} = F_{nom,0}/F_{real,0} = 1.708$, thus making the nominal and real funding ratio constraints comparable. When the regulatory "floor" on nominal constraints is k, we denote the equivalent regulatory "floor" on real constraints as $\bar{k} = k/F_{nom2real}$, since the time 0 price of buying a nominal bond with payoff $kL_{nom,T}$ is equal to the time 0 price of buying a real bond with payoff $\bar{k}L_T$.

We illustrate the differences between nominal constrained and real constrained fixed weight strategy. The rebalancing period is one year so at the beginning of each year the investor adjusts his portfolio to make sure the portfolio weight is the same as the predetermined one. Notice that this leads to "buy high and sell low" and the investor suffers losses as opposed to the dynamic portfolio choice. We run the simulation with $N = 10000$ sample paths. Figure 5.1 show the allocation to each asset with varying intial (nominal) funding ratios. We impose the assumption that investor allocates 100% to the riskless asset (nominal bond in the nominal constrained case and real bond in the real constrained case) if no strategy can satisfied the VaR constraint. When initial funding is high, the constraints are not binding, and the nominal and real constraints converge to the unconstrained strategy. When initial funding ratio is just above the regulatory required level k, the nominal constrained investor holds a large amount of nominal bond, but if initial funding is higher he turns to real bonds. This can be explained by the high correlation between real bond returns and nominal bond return. Nominal bond and real bond act as substitutes in terms of inflation hedging and since utility is in real terms the investor turns to real bonds as soon as he makes sure the nominal VaR constraint is satisfied.

We also include a sensitivity analysis later to see whether different assumptions on inflation volatility changes our conclusion. See Chapter 6 for further details.

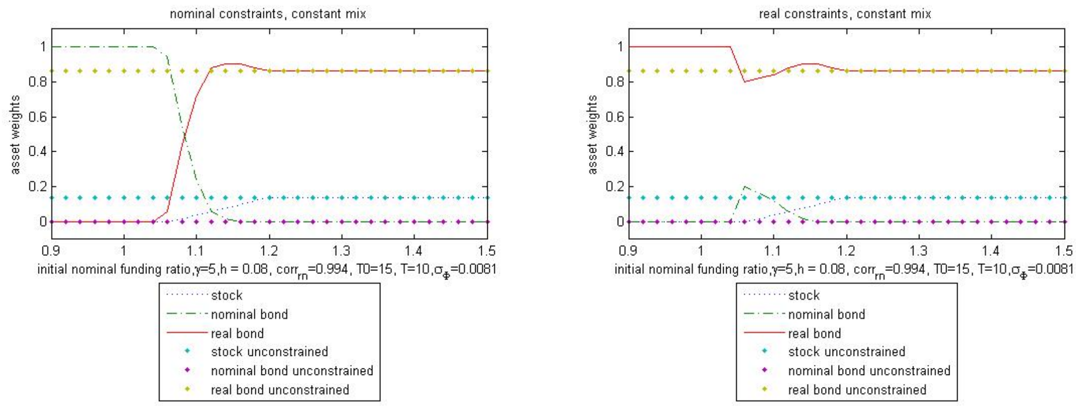


Figure 5.1: asset weights, $\gamma = 5$

Figure 5.2 shows that the nominal constrained investor suffers from approximately one more basis point of certainty equivalent loss rate compared to the real constrained case when initial funding ratio is very low. This is because the “safe” asset for a nominally constrained investor is the nominal bond, and when no strategy exists to satisfy the constraints, the said investor has to allocate fully in the safe asset. As initial funding ratio becomes larger, the certainty equivalent loss is smaller since there are less states where the constraints are met. Compared with the certainty equivalent loss in the last chapter, the certainty equivalent loss in the stochastic interest rate and inflation case is much smaller, since in the constant interest rate case an investor bound by the constraints gives away his relatively larger stock exposure for riskless cash, while the demand for stocks in comparable settings in this chapter is much smaller due to the hedging demand. The real bond and nominal bond returns are almost perfectly correlated, making them good substitutes for each other, so the utility loss from exchanging one for another is not very significant.

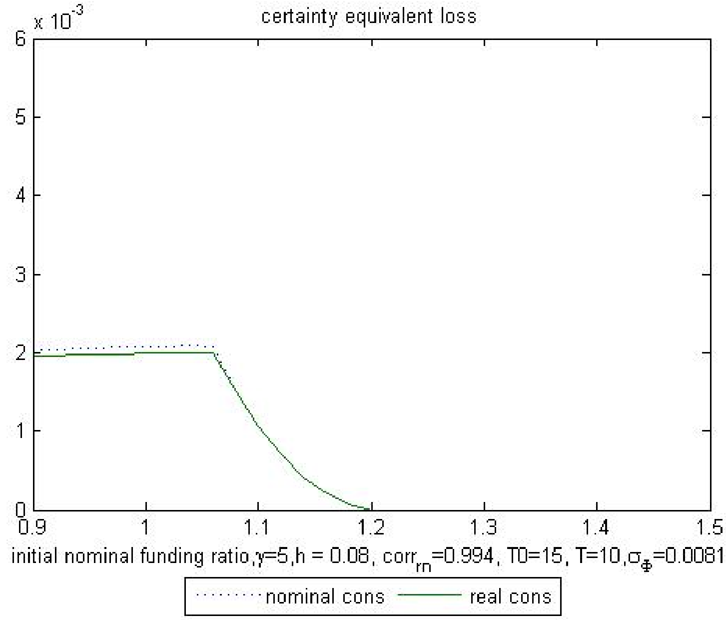


Figure 5.2: Certainty equivalent losses under the two types of constraints

The shortfall probability is the probability of having a terminal (nominal) funding ratio smaller than k. Intuitively, when initial funding ratio is low, the constrained investors are forced to hold riskless assets and can never benefit from high returns of equities, thus having a much larger probability of shortfall than the unconstrained investor. When initial funding ratio is larger than k, there exist feasbile weights to satisfy the constraints, so the shortfall probability of constrained investors are kept to the level p while the unconstrained investor suffers from being "too aggressive".

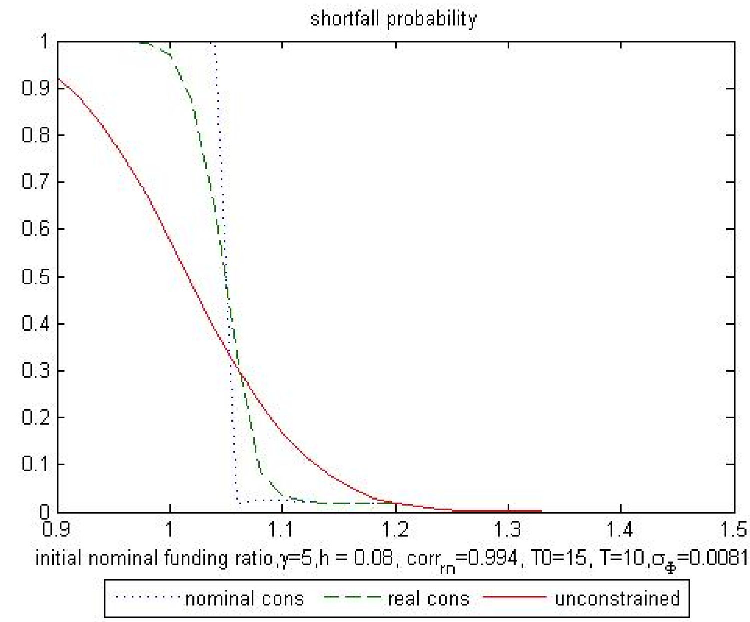


Figure 5.3: Shortfall probability under no constraint, nominal constraints and real constraints.

Expected shortfall size is the magnitude of shortfall $\max\{k - F_T, 0\}$. It is shown in the following graph that the expected shortfall size is limited when constraints are binding.

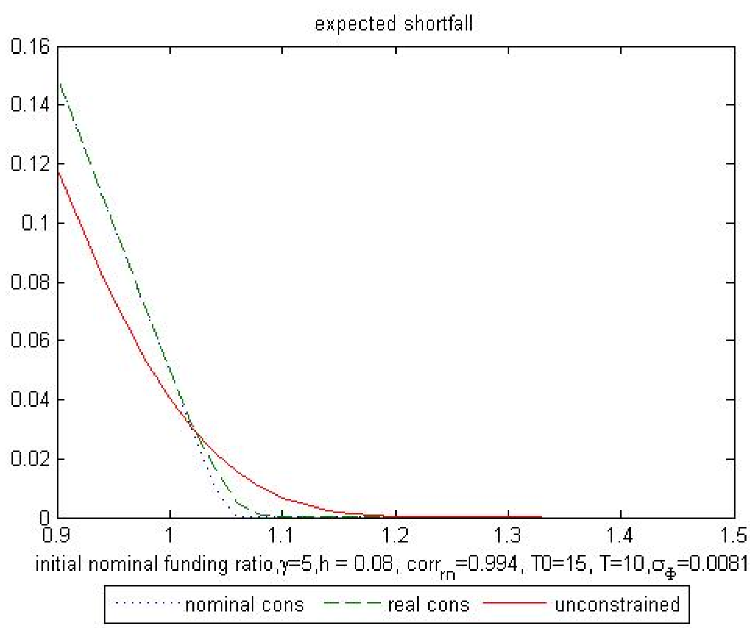


Figure 5.4: expected shortfall under no constraint, nominal constraints and real constraints.

As the constraint becomes binding, the investor allocates less to stocks and more to the hedging portfolio. In the nominal constrained case, the investor only invests in nominal bonds when the constraint is very close to initial funding ratio. One explanation is that when the real bond and nominal bond returns are highly correlated and real bond has a higher Sharpe Ratio, the investor turns to the real bond as the hedging portfolio if the VaR constraint is not very stringent.

The optimal allocation never invests in cash; in fact, if we loose the short sale constraint the investor would short nominal bonds and cash to buy real bonds and stocks. This is reasonable since when the utility and regulatory constraints are over terminal funding ratio, the bonds that can replicate liabilities are the new "risk free" asset instead of cash.

Comparing the portfolio choice of investors with different risk aversion parameters, we find that they behave similarly when constraints are binding: they have approximately the same allocations. As initial funding ratios are higher and the regulatory constraints are less binding, the investor behave more according to their own risk aversion. The certainty equivalent loss is again larger for less risk averse investors. The more risk averse the investor is, the larger the difference between CE losses from nominally and real-ly constrained strategies.

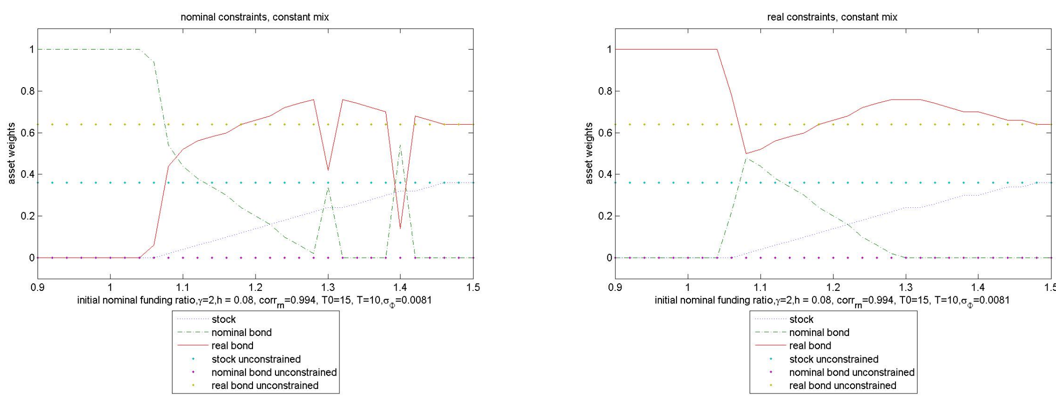


Figure 5.5: Optimal asset weights, $\gamma = 2$

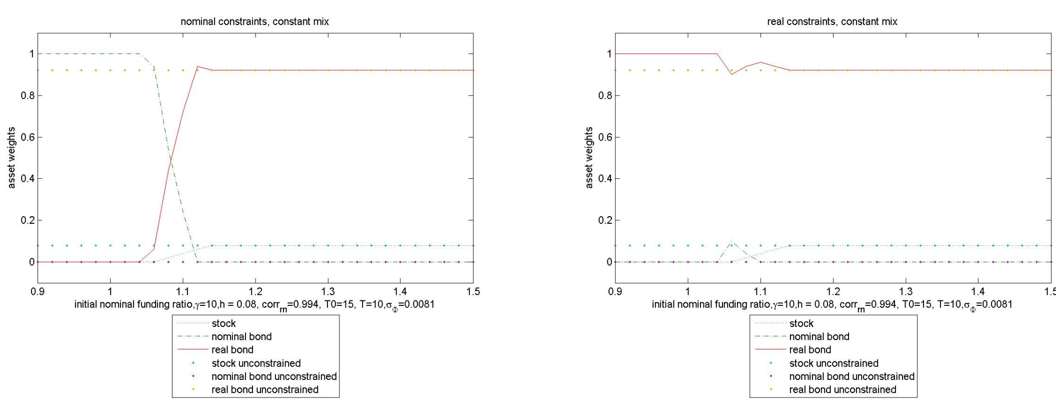


Figure 5.6: Optimal asset weights, $\gamma = 10$

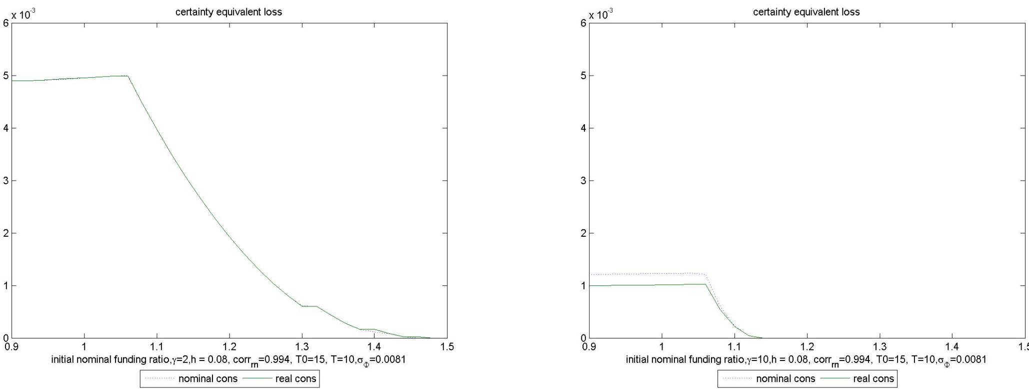


Figure 5.7: CE loss rate

One interesting observation is that in the real constrained case, there is still some demand for nominal bonds when initial funding ratio is very close to the floor k. This can be explained by the difference in bond return volatilities. The nominal bond return is slightly less volatile than the real bond return, and the correlation between these two returns is very high, leading to the supplement effects when constraints are binding.

5.2 Dynamic Asset Allocation

When investors are not bound by short sale constraints or transaction cost/ liquidity concerns, they are able to continuously trade in the given asset classes and benefit from dynamic portfolio choice using new information.

Unconstrained

Using the Martingale Approach, the optimal terminal wealth of an unconstrained investor is

$$A_T^* = [(yM_T L_T)^{-\frac{1}{\gamma}} + h]L_T$$

where y is chosen to fulfill the initial budget constraint $E[M_T A_T^*] = A_0$,

$$E[(yM_T L_T)^{-\frac{1}{\gamma}} M_T L_T + hM_T L_T] = A_0.$$

The optimal portfolio at each time t is therefore

$$
\begin{aligned}
A_t^* =& E_t[\frac{M_T}{M_t} A_T^*] \\
=& E_t \left[\frac{M_T}{M_t} \left(\frac{A_0 - E(hM_T L_T)}{E[(M_T L_T)^{1-\frac{1}{\gamma}}]} (M_T L_T)^{-\frac{1}{\gamma}} + h \right) L_T \right] \\
=& \frac{A_0 - hL_0^{T,T_0}}{M_t E[(M_T L_T)^{1-\frac{1}{\gamma}}]} E_t[(M_T L_T)^{1-\frac{1}{\gamma}}] + hL_t,
\end{aligned}
$$

and we have the optimal weight

$$
\boldsymbol{\omega}_t = [\frac{1}{\gamma}\boldsymbol{\sigma}_t^{-1}\boldsymbol{\lambda} + (1 - \frac{1}{\gamma})\boldsymbol{\sigma}_t^{-1}\boldsymbol{\sigma}_L](1 - \frac{h}{F_t}) + \boldsymbol{\sigma}_t^{-1}\boldsymbol{\sigma}_L \frac{h}{F_t}.
$$

See Appendix G for detailed derivation.

Again the allocation is dependent on current funding ratio when h is nonzero. The investor sets aside a real hedging portfolio that has payoff hL_t to guarantee the subsistence level. The rest of the portfolio wealth is invested in the performance seeking portfolio that gives the largest Sharpe ratio and the liability hedging portfolio which has the largest correlation with pension liabilities.

Long term constraints

The terminal wealth of an investor with long term constraints on real funding ratio is

$$
A_T^{VaR} = \begin{cases} (yM_T)^{-\frac{1}{\gamma}} L_T^{1-\frac{1}{\gamma}} + hL_T & \text{if } M_T < \underline{M}, \\ \bar{k} L_T & \text{if } \underline{M} \le M_T < \overline{M}, \\ (yM_T)^{-\frac{1}{\gamma}} L_T^{1-\frac{1}{\gamma}} + hL_T & \text{if } M_T \ge \overline{M} \end{cases}
$$

where $\underline{M}$ is such that $(y\underline{M})^{-\frac{1}{\gamma}} L_T^{1-\frac{1}{\gamma}} + hL_T = \bar{k}L_T$,
$\overline{M}$ is such that $\Pr_0(M_T > \overline{M}) = p$.
Since the pricing kernel is lognormally distributed, denote $log(M_T/M_t) \sim N(\mu_{M,t,T}, \sigma^2_{M,t,T})$, then $\underline{M} \equiv \frac{(\bar{k}-h)^{-\gamma}}{yL_T}$, $\overline{M} \equiv exp(\Phi^{-1}(1-p)\sigma_{M,0,T} + \mu_{M,0,T})M_0$.

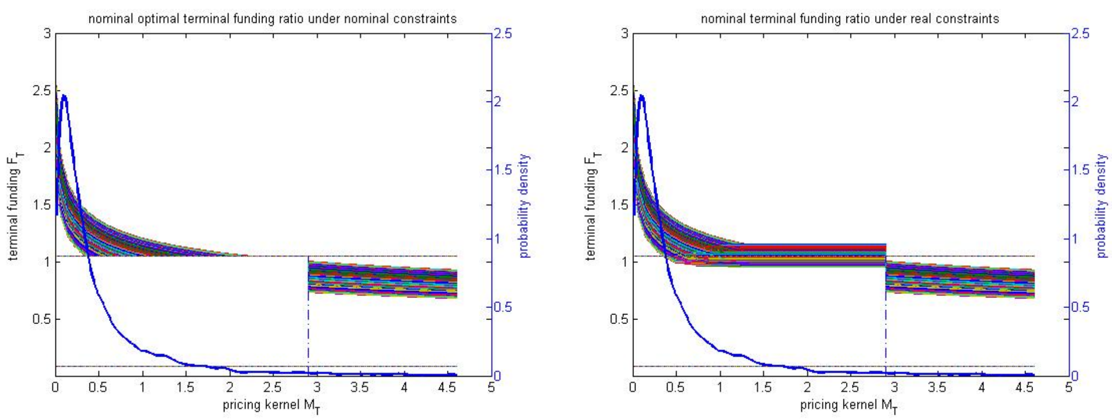
The terminal wealth of an investor with long term constraints on nominal funding ratio is

$$
A_T^{VaR,nom} = \begin{cases} (y_1 M_T)^{-\frac{1}{\gamma}} L_T^{1-\frac{1}{\gamma}} + hL_T & if\ M_T < \underline{M}, \\ kL_{nom,T} & if\ \underline{M} \le M_T < \overline{M}, \\ (y_1 M_T)^{-\frac{1}{\gamma}} L_T^{1-\frac{1}{\gamma}} + hL_T & if\ M_T \ge \overline{M} \end{cases}
$$

where $\underline{M}$ is such that $(y_1\underline{M_T})^{-\frac{1}{\gamma}} L_T^{1-\frac{1}{\gamma}} + hL_T = kL_{nom,T}$,
$\overline{M}$ is such that $\Pr_0(M_T > \overline{M}) = p$.

The upper threshold $\overline{M}$ is the same for both types of consttaints. In each state of the world, or each realizations of L_T and $L_{nom,T}$, there is a corresponding lower threshold $\underline{M}(L_T, L_{nom,T})$ which differs according to types of constraints. If $\overline{M} < \underline{M}$, the VaR constraint is never binding and $A_T^{VaR} = (yM_T)^{-\frac{1}{\gamma}} L_T^{1-\frac{1}{\gamma}} + hL_T = \overline{y}^{-\frac{1}{\gamma}}(A_T^u - hL_T) + hL_T$. Otherwise, the investor allocates in a "corridor option" to make sure that the terminal wealth in the intermediate states of the world $\underline{M} \leq M_T < \overline{M}$ is kept exactly at the required floor level ($\bar{k}L_T$ under real constraints and $kL_{nom,T}$ under nominal constraints).

Figure 5.8 shows the terminal funding ratios dependent on pricing kernel. In the real constrained case, nominal funding ratio is not guaranteed to be no smaller than k with a probability of $1 - p$.



(a) Real constrained

(b) Nominal constrained

Figure 5.8: terminal funding ratio

Figure 5.9 shows the distribution of terminal (nominal) funding ratio of an unconstrained investor, a long term nominally constrained investor and a long term real-ly constrained investor. As is demonstrated on the graph, the constraints effectively limit the probability of reaching a terminal funding ratio lower than 1.05, but also reduces probability mass in higher funding ratios.

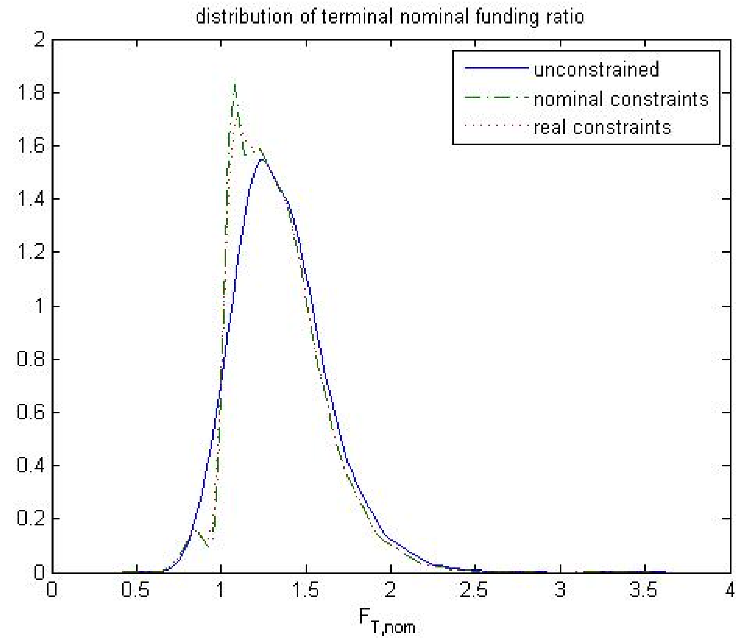


Figure 5.9: Distribution of terminal funding ratio under long term nominal constraints, long term real constraints and no constraints.

In Figure 5.10 we compare the certainty equivalent loss induced by constraints on terminal nominal and real funding ratio. The loss is measured against the theoretical optimal dynamic strategy where the investor continuously rebalances and is not bound by long only constraint. The nominal constraints lead to slightly higher certainty equivalent losses than real ones, approximately one basis points for the same initial funding ratios. Compared to the fixed weight strategy, the magnitude of the losses in the dynamic asset allocation case is larger when initial funding ratio is low and smaller when initial funding ratio is high since the dynamic investor is allowed to go short in order to make sure he satisfies the long term constraint.

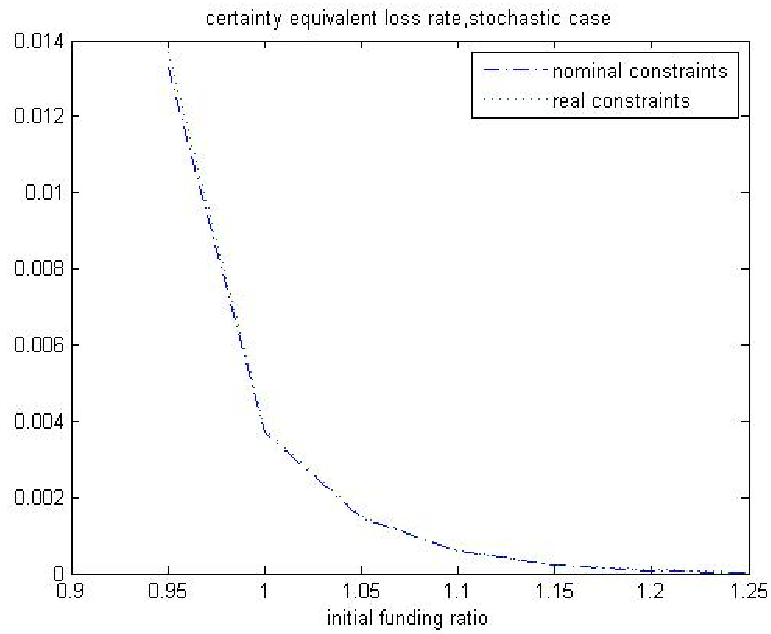


Figure 5.10: Certainty equivalent loss of long term nominal and real constraints.

We show the benefits of the constraints in Figures 5.11 and 5.12. Compared with the unconstrained case, both the real constraints and nominal constraints significantly lower the shortfall probability and expected shortfall size of terminal (nominal) funding ratio. The nominal constraints are more effective in reducing the probability and magnitude of nominal funding shortfalls.

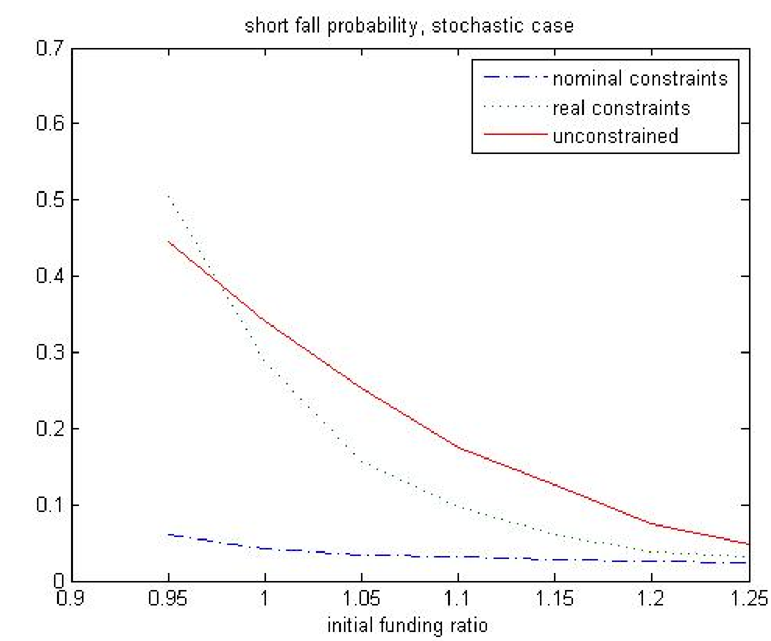


Figure 5.11: Shortfall probability under long term nominal and real constraints and no constraints.

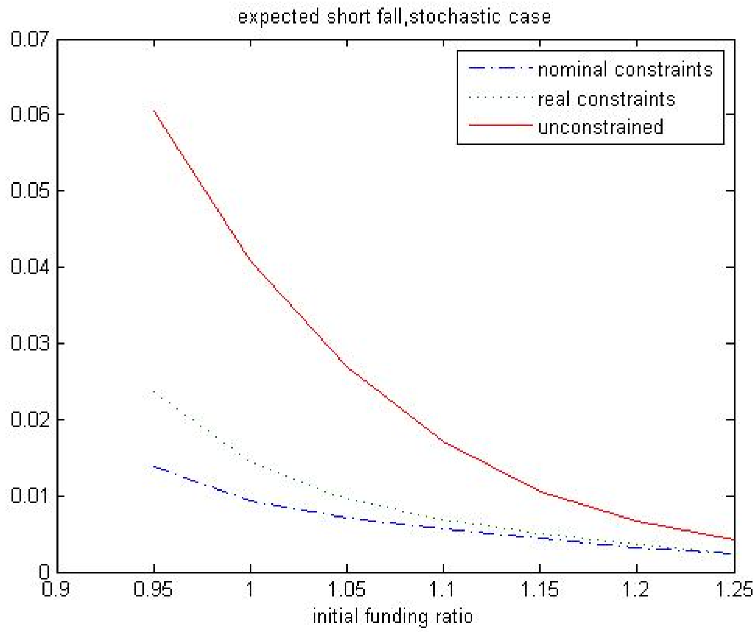


Figure 5.12: Expected shortfall under long term nominal and real constraints and no constraints.

Short term constraints

With short term constraints the problem can only be solved using dynamic programming methods. However, when there are too many state variables or asset classes it suffers from the curse of dimensionality, making it infeasible. It is therefore necessary to use approximations and make certain assumptions to circumvent this problem.

One of the possible solutions is the simulation based dynamic programming method by Brandt, Goyal, Santa-Clara and Stroud (2005) to solve for optimal portfolio weights. The main idea of Brandt et al. (2005) is to use Taylor series expansions to approximate the value function at each time point, and then solve for the FOC which is a polynominal containing optimal weight and some conditional expectations. They used a fourth order Taylor expansion, which means the optimal weight has to be solved iteratively given a starting value. The starting value of the weight is obtained by using a second order expansion of the value function. However, this iteration may not converge to optimal and is infeasible in the presence of probabilistic constraints as in our case. Instead we find the optimal portfolio through a grid search across all feasible portfolio, as proposed in van Binsbergen and Brandt (2008).

As soon as the conditional expectations mentioned above are known, we can solve for the optimal weights numerically using the grid search. Brandt et al. (2005) proposed to regress the realization of time $t + 1$ value of the expressions in the conditional expectation on some polynominals of the state varibales at time t accross the simulated sample paths. The fitted values of these regressions are used as estimates of the conditional expectations.

This approach avoids grid search over all state varibles, however is still time-consuming.

Koijen, Nijman and Werker (2007) improves the optimization by parameterizing the regression coefficients as an affine expansion of the portfolio weights, reducing the number of cross-sectional regressions. The linearity of the coefficients makes it possible to use KarushKuhnTucker conditions instead of iteratively solve for the FOC when finding optimal weights. However, with nonlinear probability constraints in our problem, KKT conditions can not be applied. We therefore relegate this challenge to future research on this topic.

Chapter 6

Sensitivity Analysis

6.1 Sensitivity to assumptions on inflation volatilities

We first investigate the differences between regulatory constraints on nominal and real funding ratio under different assumptions on inflation volatility for a fixed weight investor. The pattern of the constrained investor behavior is similar: when initial funding ratio is close to the regulatory floor, the investor takes less stock exposure than the unconstrained one, and turns to nominal bonds if the constraints are nominal.

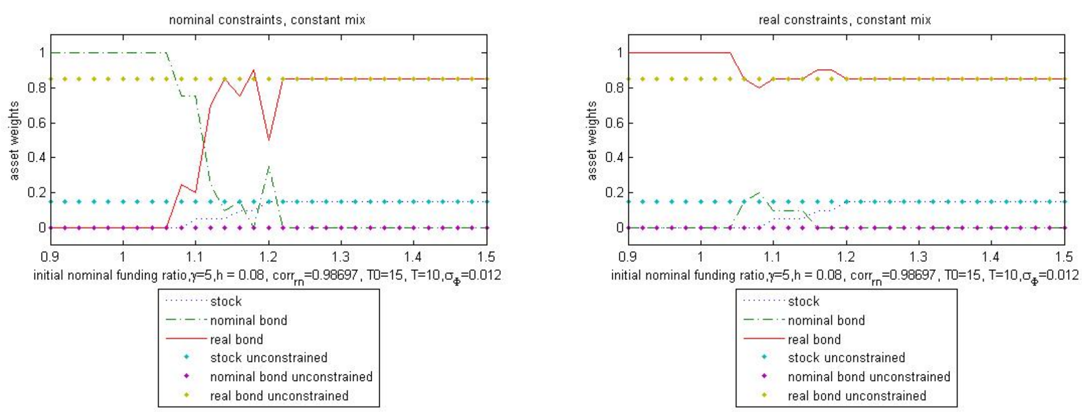


Figure 6.1: Optimal asset weights, left panel: nominal constraints; right panel: real constraints. $\sigma_\Phi = 0.012$.

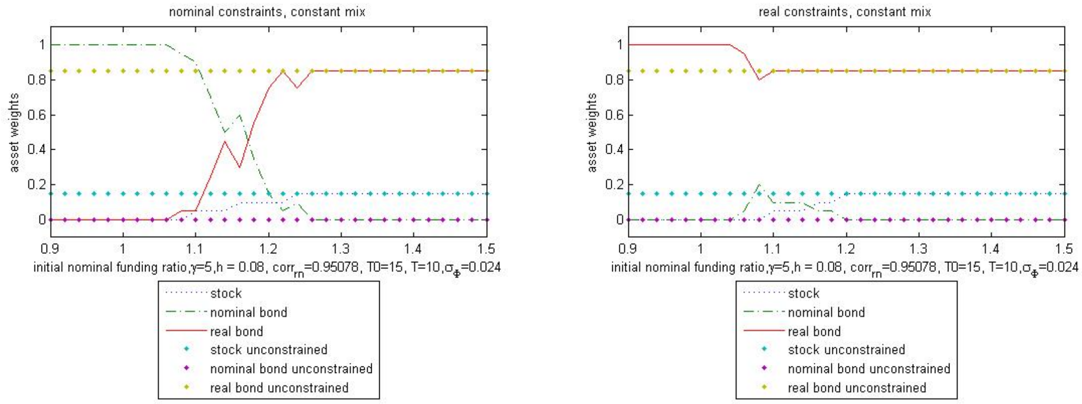


Figure 6.2: Optimal asset weights, left panel: nominal constraints; right panel: real constraints. $\sigma_\Phi = 0.024$.

As inflation becomes more volatile, the correlation between real and nominal bonds becomes smaller, and the investor can only fully replaces his nominal bond portfolio with real bonds when initial funding ratio is much higher. Nominal constraints incur slightly more certainty equivalent utility loss compared with real constraints. The difference between the CE losses incurred by nominal and real constraints is larger when the correlation between real and nominal returns is lower since the nominal constrained investor cannot use real bonds as a perfect substitute of nominal bonds. Figure 6.3 compares the certainty equivalent loss rates under different inflation volatilities. The certainty equivalent loss is smaller when initial funding ratio is higher, since there are less states where the constraints are binding.

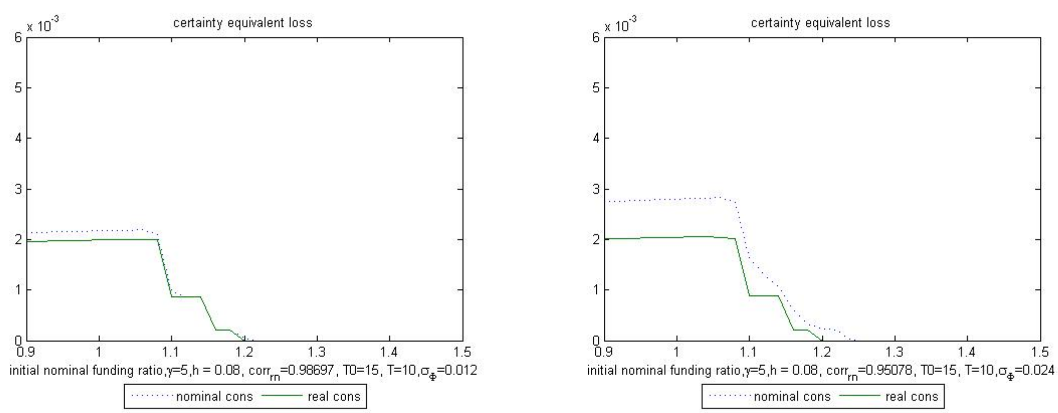


Figure 6.3: Certainty equivalent losses, left panel: $\sigma_\Phi = 0.012$. Right panel: $\sigma_\Phi = 0.024$.

6.2 Sensitivity to the inflation risk premium

The base case parameters used in our analysis as in Munk et al. (2004) assumes an inflation risk premium of 0. However, in reality investors require higher returns on nominal bond for bearing inflation risk. As a result, a negative inflation risk premium is expected in our model setting. Changing from $\lambda_\Phi = 0$ in the base case to $\lambda_\Phi = -0.02$ and $\lambda_\Phi = -0.05$, the optimal asset weights vary as initial funding ratio varies. When initial funiding is lower than 1.05, the investor holds no stocks at all since he is not allowed to take risks; as initial funding grows and becomes less binding, the stock exposure increases until it reaches the optimal level without constraints.

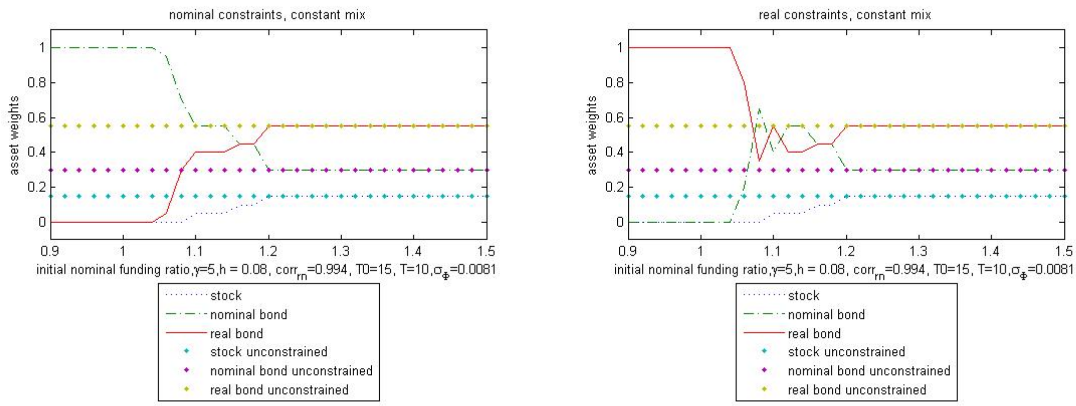


Figure 6.4: Optimal asset weights, left panel: nominal constraints; right panel: real constraints. $\lambda_\Phi = -0.02$.

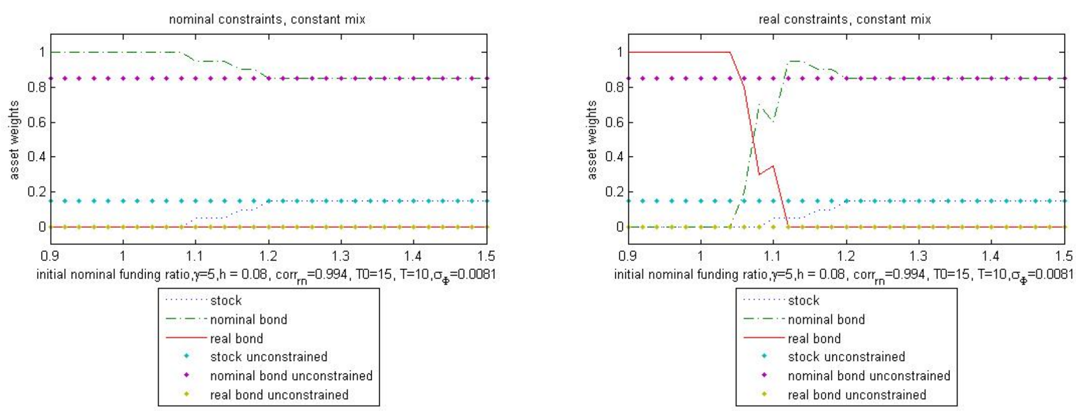


Figure 6.5: Optimal asset weights, left panel: nominal constraints; right panel: real constraints. $\lambda_\Phi = -0.05$.

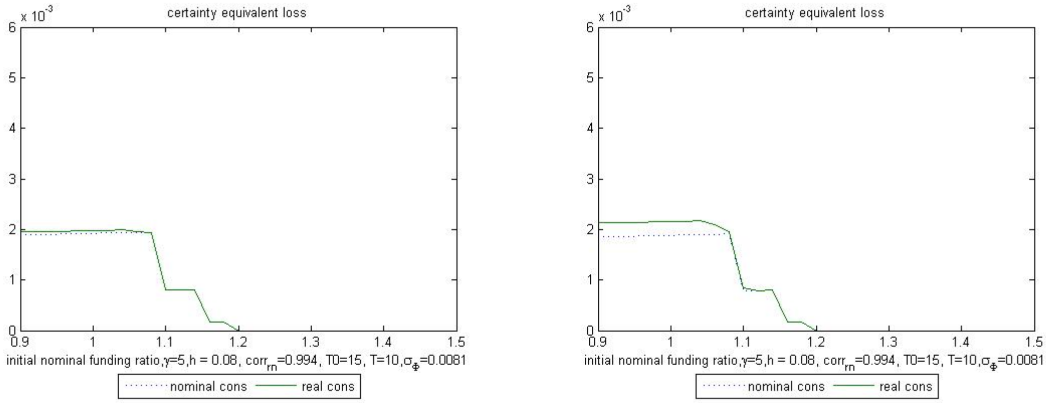


Figure 6.6: Certainty equivalent loss rate, left panel: $\lambda_\Phi = -0.02$. Right panel: $\lambda_\Phi = -0.05$.

The reversed pattern of investment behaviors under different inflation risk premium does not invalidate our earlier conclusions. Instead, it reinforces our conclusion that, when funding ratio constraints are binding, the investors are forced to hold assets in line with the constraints. In Figure 6.5, due to the raised inflation risk premium, an unconstrained investor prefers nominal bonds to real bonds. With binding nominal constraints, the investor allocates even more to nominal bonds. With binding nominal constraints, the investor allocates in real bonds even if the unconstrained demand for real bonds is zero.

Chapter 7

Conclusions and Recommendations

In this thesis we look at the implications of explicitly imposing regulatory constraints on funding ratios on pension funds asset allocation decisions. These constraints as a risk management measure help regulators control the risk of underfunding of defined benefit pension plans. However, such constraints as set by the Dutch pension regulations FTK are over nominal and short term funding ratio, while pension funds typically have long term goals to provide indexed (real) benefits. As we have shown in Figure 4.9, such conflict of interest incurs utility losses on pension investors. Under short term constraints as opposed to long term constraints, this utility loss is higher. Less risk averse investors suffer more from this utility loss since they would have invested in more risky assets without the constraints.

When the constraints on nominal funding ratios are binding, the investor is forced to hold nominal bonds that are otherwise undesirable because of their real ambitions. This is shown in Figure 5.1. Furthermore, comparing between Figures 5.1, 5.5 and 5.6 shows that constrained investors with different levels of risk aversion invest in similar strategies when constraints are binding, deviating from their separate levels of risk preference.

Another problem of the VaR-type regulatory constraints is that when pension funds do get in trouble, they are required not to take further risks, thus having to invest in 'safe' assets and stuck in the underfunding status. On the other hand, when initial funding ratio is high enough for feasible portfolio to exist, the constraints efficiently limit the shortfall probability to the required level, as is shown in e.g. Figures 5.3 and 5.4.

The conclusions we have drawn depend on the simplified setting in our problem. For example, we assume a complete market with real bonds with all durations, while in reality there may not be enough supply for index-linked bonds. Transaction costs and liquidity constraints might also prevent investors to benefit from dynamic asset allocation. It is also interesting to look at situations where liabilities are not completely hedgeable, which is the case in reality where longevity risk is not negligible.

Appendix

A Base case parameter settings

Parameter	Estimate
Interest rate process	
a	0.0395
$b + \boldsymbol{\sigma}_r' \boldsymbol{\lambda}/a$	0.0369
σ_r	0.0195
Price index process	
φ	0.0357
σ_Φ	0.0081
Stock price process	
σ_s	0.1468
Correlation parameters	
$\rho_{r\Phi}$	-0.0032
ρ_{sr}	-0.0845
$\rho_{r\Phi}$	-0.0678
Prices of risk	
λ_r	-0.2747
λ_Φ	0
λ_s	0.343
Pension fund parameters	
T	10
T_0, τ_1, τ_2	15
γ	5
$F_{nom,0}$	1.1
Regulatory constraints	
p	2.5%
k	1.05

B Normalization of Brownian motions

In order to simplify the calculations, we first transform $\{z_{r,t}, z_{s,t}, z_{\Phi,t}\}$ with respective pairwise correlation $\rho_{rs}, \rho_{r\Phi}, \rho_{s\Phi}$ into an orthogonal three-dimensional standard Brownian Motion and the corresponding volatility vectors as well as the prices of risk. This transformation is not unique and might have infinite other equivalent alternatives.

Denote a lower triangular matrix $P = \begin{pmatrix} 1 & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$. We solve for each element of P such that $P \begin{pmatrix} z_{r,t} \\ z_{s,t} \\ z_{\Phi,t} \end{pmatrix} = \boldsymbol{z}_t, E[\boldsymbol{z}_t] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, E[\boldsymbol{z}_t \boldsymbol{z}_t'] = \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix}$.

Since $E[z_{r,t}] = E[z_{s,t}] = E[z_{\Phi,t}] = 0, E[(z_{r,t}, z_{s,t}, z_{\Phi,t})(z_{r,t}, z_{s,t}, z_{\Phi,t})'] = t \begin{pmatrix} 1 & \rho_{rs} & \rho_{r\Phi} \\ \rho_{rs} & 1 & \rho_{s\Phi} \\ \rho_{r\Phi} & \rho_{s\Phi} & 1 \end{pmatrix}$, it is easy to solve that

$$P = \begin{pmatrix} 1 & 0 & 0 \\ \frac{-\rho_{rs}}{\sqrt{1-\rho_{rs}^2}} & \frac{1}{\sqrt{1-\rho_{rs}^2}} & 0 \\ \frac{\rho_{rs}\rho_{s\Phi}-\rho_{r\Phi}}{d} & \frac{\rho_{rs}\rho_{r\Phi}-\rho_{s\Phi}}{d} & \frac{1-\rho_{rs}^2}{d} \end{pmatrix}, d = \sqrt{(1-\rho_{rs}^2)(1-\rho_{rs}^2-\rho_{s\Phi}^2-\rho_{r\Phi}^2+2\rho_{rs}\rho_{s\Phi}\rho_{r\Phi})}.$$

Furthermore, since $(\boldsymbol{\sigma}_r, \boldsymbol{\sigma}_s, \boldsymbol{\sigma}_\Phi)' d\boldsymbol{z}_t \equiv (\sigma_r dz_{r,t}, \sigma_s dz_{s,t}, \sigma_\Phi dz_{\Phi,t})', (\boldsymbol{\sigma}_r, \boldsymbol{\sigma}_s, \boldsymbol{\sigma}_\Phi)' \boldsymbol{\lambda} \equiv (\sigma_r \lambda_r, \sigma_s \lambda_s, \sigma_\Phi \lambda_\Phi)'$, the volatility vectors can be written as $(\boldsymbol{\sigma}_r, \boldsymbol{\sigma}_s, \boldsymbol{\sigma}_\Phi) \equiv (P')^{-1} diag(\sigma_r, \sigma_s, \sigma_\Phi)$, and the price of risk vector is $\boldsymbol{\lambda} \equiv P(\lambda_r, \lambda_s, \lambda_\Phi)'$.

C Verification of the pricing kernel

To verify the pricing kernel, first we show that the discounted price of the stock index is a martingale under Q.

$$dS_t = S_t(r_t dt + \boldsymbol{\sigma}_s' d\tilde{\boldsymbol{z}}_t),$$

or equivalently,

$$\begin{aligned} S_T &= S_t \exp\left(\int_t^T (r_u - \frac{1}{2}||\boldsymbol{\sigma}_s||^2) du + \int_t^T \boldsymbol{\sigma}_s' d\tilde{\boldsymbol{z}}_u \right) \\ &= S_t \exp\left(\int_t^T r_u du - \frac{1}{2}||\boldsymbol{\sigma}_s||^2 (T-t) + \boldsymbol{\sigma}_s' (\tilde{\boldsymbol{z}}_T - \tilde{\boldsymbol{z}}_t) \right). \end{aligned}$$

For fixed T and t, $\boldsymbol{\sigma}_s'(\tilde{\boldsymbol{z}}_T - \tilde{\boldsymbol{z}}_t)$ is Gaussian with mean zero and variance $T - t$. Using the expression for expectations of log-normal random variables $E[e^Z] = e^{\mu + \frac{1}{2}\sigma^2}$ when $Z \sim N(\mu, \sigma^2)$, we obtain

$$\begin{aligned}E^Q\left[S_T\exp\left(-\int_t^T r_u du\right)\middle|\mathcal{F}_t\right] =&S_tE^Q\left[\exp\left(\log(\frac{S_T}{S_t})-\int_t^T r_u du\right)\middle|\mathcal{F}_t\right]\\ =&S_tE^Q\left[\exp\left(-\frac{1}{2}||\boldsymbol{\sigma}_s||^2(T-t)+\boldsymbol{\sigma}_s'(\tilde{\boldsymbol{z}}_T-\tilde{\boldsymbol{z}}_t)\right)\middle|\mathcal{F}_t\right]\\ =&S_t\exp\left(-\frac{1}{2}||\boldsymbol{\sigma}_s||^2(T-t)+\frac{1}{2}Var^Q[\boldsymbol{\sigma}_s'(\tilde{\boldsymbol{z}}_T-\tilde{\boldsymbol{z}}_t)|\mathcal{F}_t]\right)\\ =&S_t.\end{aligned}$$

Next we show that if the expression for the pricing kernel is correct, then $E^P[M_TS_T|\mathcal{F}_t] = M_tS_t$. Since

$$M_T = M_t\exp\left(-\int_t^T r_u du-\boldsymbol{\lambda}'(\boldsymbol{z}_T-\boldsymbol{z}_t)-\frac{1}{2}||\boldsymbol{\lambda}||^2(T-t)\right)$$

we have

$$\begin{aligned}E^P[M_TS_T|\mathcal{F}_t] =&M_tS_tE^P\left[\frac{M_T}{M_t}\cdot\frac{S_T}{S_t}\middle|\mathcal{F}_t\right]\\ =&M_tS_tE^P\left[\exp\left(-\int_t^T r_u du-\boldsymbol{\lambda}'(\boldsymbol{z}_T-\boldsymbol{z}_t)-\frac{1}{2}||\boldsymbol{\lambda}||^2(T-t)\right.\right.\\ &\left.\left.+\int_t^T r_u du+(\boldsymbol{\sigma}_s'\boldsymbol{\lambda}-\frac{1}{2}||\boldsymbol{\sigma}_S||^2)(T-t)+\boldsymbol{\sigma}_s'(\boldsymbol{z}_T-\boldsymbol{z}_t\right)\middle|\mathcal{F}_t\right]\\ =&M_tS_tE^P\left[\exp\left(-\frac{1}{2}||\boldsymbol{\lambda}-\boldsymbol{\sigma}_s||^2(T-t)-(\boldsymbol{\lambda}-\boldsymbol{\sigma}_s)'(\boldsymbol{z}_T-\boldsymbol{z}_t)\right)\middle|\mathcal{F}_t\right]\\ =&M_tS_t\exp\left(-\frac{1}{2}||\boldsymbol{\lambda}-\boldsymbol{\sigma}_s||^2(T-t)+\frac{1}{2}Var[(\boldsymbol{\lambda}-\boldsymbol{\sigma}_s)'(\boldsymbol{z}_T-\boldsymbol{z}_t)|\mathcal{F}_t]\right)\\ =&M_tS_t.\end{aligned}$$

Therefore the equation for the pricing kernel $\{M_t\}$ is correct.

Furthermore, the pricing kernel at a given time is lognormally distributed,

$$\log(\frac{M_T}{M_t}) = -\int_t^T r_u du-\boldsymbol{\lambda}'(\boldsymbol{z}_T-\boldsymbol{z}_t)-\frac{1}{2}||\boldsymbol{\lambda}||^2(T-t)\sim N(\mu_{M,t,T},\sigma^2_{M,t,T}),$$

with conditional mean

$$\begin{aligned}\mu_{M,t,T} &= E\left[-\int_t^T r_u du-\frac{1}{2}||\boldsymbol{\lambda}||^2(T-t)\middle|\mathcal{F}_t\right]\\ &= -(r_t-(b+\frac{\boldsymbol{\sigma}_r'\boldsymbol{\lambda}}{a}))\alpha(T-t)-(b+\frac{\boldsymbol{\sigma}_r'\boldsymbol{\lambda}}{a})(T-t)-\frac{1}{2}||\boldsymbol{\lambda}||^2(T-t),\end{aligned}$$

and conditional variance

$$\begin{aligned}\sigma_{M,t,T}^2 &= Var\left[-\int_t^T r_u du - \boldsymbol{\lambda}'(\boldsymbol{z}_T - \boldsymbol{z}_t)\middle|\mathcal{F}_t\right] \\ &= Var\left[-\int_t^T r_u du\middle|\mathcal{F}_t\right] + ||\boldsymbol{\lambda}||^2(T-t) + 2Cov\left[-\int_t^T r_u du, -\boldsymbol{\lambda}'(\boldsymbol{z}_T - \boldsymbol{z}_t)\middle|\mathcal{F}_t\right] \\ &= -\frac{||\boldsymbol{\sigma}_r||^2}{2a}\alpha(T-t)^2 - \frac{||\boldsymbol{\sigma}_r||^2}{a^2}(\alpha(T-t)-(T-t)) + ||\boldsymbol{\lambda}||^2(T-t) + 2\frac{\boldsymbol{\sigma}_r'\boldsymbol{\lambda}}{a}[(T-t)-\alpha(T-t)].\end{aligned}$$

D Price, dynamics and term structure of the nominal bond

This derivation is due to Mamon(2004).

Under the risk neutral probability measure Q,

$$dr_t = a(b - r_t)dt + \boldsymbol{\sigma}_r' d\tilde{\boldsymbol{z}}_t.$$

Multiply by e^{at} on both sides of the equation and apply Itô's lemma, we have

$$d(e^{at}r_t) = abe^{at}dt + e^{at}\boldsymbol{\sigma}_r' d\tilde{\boldsymbol{z}}_t.$$

Take the integral from $s = 0$ to $s = t$, we obtain

$$r_t = e^{-at}\left[r_0 + \int_0^t abe^{as}ds + \boldsymbol{\sigma}_r'\int_0^t e^{as}d\tilde{\boldsymbol{z}}_s\right],$$

and it follows that for $u > t$,

$$r_u = e^{-au}\left[e^{at}r_t + \int_t^u abe^{as}ds + \boldsymbol{\sigma}_r'\int_t^u e^{as}d\tilde{\boldsymbol{z}}_s\right].$$

Note that when $\delta(t)$ is a deterministic function of t and Z_t is a Brownian motion, then $\int_0^t \delta(u)dZ_u$ is Gaussian since by definition the stochastic integral is $\lim_{|\pi|\to 0}\sum_{i=0}^{n-1}\delta(u_i)(Z_{u_{i+1}} - Z_{u_i})$ (the limit is taken in terms of finer partition) and the increment $(Z_{u_{i+1}} - Z_{u_i}) \sim N(0, u_{i+1} - u_i)$.

Therefore r_t is a Gaussian random variable under Q, $r_t \sim N(\mu_t, \sigma_t^2)$, where

$$\mu_t = E^Q[r_t] = r_0 + \int_0^t a(b - E^Q[r_u])du = e^{-at}[r_0 + b(e^{at} - 1)].$$

The last equality is easily obtained by solving the ODE $\frac{d}{dt}\mu_t = a(b - \mu_t)$.

Furthermore, by Itô's isometry $\left(E\left[(\int_0^t X_u dZ_u)^2\right] = E\left[\int_0^t X_u^2 du\right]\right)$,

$$\begin{aligned}\sigma_t^2 &= Var^Q[r_t] = E^Q\left[\left(e^{-at}\boldsymbol{\sigma}_r'\int_0^t e^{au}d\tilde{\boldsymbol{z}}_u\right)^2\right] \\ &= e^{-2at}E^Q\left[||\boldsymbol{\sigma}_r||^2\int_0^t e^{2au}dt\right] \\ &= ||\boldsymbol{\sigma}_r||^2\left(\frac{1-e^{-2at}}{2a}\right)\end{aligned}$$

Again note that if r_t is Gaussian for all t, then $-\int_t^\tau r_u du$ is also Gaussian, with conditional mean

$$\mu_{r,t,\tau} = E^Q\left[-\int_t^\tau r(u)du\middle|\mathcal{F}_t\right] = -\int_t^\tau e^{-au}\left[e^{at}r_t + \int_t^u abe^{as}ds\right]du = -(r_t-b)\alpha(\tau-t) - b(\tau-t),$$

where $\alpha(s) = \frac{1-e^{-as}}{a}$, and conditional variance

$$\begin{aligned}\sigma_{r,t,\tau}^2 = Var^Q\left[-\int_t^\tau r_u du\middle|\mathcal{F}_t\right] &= Cov^Q\left[\int_t^\tau r_u du, \int_t^\tau r_s ds\middle|\mathcal{F}_t\right] \\ &= E^Q\left[\left(\int_t^\tau r_u du - E^Q\left[\int_t^\tau r_u du\middle|\mathcal{F}_t\right]\right)\left(\int_t^\tau r_s ds - E^Q\left[\int_t^\tau r_s ds\middle|\mathcal{F}_t\right]\right)\middle|\mathcal{F}_t\right] \\ &= \int_t^\tau\int_t^\tau E^Q\left[(r_u - E^Q[r_u|\mathcal{F}_t])(r_s - E^Q[r_s|\mathcal{F}_t])\middle|\mathcal{F}_t\right]duds \\ &= \int_t^\tau\int_t^\tau Cov^Q[r(u), r(s)|\mathcal{F}_t]duds \\ &= \int_t^\tau\int_t^\tau \frac{||\boldsymbol{\sigma}_r||^2}{2a}e^{-a(u+s)}(e^{2a\min\{u,s\}} - e^{2at})duds \\ &= \frac{||\boldsymbol{\sigma}_r||^2}{2a^3}(2a(\tau-t) - 3 + 4e^{-a(\tau-t)} - e^{-2a(\tau-t)}) \\ &= -\frac{||\boldsymbol{\sigma}_r||^2}{2a}\alpha(\tau-t)^2 - \frac{||\boldsymbol{\sigma}_r||^2}{a^2}(\alpha(\tau-t) - (\tau-t)).\end{aligned}$$

The fifth equality comes from Itô's isometry in the following equation, when $u, s > t$,

$$\begin{aligned}Cov^Q[r(u), r(s)|\mathcal{F}_t] &= E^Q\left[e^{-au}\int_t^u \boldsymbol{\sigma}_r' e^{av}d\tilde{\boldsymbol{z}}_v \cdot e^{-as}\int_t^s \boldsymbol{\sigma}_r' e^{av}d\tilde{\boldsymbol{z}}_v\middle|\mathcal{F}_t\right] \\ &= ||\boldsymbol{\sigma}_r||^2 e^{-a(u+s)}\int_t^{min\{u,s\}} e^{2av}dv \\ &= \frac{||\boldsymbol{\sigma}_r||^2}{2a}e^{-a(u+s)}(e^{2a\min\{u,s\}} - e^{2at}).\end{aligned}$$

The time t price of a risk-free bond that pays one unit of cash at maturity τ is

$$
\begin{aligned}
B(t,\tau) =& E^Q\left[1\cdot\exp\left(-\int_t^\tau r_u du\right)\bigg|\mathcal{F}_t\right]\\
=&\exp\left(E^Q\left[-\int_t^\tau r(u)du\bigg|\mathcal{F}_t\right]+\frac{1}{2}Var^Q\left[-\int_t^\tau r_u du\bigg|\mathcal{F}_t\right]\right)\\
=&\exp\left(-(r_t-b)\alpha(\tau-t)-b(\tau-t)-\frac{||\boldsymbol{\sigma}_r||^2}{4a}\alpha(\tau-t)^2-\frac{||\boldsymbol{\sigma}_r||^2}{2a^2}(\alpha(\tau-t)-(\tau-t))\right)\\
=&\exp\left(-\alpha(\tau-t)r_t+(\alpha(\tau-t)-(\tau-t))(b-\frac{||\boldsymbol{\sigma}_r||^2}{2a^2})-\frac{||\boldsymbol{\sigma}_r||^2}{4a}\alpha(\tau-t)^2\right)
\end{aligned}
$$

Denote $R_\infty = b-\frac{||\boldsymbol{\sigma}_r||^2}{2a^2}, \beta_1(s) = (\alpha(s)-s)R_\infty-\frac{||\boldsymbol{\sigma}_r||^2}{4a}\alpha(s)^2$, then the bond price can be written as

$$B(t,\tau)=\exp(-\alpha(\tau-t)r_t+\beta_1(\tau-t)).$$

Applying Itô's lemma, the bond price dynamics (under Q) can be written as:

$$
\begin{aligned}
dB(t,\tau) =& B(t,\tau)(-d\alpha(\tau-t)r_t-\alpha(\tau-t)dr_t+d\beta_1(\tau-t))+\frac{1}{2}B(t,\tau)\alpha(\tau-t)^2(dr_t)^2\\
=& B(t,\tau)[-(a\alpha(\tau-t)-1)r_t dt-\alpha(\tau-t)(a(b-r_t)dt+\boldsymbol{\sigma}_{\boldsymbol{r}}'d\tilde{\boldsymbol{z}}_t)\\
&+a\alpha(\tau-t)R_\infty dt-\frac{||\boldsymbol{\sigma}_r||^2}{2a}\alpha(\tau-t)(a\alpha(\tau-t)-1)dt]+\frac{1}{2}B(t,\tau)\alpha(\tau-t)^2||\boldsymbol{\sigma}_r||^2dt\\
=& B(t,\tau)[r_t dt-\alpha(\tau-t)\boldsymbol{\sigma}_r'd\tilde{\boldsymbol{z}}_t].
\end{aligned}
$$

Furthermore, the nominal term structure can be derived from the bond price. Denote with $R(t,\tau)$ the time t nominal interest rate with time to maturity $\tau-t$,

$$
\begin{aligned}
R(t,\tau) =& -\frac{1}{\tau-t}\log[B(t,\tau)] = -\frac{1}{\tau-t}[-\alpha(\tau-t)r_t+\beta_1(\tau-t)]\\
=& R_\infty+\frac{\alpha(\tau-t)}{\tau-t}(r_t-R_\infty)+\frac{||\boldsymbol{\sigma}_r||^2}{4a(\tau-t)}\alpha(\tau-t)^2.
\end{aligned}
$$

Since

$$\lim_{\tau\to t}\frac{\alpha(\tau-t)}{\tau-t}=1,\ \lim_{\tau\to\infty}\frac{\alpha(\tau-t)}{\tau-t}=0, \lim_{\tau\to t}\frac{\alpha(\tau-t)^2}{\tau-t}=0,\ \lim_{\tau\to\infty}\frac{\alpha(\tau-t)^2}{\tau-t}=0,$$

we can verify that $R(t,t)=r_t, R(t,\infty)=R_\infty$.

E Price, dynamics and term structure of the real bond

The time t price of a zero-coupon bond that pays Φ_τ at maturity τ is

$$I(t,\tau) = E^Q\left[\exp\left(-\int_t^\tau r_u du\right)\cdot\Phi_\tau\bigg|\mathcal{F}_t\right]$$

Under the risk neutral probability Q,

$$d\Phi_t = \Phi_t(\varphi dt + \boldsymbol{\sigma}'_\Phi d\tilde{\boldsymbol{z}}_t),$$

or equivalently,

$$\Phi_\tau = \Phi_t \exp\left[\int_t^\tau (\varphi - \frac{1}{2}||\boldsymbol{\sigma}_\Phi||^2)du + \boldsymbol{\sigma}'_\Phi d\tilde{\boldsymbol{z}}_u\right].$$

Similar to the derivation above, $\log\frac{\Phi_\tau}{\Phi_t}$ is also normally distributed with conditional mean

$$\mu_{\Phi,t,\tau} = E^Q\left[\int_t^\tau (\varphi - \frac{1}{2}||\boldsymbol{\sigma}_\Phi||^2)du\middle|\mathcal{F}_t\right] = (\varphi - \frac{1}{2}||\boldsymbol{\sigma}_\Phi||^2)(\tau - t)$$

and variance

$$\sigma^2_{\Phi,t,\tau} = Var^Q\left[\boldsymbol{\sigma}_\Phi(\tilde{\boldsymbol{z}}_\tau - \tilde{\boldsymbol{z}}_t)\middle|\mathcal{F}_t\right] = ||\boldsymbol{\sigma}_\Phi||^2(\tau - t).$$

Since the sum of two Gaussian random variables are still Gaussian, we have

$$\begin{aligned}
I(t,\tau) =&\Phi_t E^Q\left[\exp\left(-\int_t^\tau r_u du + \log\left(\frac{\Phi_\tau}{\Phi_t}\right)\right)\middle|\mathcal{F}_t\right] \\
=&\Phi_t \exp\left[\mu_{r,t,\tau} + \mu_{\Phi,t,\tau} + \frac{1}{2}\left(\sigma^2_{r,t,\tau} + \sigma^2_{\Phi,t,\tau} + 2Cov(-\int_t^\tau r_u du, \log\frac{\Phi_\tau}{\Phi_t}|\mathcal{F}_t)\right)\right] \\
=&\Phi_t \exp\left[-(r_t - b)\alpha(\tau - t) - b(\tau - t) + (\varphi - \frac{1}{2}||\boldsymbol{\sigma}_\Phi||^2)(\tau - t) + \frac{1}{2}\left(-\frac{||\boldsymbol{\sigma}_r||^2}{2a}\alpha(\tau - t)^2\right.\right. \\
&\left.\left. - \frac{||\boldsymbol{\sigma}_r||^2}{a^2}(\alpha(\tau - t) - (\tau - t))\right) + \frac{1}{2}||\boldsymbol{\sigma}_\Phi||^2(\tau - t) + \frac{1}{a}\boldsymbol{\sigma}'_r\boldsymbol{\sigma}_\Phi[\alpha(\tau - t) - (\tau - t)]\right] \\
=&\Phi_t \exp\left[-r_t\alpha(\tau - t) + b\alpha(\tau - t) - b(\tau - t) + \varphi(\tau - t) - \frac{||\boldsymbol{\sigma}_r||^2}{4a}\alpha(\tau - t)^2\right. \\
&\left. - \frac{||\boldsymbol{\sigma}_r||^2}{2a^2}\alpha(\tau - t) + \frac{||\boldsymbol{\sigma}_r||^2}{2a^2}(\tau - t) + \frac{1}{a}\boldsymbol{\sigma}'_r\boldsymbol{\sigma}_\Phi[\alpha(\tau - t) - (\tau - t)]\right],
\end{aligned}$$

where the covariance term is

$$\begin{aligned}
Cov(-\int_t^\tau r_u du, \log\frac{\Phi_\tau}{\Phi_t}|\mathcal{F}_t) =&E^Q\left[-\left(\int_t^\tau r_u du - E^Q\left[\int_t^\tau r_u du\right]\right)\left(\log\left(\frac{\Phi_\tau}{\Phi_t}\right) - E^Q\left[\log\left(\frac{\Phi_\tau}{\Phi_t}\right)\right]\right)\right] \\
=&E^Q\left[\int_t^\tau -(r_u - E[r_u])du \int_t^\tau \boldsymbol{\sigma}'_\Phi d\tilde{\boldsymbol{z}}_s\right] \\
=&\int_t^\tau -E^Q\left[\int_t^u e^{a(s-u)}\boldsymbol{\sigma}'_r d\tilde{\boldsymbol{z}}_s \int_t^\tau \boldsymbol{\sigma}'_\Phi d\tilde{\boldsymbol{z}}_s\right] du \\
=&\int_t^\tau -\boldsymbol{\sigma}'_r\boldsymbol{\sigma}_\Phi \int_t^u e^{a(s-u)} ds du \\
=&-\boldsymbol{\sigma}'_r\boldsymbol{\sigma}_\Phi \int_t^\tau \frac{1 - e^{a(t-u)}}{a} du \\
=&-\frac{\boldsymbol{\sigma}'_r\boldsymbol{\sigma}_\Phi}{a}((\tau - t) + \frac{e^{a(t-\tau)} - 1}{a}) = \frac{\boldsymbol{\sigma}'_r\boldsymbol{\sigma}_\Phi}{a}[\alpha(\tau - t) - (\tau - t)].
\end{aligned}$$

Note that $\int_0^s \alpha(s)ds = \frac{s-\alpha(s)}{a}$, $\int_0^s \alpha(s)^2 ds = \frac{s-\alpha(s)}{a^2} - \frac{\alpha(s)^2}{2a}$, and denote

$$\begin{aligned}\beta_2(s) =& b\alpha(s) - bs + \varphi s - \frac{||\boldsymbol{\sigma}_r||^2}{4a}\alpha(s)^2 - \frac{||\boldsymbol{\sigma}_r||^2}{2a^2}\alpha(s) + \frac{||\boldsymbol{\sigma}_r||^2}{2a^2}s + \frac{1}{a}\boldsymbol{\sigma}_r'\boldsymbol{\sigma}_\Phi[\alpha(s) - s]\\ =& b\alpha(s) + (\varphi - b)s + \frac{1}{2}\left(||\boldsymbol{\sigma}_r||^2 \int_0^s \alpha(u)^2 du - 2\boldsymbol{\sigma}_r'\boldsymbol{\sigma}_\Phi \int_0^s \alpha(u)du\right)\\ =& b\alpha(s) + (\varphi - b - \frac{||\boldsymbol{\sigma}_\Phi||^2}{2})s + \frac{1}{2}\int_0^s ||\alpha(u)\boldsymbol{\sigma}_r - \boldsymbol{\sigma}_\Phi||^2 du,\end{aligned}$$

we can rewrite the real bond price as

$$I(t, \tau) = \Phi_t \exp(-r_t\alpha(\tau - t) + \beta_2(\tau - t)) := \Phi_t \exp(A_t).$$

Applying Itô's lemma, the dynamics (under Q) of the real bond price is

$$\begin{aligned}dI(t, \tau) =& d\Phi_t \exp(A_t) + \Phi_t(\exp(A_t)dA_t + \frac{1}{2}\exp(A_t)dA_t^2) + d\Phi_t(\exp(A_t)dA_t + \frac{1}{2}\exp(A_t)dA_t^2)\\ =& I(t, \tau)\left((\varphi dt + \boldsymbol{\sigma}_\Phi' d\tilde{\boldsymbol{z}}_t) + dA_t + \frac{1}{2}dA_t^2 + (\varphi dt + \boldsymbol{\sigma}_\Phi' d\tilde{\boldsymbol{z}}_t)(dA_t + \frac{1}{2}dA_t^2)\right)\\ =& I(t, \tau)\Bigg((\varphi dt + \boldsymbol{\sigma}_\Phi' d\tilde{\boldsymbol{z}}_t) + d(-r_t\alpha(\tau - t) + \beta_2(\tau - t)) + \frac{1}{2}\alpha(\tau - t)^2 dr_t^2\\ &+ \boldsymbol{\sigma}_\Phi' d\tilde{\boldsymbol{z}}_t(-\alpha(\tau - t)dr_t + \frac{1}{2}\alpha(\tau - t)^2 dr_t^2)\Bigg)\\ =& I(t, \tau)((r_t dt - (\alpha(\tau - t)\boldsymbol{\sigma}_r - \boldsymbol{\sigma}_\Phi)' d\tilde{\boldsymbol{z}}_t)\end{aligned}$$

Denote with $R_I(t, \tau)$ the time t real yield with time to maturity $\tau - t$. By definition this is the yield of holding the real bond with real price $I(t, \tau)/\Phi(t)$ and real payoff of 1 from time t to τ,

$$\begin{aligned}R_I(t, \tau) =& -\frac{1}{\tau - t}\log[I(t, \tau)/\Phi_t] = -\frac{1}{\tau - t}[-\alpha(\tau - t)r_t + \beta_2(\tau - t)]\\ =& R_\infty - \varphi + \frac{1}{a}\boldsymbol{\sigma}_\Phi'\boldsymbol{\sigma}_r + \frac{\alpha(\tau - t)}{\tau - t}(r_t - R_\infty - \frac{1}{a}\boldsymbol{\sigma}_\Phi'\boldsymbol{\sigma}_r) + \frac{||\boldsymbol{\sigma}_r||^2}{4a(\tau - t)}\alpha(\tau - t)^2\end{aligned}$$

Since

$$\lim_{\tau\to t}\frac{\alpha(\tau - t)}{\tau - t} = 1, \lim_{\tau\to\infty}\frac{\alpha(\tau - t)}{\tau - t} = 0, \lim_{\tau\to t}\frac{\alpha(\tau - t)^2}{\tau - t} = 0, \lim_{\tau\to\infty}\frac{\alpha(\tau - t)^2}{\tau - t} = 0,$$

we can verify that $R_I(t, t) = r_t - \varphi, R_I(t, \infty) = R_\infty - \varphi + \frac{1}{a}\boldsymbol{\sigma}_\Phi'\boldsymbol{\sigma}_r$.

F Solution to fixed-weight problem

Suppose that at time 0 the fund manager chooses a fixed weight for the entire investment horizon and continuously rebalances to this position. The unconstrained problem is then

$$\max_{\omega} E[U(A_T/L_T)]$$
$$s.t. A_T = A_0 \exp[\int_0^T (r_u + \boldsymbol{\omega'\sigma'_u\lambda} - \frac{1}{2}||\boldsymbol{\omega'\sigma'_u}||^2)du + \boldsymbol{\omega'\sigma'}_u d\boldsymbol{z}_u],$$

where $\boldsymbol{\sigma}_t = (\boldsymbol{\sigma}_S, \boldsymbol{\sigma}_B, \boldsymbol{\sigma}_I) = (\boldsymbol{\sigma}_S, -\alpha(T-t)\boldsymbol{\sigma}_r, \boldsymbol{\sigma}_\Phi - \alpha(T-t)\boldsymbol{\sigma}_r)$.

The liability is a lump-sum payment indexed to CPI: $L_t = I(t, T_0)$. For nominal liabilities, $L_{nom,t} = B(t, T_0)$.

$$L_T = L_t \exp[\int_t^T (r_u + \boldsymbol{\sigma}'_{L,u}\boldsymbol{\lambda} - \frac{1}{2}||\boldsymbol{\sigma}_{L,u}||^2)du + \int_t^T \boldsymbol{\sigma}'_{L,u} d\boldsymbol{z}_t]$$

The funding ratio is therefore log-normally distributed: $\log(F_T/F_t) \sim N(\mu_{F,t,T}, \sigma^2_{F,t,T})$, where

$$\mu_{F,t,T} = \mu_{A,t,T} - \mu_{L,t,T} = \int_t^T (\boldsymbol{\omega'\sigma'_u\lambda} - \boldsymbol{\sigma}'_{L,u}\boldsymbol{\lambda} - \frac{1}{2}||\boldsymbol{\omega'\sigma'_u}||^2 + \frac{1}{2}||\boldsymbol{\sigma}_{L,u}||^2)du$$
$$\sigma^2_{F,t,T} = \sigma^2 A, t, T + \sigma^2 L, t, T - 2Cov(A, L) = \int_t^T ||\boldsymbol{\omega'\sigma'_u} - \boldsymbol{\sigma}_{L,u}||^2 du$$

When $h = 0$, $U(F_T)$ is also lognormally distributed. Writing down the expectation of the terminal utility and taking first derivatives with respect to ω, the optimal fixed weight without constraints is

$$\omega = \left(\int_0^T \boldsymbol{\sigma}'_u\boldsymbol{\sigma}_u du\right)^{-1} \left(\frac{1}{\gamma}\int_0^T \boldsymbol{\sigma}'_u\boldsymbol{\lambda} du + (1-\frac{1}{\gamma})\boldsymbol{\sigma}'_u\boldsymbol{\sigma}_{L,u} du\right)$$

G Solution to unconstrained problem

For simplification, write from now on $E[X|\mathcal{F}_t] := E_t[X]$.

The following solution is due to Cox and Huang(1989).

The value of all liabilities between T_1 and T_2 evaluated at time t is

$$L_t^{T_1,T_2} = E_t^Q[\int_{(T_1,T_2]} e^{-\int_t^s r_u du} dC_s]$$

where $\{C_t\}$ is a stream of fully-indexed liability payments. For simplicity write $L_t^{t,T_0} := L_t$.

It is easy to see that $M_t L_t^{T_1,T_2}$ is a martingale under P when T_1, T_2 are fixed for all t. Note that A_t incorporates liability payments up to time t, thus it is not a self-financing portfolio. Instead,

$A_t - L_t^{t,T_0}$ (the surplus) is self-financing and $M_t(A_t - L_t^{t,T_0})$ is a martingale under P, resulting in the following budget constraint

$$E_t[M_T(A_T - L_T^{T,T_0})] = M_t(A_t - L_t^{t,T_0})$$

or equivalently,

$$E_t[M_T A_T] = M_t A_t - M_t L_t^{t,T_0} + M_t L_t^{T,T_0} = M_t(A_t - L_t^{t,T}).$$

To simplify the notation we assume liability is a one-off payment at maturity T_0. The value of real liabilities evaluated at time t is L_t, the time t price of a default free bond that pays one unit of price index Φ_{T_0} at maturity. Under this assumption we can write $L_t^{t,T_0} = L_t$ and $L_t^{t,T} = 0$ when $T < T_0$.

The pension fund has utility function defined over the excess of real funding ratio above a certain subsistence level h,

$$U(F_T) = \begin{cases} (\frac{A_T}{L_T} - h)^{1-\gamma}/(1-\gamma) & \text{if} \frac{A_T}{L_T} > h \\ -\infty & \text{if} \frac{A_T}{L_T} \leq h \end{cases}$$

The unconstrained dynamic optimization problem is

$$\max_{A_T} E_0 \left[\frac{(A_T/L_T - h)^{1-\gamma}}{1-\gamma} \right]$$
$$s.t. E_0[M_T A_T] = A_0$$

Solution:

For each state of the world, $\{L_t\}$ is exogenously given. The Lagrangian can be written for the maximization problem:

$$\mathcal{L} = \frac{(A_T/L_T - h)^{1-\gamma}}{1-\gamma} - y(M_T A_T - A_0),$$

where the constant y is the same in all states of the world.

The first order optimality condition is

$$\frac{1}{L_T} \left(\frac{A_T^*}{L_T} - h \right)^{-\gamma} - y M_T = 0.$$

So the optimal portfolio at time T is

$$A_T^* = [(y M_T L_T)^{-\frac{1}{\gamma}} + h] L_T$$

where y is chosen to fulfill the initial budget constraint $E_0[M_T A_T^*] = A_0$:

$$E_0[(y M_T L_T)^{-\frac{1}{\gamma}} M_T L_T + h M_T L_T] = A_0.$$

From the martingale property at time t

$$E_t[M_T A_T] = M_t(A_t),$$

the optimal portfolio at each time t is

$$\begin{aligned}
A_t^* =& E_t[\frac{M_T}{M_t} A_T^*] \\
=& E_t\left[\frac{M_T}{M_t}\left(\frac{A_0 - E_0(hM_T L_T)}{E_0[(M_T L_T)^{1-\frac{1}{\gamma}}]}(M_T L_T)^{-\frac{1}{\gamma}} + h\right) L_T\right] \\
=& \frac{A_0 - hL_0}{M_t E_0[(M_T L_T)^{1-\frac{1}{\gamma}}]} E_t[(M_T L_T)^{1-\frac{1}{\gamma}}] + hL_t.
\end{aligned}$$

Using Itô 's lemma

$$d(A_t^* - hL_t) = (A_t^* - hL_t)[(...)dt + (\frac{1}{\gamma}\boldsymbol{\lambda} + (1-\frac{1}{\gamma})\boldsymbol{\sigma}_L)' d\boldsymbol{z_t}]$$

Since the value of the financial portfolio evolves as

$$dA_t = A_t[(r_t + \boldsymbol{\omega}_t'\boldsymbol{\lambda}')dt + \boldsymbol{\omega}_t'\boldsymbol{\sigma}_t' d\boldsymbol{z}_t]$$

we have the optimal weight is

$$\boldsymbol{\omega}_t = [\frac{1}{\gamma}\boldsymbol{\sigma}_t^{-1}\boldsymbol{\lambda} + (1-\frac{1}{\gamma})\boldsymbol{\sigma}_t^{-1}\boldsymbol{\sigma}_L](1-\frac{h}{F_t}) + \boldsymbol{\sigma}_t^{-1}\boldsymbol{\sigma}_L\frac{h}{F_t}$$

H Solution to long term constrained problem

$$U(F_T) = \begin{cases} (\frac{A_T}{L_T} - h)^{1-\gamma}/(1-\gamma) & \text{if} \frac{A_T}{L_T} > h \\ -\infty & \text{if} \frac{A_T}{L_T} \leq h \end{cases}$$

$$\begin{gathered}
\max_{A_T} E_0[U(F_T)] \\
s.t. E[M_T A_T] = A_0 \\
\text{Pr}_0[\frac{A_T}{L_T^{Nom}} < k] \leq p
\end{gathered}$$

For each state of the world, L_T and L_T^{Nom} are exogenous of the asset allocation, $\tilde{U}(A_T) := U(A_T/L_T)$.

$$\begin{aligned}
\tilde{U}' &= \left(\frac{A_T}{L_T} - h\right)^{-\gamma}\frac{1}{L_T}, \tilde{U}' > 0 \text{ when } \frac{A_T}{L_T} > h \\
\tilde{U}'' &= -\gamma\left(\frac{A_T}{L_T} - h\right)^{-\gamma-1}\frac{1}{L_T^2}, \tilde{U}'' < 0 \text{ when } \frac{A_T}{L_T} > h
\end{aligned}$$

Define $G(\cdot)$ to be the inverse function of $\tilde{U}'$:

$$G(x) = L_T[(L_T x)^{-\frac{1}{\gamma}} + h].$$

It is strictly decreasing and continuous.

For each given state of the world, thus given (L_T, L_T^{Nom}), define $\underline{M}(L_T, L_T^{Nom})$ and $\overline{M}(L_T, L_T^{Nom})$ as follows,

$$\underline{M}(L_T, L_T^{Nom}) \equiv \tilde{U}'(kL_T^{Nom})/y_1,$$
$$\overline{M}(L_T, L_T^{Nom}) \text{ is such that } \Pr(M_T > \overline{M}) \equiv p,$$

$y_1 \geq 0$ solves the budgets constraint$E[M_T A_T^{VaR}(y_1, L_T, L_T^{Nom})] = A_0$, A_T^{VaR} is the time T optimal wealth in the long term VaR constrained problem.

For simplicity write $\underline{M}(L_T, L_T^{Nom})$ and $\overline{M}(L_T, L_T^{Nom})$ as $\underline{M}$ and $\overline{M}$.

If $\overline{M} < \underline{M}$, then

$$G(y_1\overline{M}) > G(y_1\underline{M}) = kL_T^{Nom}$$
$$\Pr_0[G(y_1 M_T) \geq kL_T^{Nom}] = Pr_0[M_T \leq \underline{M}] > Pr_0[M_T \leq \overline{M}] \equiv 1 - p.$$

Therefore the VaR constrained optimal wealth is the same as the unconstrained one,

$$A_T^{VaR} = A_T^u = G(y_1 M_T).$$

If $\overline{M} \geq \underline{M}$, then we have the following proposition:

Proposition 1. *When $\overline{M} \geq \underline{M}$, the optimal wealth of the VaR agent is*

$$A_T^{VaR} = \begin{cases} G(y_1 M_T) & \text{if } M_T < \underline{M}, \\ kL_T^{Nom} & \text{if } \underline{M} \leq M_T < \overline{M}, \\ G(y_1 M_T) & \text{if } M_T \geq \overline{M} \end{cases}$$

Proof of Proposition 1.

Note that the probability constraint $Pr_0[\frac{A_T}{L_T^{Nom}} < k] \leq p$ can be rewritten as

$$E_0[1_{A_T \geq kL_T^{Nom}}] \geq 1 - p.$$

Thus for each state of the world we can write the Lagrangian as

$$\mathcal{L}(A_T, y_1, y_2, M_T, L_T, L_T^{Nom}) = \tilde{U}(A_T) - y_1 M_T A_T + y_2 1_{A_T \geq kL_T^{Nom}}$$

where y_1 is defined above and y_2 is defined in the following lemma.

Lemma 1. *For given L_T, L_T^{Nom} and $\forall M_T$, A_T^{VaR} in Proposistion 1 maximizes the Lagrangian $\mathcal{L}(A_T, y_1, y_2, M_T, L_T, L_T^{Nom})$:*

$$\max_{A_T}\{\tilde{U}(A_T) - y_1 M_T A_T + y_2 1_{A_T \geq kL_T^{Nom}}\} = \tilde{U}(A_T^{VaR}) - y_1 M_T A_T^{VaR} + y_2 1_{A_T^{VaR} \geq kL_T^{Nom}}$$

where $y_2 \equiv \tilde{U}(G(y_1\overline{M})) - y_1\overline{M}G(y\overline{M}) - \tilde{U}(kL_T^{Nom}) + y_1\overline{M}(kL_T^{Nom}) \geq 0$.

Next, we show that for any other optimal solution A_T that satisfies the budget constraint and the VaR constraint, its expected utility is no larger than the expected utility of A_T^{VaR}.

$$
\begin{aligned}
&E[\tilde{U}(A_T^{VaR})] - E[\tilde{U}(A_T)] \\
=&E[\tilde{U}(A_T^{VaR})] - E[\tilde{U}(A_T)] - y_1(A_0) + y_1(A_0-) + y_2(1-p) - y_2(1-p) \\
\geq&E[\tilde{U}(A_T^{VaR})] - E[\tilde{U}(A_T)] - y_1 E[M_T A_T^{VaR}] + y_1 E[M_T A_T] + y_2 E[1_{A_T^{VaR} \geq kL_T^{Nom}}] - y_2 E[1_{A_T \geq kL_T^{Nom}}] \geq 0,
\end{aligned}
$$

where the first inequality follows from $y_1 \geq 0, y_2 \geq 0$ and the budget constraint $E[M_T(A_T^{VaR})] = E[M_T A_T] = A_0$ and the VaR constraint where holding with equality for A_T^{VaR} and inequality for A_T:

$$
E[1_{A_T^{VaR} \geq kL_T^{Nom}}] = Pr[M_T \leq \overline{M}] = 1-p, E[1_{A_T \geq kL_T^{Nom}}] \geq 1-p.
$$

The second inequality follows from Lemma 1. Therefore A_T^{VaR} is optimal.

End of proof of Proposition 1.

Proof of Lemma 1. First notice that y_2 is chosen such that $\mathcal{L}(A_T^{VaR}, y_1, y_2, M_T, L_T, L_T^{Nom})$ is continuous in M_T:

$$
\mathcal{L}(A_T^{VaR}, y_1, y_2, \overline{M}^-, L_T, L_T^{Nom}) \equiv \mathcal{L}(A_T^{VaR}, y_1, y_2, \overline{M}, L_T, L_T^{Nom})
$$

where $\overline{M}^-$ means taking the limit from the left hand side of $\overline{M}$ .That is,

$$
\tilde{U}(kL_T^{Nom}) - y_1\overline{M}(kL_T^{Nom}) + y_2 \equiv \tilde{U}(G(y_1\overline{M})) - y_1\overline{M}G(y_1\overline{M}).
$$

Therefore, we have

$$
\begin{aligned}
y_2 &= \tilde{U}(G(y_1\overline{M})) - y_1\overline{M}G(y\overline{M}) - \tilde{U}(kL_T^{Nom}) + y_1\overline{M}(kL_T^{Nom}) \\
&= \tilde{U}(G(y_1\overline{M})) - y_1\overline{M}G(y_1\overline{M}) - \tilde{U}(G(y_1\underline{M})) - y_1\overline{M}G(y_1\underline{M}) \\
&= [\tilde{U}(G(y_1\overline{M})) - y_1\overline{M}G(y_1\overline{M}) + y_1 k L_T^{Nom}\overline{M}] - [\tilde{U}(G(y_1\underline{M})) - y_1\underline{M}G(y_1\underline{M}) + y_1 k L_T^{Nom}\underline{M}] \\
&\geq 0
\end{aligned}
$$

where the inequality follows from $\underline{M} \leq \overline{M}$ and

$$
\frac{\partial}{\partial M_T}\left(\tilde{U}(G(y_1 M_T)) - y_1 M_T G(y_1 M_T) + y_1 k L_T^{nom} M_T\right) = -y_1 G(y M_T) + y_1 k L_T^{Nom} \geq 0.
$$

Then, to solve the piecewise problem of $\max_{A_T}\{\mathcal{L}(A_T, y_1, y_2, M_T, L_T, L_T^{Nom})\}$, we use the first order condition and notice the discontinuity at the point where $A_T = kL_T^{Nom}$. The local maxima can only be achieved at either $\tilde{U}(A_T^*)' = y_1 M_T$ and $A_T^{**} = kL_T^{Nom}$. To see which one is the global maximum for different values of M_T, we compare the value and Lagrangian value of A_T^* and A_T^{**}.

When $M_T < \underline{M}$, we have

$$
A_T^* = G(y_1 M_T) > G(y_1 \underline{M}_T) = A_T^{**}
$$

and since $\frac{\partial}{\partial A_T}\left(\tilde{U}(A_T) - y_1 M_T A_T\right) = \tilde{U}'(A_T) - y_1 M_T > 0$ when $A_T < A_T^*$,

$$\begin{aligned}\mathcal{L}(A_T^*, y_1, y_2, M_T, L_T, L_T^{Nom}) &= \tilde{U}(A_T^*) - y_1 M_T A_T^* + y_2 \\ &> \tilde{U}(A_T^{**}) - y_1 M_T A_T^{**} + y_2 \\ &= \mathcal{L}(A_T^{**}, y_1, y_2, M_T, L_T, L_T^{Nom})\end{aligned}$$

so A_T^* is the global maximum.

When $\underline{M} \leq M_T < \overline{M}$, we have

$$A_T^* = G(y_1 M_T) \leq G(y_1 \underline{M}_T) = A_T^{**}$$

and

$$\begin{aligned}&\mathcal{L}(A_T^{**}, y_1, y_2, M_T, L_T, L_T^{Nom}) = \tilde{U}(A_T^{**}) - y_1 M_T A_T^{**} + y_2 \\ &= \tilde{U}(G(y_1 \underline{M}_T)) + \tilde{U}(G(y_1 \overline{M})) - y_1 \overline{M} G(y_1 \overline{M}) - \tilde{U}(k L_T^{Nom}) + y_1 \overline{M}(k L_T^{Nom}) - y_1 M_T A_T^{**} \\ &> \tilde{U}(A_T^*) - y_1 M_T A_T^* = \mathcal{L}(A_T^*, y_1, y_2, M_T, L_T, L_T^{Nom})\end{aligned}$$

where the inequality comes from $M_T < \overline{M}$ and the fact that when $\underline{M} \leq M_T$,

$$\frac{\partial}{\partial M_T}\{\tilde{U}(G(y_1 M_T)) - y_1 M_T G(y_1 M_T) + y_1 G(y_1 \underline{M}_T) M_T\} = -y_1 G(y_1 M_T) + y_1 G(y_1 \underline{M}_T) \geq 0.$$

so A_T^{**} is the global maximum.

When $M_T \geq \overline{M}$, the inequalities in the previous case is reversed, so A_T^* is the global maximum.

Therefore A_T^{VaR} given in proposition 1 solves the maximization problem of the Lagrangian.

End of proof of Lemma 1.

I Optimal allocation with constant interest rate and no inflation

As derived in Appendix H the optimal wealth in the long term constrained problem for the Merton case is as follows,

$$A_T^{VaR} = \begin{cases} (yM_T)^{-\frac{1}{\gamma}} L_T^{1-\frac{1}{\gamma}} + hL_T & if\ M_T < \min\{\underline{M}, \overline{M}\}, \\ kL_T & if\ \min\{\underline{M}, \overline{M}\} \leq M_T < \overline{M}, \\ (yM_T)^{-\frac{1}{\gamma}} L_T^{1-\frac{1}{\gamma}} + hL_T & if\ M_T \geq \overline{M} \end{cases}$$

where $L_t = e^{-r(T_0 - t)}$, $\underline{M}$ is such that $(y\underline{M})^{-\frac{1}{\gamma}} L_T^{1-\frac{1}{\gamma}} + hL_T \equiv kL_T$, $\overline{M}$ is such that $\Pr_0[M_T > \overline{M}] \equiv p$.

First note that $\underline{M} \equiv \frac{1}{y}e^{r(T_0-T)}(k-h)^{-\gamma}$, $\overline{M} \equiv \exp(\lambda_s N^{-1}(1-p)\sqrt{T}-(r+\lambda_s^2/2)T)$, $\underline{M}$ and $\overline{M}$ are constant through time and holds for all states of the world. Using the Martingale property of $\{M_t A_t^{VaR}\}$,

$$
\begin{aligned}
A_t^{VaR} &= \frac{1}{M_t}E_t[A_T^{VaR}M_T] \\
&= \frac{1}{M_t}E_t[(y^{-\frac{1}{\gamma}}(M_TL_T)^{1-\frac{1}{\gamma}} + hM_TL_T)1_{M_T<\min\{\underline{M},\overline{M}\}}] + \frac{1}{M_t}E_t[(kM_TL_T)1_{\min\{\underline{M},\overline{M}\}\le M_T<\overline{M}}] \\
&+ \frac{1}{M_t}E_t[(y^{-\frac{1}{\gamma}}(M_TL_T)^{1-\frac{1}{\gamma}} + hM_TL_T)1_{M_T\ge\overline{M}}]
\end{aligned}
$$

Denote $-\log(\frac{M_T}{M_t}) := Y_{t,T}$, which is normally distributed with $\mu_Y = r(T-t) + \frac{1}{2}\lambda_s^2(T-t)$, $\sigma_Y^2 = \lambda_s^2(T-t)$,

$$
\begin{aligned}
A_t^{VaR} &= y^{-\frac{1}{\gamma}}M_t^{-\frac{1}{\gamma}}L_t^{1-\frac{1}{\gamma}}E_t[e^{r(1-\frac{1}{\gamma})(T-t)-Y(1-\frac{1}{\gamma})}1_{Y>-\log(\frac{\min\{\underline{M},\overline{M}\}}{M_t})}] + hL_tE_t[e^{r(T-t)-Y}1_{Y>-\log(\frac{\min\{\underline{M},\overline{M}\}}{M_t})}] \\
&+ kLtE_t[e^{r(T-t)-Y}1_{-\log(\frac{\overline{M}}{M_t})<Y\le-\log(\frac{\min\{\underline{M},\overline{M}\}}{M_t})}] + y^{-\frac{1}{\gamma}}M_t^{-\frac{1}{\gamma}}L_t^{1-\frac{1}{\gamma}}E_t[e^{r(1-\frac{1}{\gamma})(T-t)-Y(1-\frac{1}{\gamma})}1_{Y\le-\log(\frac{\overline{M}}{M_t})}] \\
&+ hL_tE_t[e^{r(T-t)-Y}1_{Y\le-\log(\frac{\overline{M}}{M_t})}] \\
&= y^{-\frac{1}{\gamma}}M_t^{-\frac{1}{\gamma}}L_t^{1-\frac{1}{\gamma}}E_t[e^{r(1-\frac{1}{\gamma})(T-t)-Y(1-\frac{1}{\gamma})}] \\
&- y^{-\frac{1}{\gamma}}M_t^{-\frac{1}{\gamma}}L_t^{1-\frac{1}{\gamma}}\int_{-\infty}^{-\log(\frac{\min\{\underline{M},\overline{M}\}}{M_t})} e^{r(1-\frac{1}{\gamma})(T-t)-Y(1-\frac{1}{\gamma})}\frac{1}{\sqrt{2\pi\sigma_Y^2}}e^{-\frac{(Y-\mu_Y)^2}{2\sigma_Y^2}}dY \\
&+ hL_tE_t[e^{r(T-t)-Y}] - hL_t\int_{-\infty}^{-\log(\frac{\min\{\underline{M},\overline{M}\}}{M_t})} e^{r(T-t)-Y}\frac{1}{\sqrt{2\pi\sigma_Y^2}}e^{-\frac{(Y-\mu_Y)^2}{2\sigma_Y^2}}dY \\
&+ kL_t\int_{-\log(\frac{\overline{M}}{M_t})}^{-\log(\frac{\min\{\underline{M},\overline{M}\}}{M_t})} e^{r(T-t)-Y}\frac{1}{\sqrt{2\pi\sigma_Y^2}}e^{-\frac{(Y-\mu_Y)^2}{2\sigma_Y^2}}dY \\
&+ y^{-\frac{1}{\gamma}}M_t^{-\frac{1}{\gamma}}L_t^{1-\frac{1}{\gamma}}\int_{-\infty}^{-\log(\frac{\overline{M}}{M_t})} e^{r(1-\frac{1}{\gamma})(T-t)-Y(1-\frac{1}{\gamma})}\frac{1}{\sqrt{2\pi\sigma_Y^2}}e^{-\frac{(Y-\mu_Y)^2}{2\sigma_Y^2}}dY \\
&+ hL_t\int_{-\infty}^{-\log(\frac{\overline{M}}{M_t})} e^{r(T-t)-Y}\frac{1}{\sqrt{2\pi\sigma_Y^2}}e^{-\frac{(Y-\mu_Y)^2}{2\sigma_Y^2}}dY \\
&= y^{-\frac{1}{\gamma}}M_t^{-\frac{1}{\gamma}}L_t^{1-\frac{1}{\gamma}}e^{\frac{1}{2}(-\frac{1}{\gamma})(1-\frac{1}{\gamma})\lambda_s^2(T-t)} - y^{-\frac{1}{\gamma}}M_t^{-\frac{1}{\gamma}}L_t^{1-\frac{1}{\gamma}}e^{\frac{1}{2}(-\frac{1}{\gamma})(1-\frac{1}{\gamma})\lambda_s^2(T-t)}N(-d_2(\min\{\underline{M},\overline{M}\})) \\
&+ hL_t - hL_tN(-d_1(\min\{\underline{M},\overline{M}\})) + kL_t[N(-d_1(\min\{\underline{M},\overline{M}\})) - N(-d_1(\overline{M}))] \\
&+ y^{-\frac{1}{\gamma}}M_t^{-\frac{1}{\gamma}}L_t^{1-\frac{1}{\gamma}}e^{\frac{1}{2}(-\frac{1}{\gamma})(1-\frac{1}{\gamma})\lambda_s^2(T-t)}N(-d_2(\overline{M})) + hL_tN(-d_1(\overline{M}))
\end{aligned}
$$

where $N(\cdot)$ is the cumulative density function of the standard normal distribution,

$$
\begin{aligned}
d_1(x) &= \frac{\log(\frac{x}{M_t}) + \mu_Y - \sigma_Y^2}{\sigma_Y}, \\
d_2(x) &= d_1(x) + \frac{1}{\gamma}\sigma_Y.
\end{aligned}
$$

Applying Itô's lemma,

$$
\begin{aligned}
dA_t^{VaR} &= y^{-\frac{1}{\gamma}} L_t^{1-\frac{1}{\gamma}} e^{\frac{1}{2}(-\frac{1}{\gamma})(1-\frac{1}{\gamma})\lambda_s^2(T-t)} dM_t^{-\frac{1}{\gamma}} - y^{-\frac{1}{\gamma}} L_t^{1-\frac{1}{\gamma}} e^{\frac{1}{2}(-\frac{1}{\gamma})(1-\frac{1}{\gamma})\lambda_s^2(T-t)} N(-d_2(\min\{\underline{M}, \overline{M}\})) dM_t^{-\frac{1}{\gamma}} \\
&+ y^{-\frac{1}{\gamma}} L_t^{1-\frac{1}{\gamma}} e^{\frac{1}{2}(-\frac{1}{\gamma})(1-\frac{1}{\gamma})\lambda_s^2(T-t)} N(-d_2(\overline{M})) dM_t^{-\frac{1}{\gamma}} \\
&- y^{-\frac{1}{\gamma}} M_t^{-\frac{1}{\gamma}} L_t^{1-\frac{1}{\gamma}} e^{\frac{1}{2}(-\frac{1}{\gamma})(1-\frac{1}{\gamma})\lambda_s^2(T-t)} \phi(-d_2(\min\{\underline{M}, \overline{M}\})) \frac{d(-d_2)}{dM_t} dM_t \\
&- hL_t \phi(-d_1(\min\{\underline{M}, \overline{M}\})) \frac{d(-d_1)}{dM_t} M_t + kL_t[\phi(-d_1(\min\{\underline{M}, \overline{M}\})) - \phi(-d_1(\overline{M}))] \frac{d(-d_1)}{dM_t} dM_t \\
&+ y^{-\frac{1}{\gamma}} M_t^{-\frac{1}{\gamma}} L_t^{1-\frac{1}{\gamma}} e^{\frac{1}{2}(-\frac{1}{\gamma})(1-\frac{1}{\gamma})\lambda_s^2(T-t)} \phi(-d_2(\overline{M})) \frac{d(-d_2)}{dM_t} dM_t + hL_t \phi(-d_1(\overline{M})) \frac{d(-d_1)}{dM_t} dM_t \\
&= A_t^{VaR}(...)dt + A_t^{VaR} \left[1 - \frac{h}{F_t} - \frac{k-h}{F_t} \left(N(-d_1(\min\{\underline{M}, \overline{M}\})) - N(-d_1(\overline{M}))\right)\right] \frac{\lambda_s}{\gamma} dz \\
&+ A_t^{VaR} \left[\frac{1}{F_t} y^{-\frac{1}{\gamma}} M_t^{-\frac{1}{\gamma}} L_t^{-\frac{1}{\gamma}} e^{\frac{1}{2}(-\frac{1}{\gamma})(1-\frac{1}{\gamma})\lambda_s^2(T-t)} [\phi(-d_2(\overline{M})) - \phi(-d_2(\min\{\underline{M}, \overline{M}\}))] \right. \\
&\left. + \frac{k-h}{F_t} [\phi(-d_1(\min\{\underline{M}, \overline{M}\})) - \phi(-d_1(\overline{M}))] \right] \frac{(-\lambda_s)}{\lambda_s \sqrt{T-t}} dz \\
&= A_t^{VaR}(...)dt + A_t^{VaR} \omega_t \sigma_s dz
\end{aligned}
$$

Equating the diffusion terms of last equation yields,

$$
\begin{aligned}
\omega_t &= \frac{\lambda_s}{\gamma \sigma_s} \left[1 - \frac{h}{F_t} - \frac{k-h}{F_t} \left(N(-d_1(\min\{\underline{M}, \overline{M}\})) - N(-d_1(\overline{M}))\right)\right] \\
&- \frac{1}{\sigma_s \sqrt{T-t}} \left[\frac{1}{F_t} y^{-\frac{1}{\gamma}} M_t^{-\frac{1}{\gamma}} L_t^{-\frac{1}{\gamma}} e^{\frac{1}{2}(-\frac{1}{\gamma})(1-\frac{1}{\gamma})\lambda_s^2(T-t)} [\phi(-d_2(\overline{M})) - \phi(-d_2(\min\{\underline{M}, \overline{M}\}))] \right. \\
&\left. + \frac{k-h}{F_t} [\phi(-d_1(\min\{\underline{M}, \overline{M}\})) - \phi(-d_1(\overline{M}))] \right]
\end{aligned}
$$

and $\phi(\cdot)$ is the standard normal probability distribution function.

References

Basak, S. and A. Shapiro, 2001, “Value-At-Risk-Based Risk Management: Optimal Policies and Asset Prices”, *Review of Financial Studies*, 14(2), 371-405

Basak, S., 2002, “A Comparative Study Of Portfolio Insurance”, *Journal of Economic Dynamics and Control*, 26(7-8), 1217-1241

Basak, S. and G. Chabakauri 2010, “Dynamic Mean-Variance Asset Allocation”, *Review of Financial Studies*, 23(8), 2970-3016

Brandt, M., 2005, “Portfolio Choice Problems”

Brennan, M. and Y. Xia 2002, “Dynamic Asset Allocation Under Inflation”, *Journal of Finance* 57(3), 1201-1238

Bikker, J.A. and P.A.J. Vlaar, 2007, “Conditional Indexation In Defined Benefit Pension Plans in the Netherlands”, *Geneva Papers on Risk and Insurance - Issues and Practice* 32, 494-515

Cairns, A.J.G., 2000, “Some Notes On The Dynamics and Optimal Control Of Stochastic Pension Fund Models In Continuous Time”, ASTIN Bulletin 30, 19-55

Campbell, J.Y. and L. Viceira 2002, “Strategic Asset Allocation: Portfolio Choice For Long Term Investors“

Cuoco, D., H. He and S. Issaenko, 2008, “Optimal Dynamic Trading Strategies With Risk Limits”, *Operations Research*, 56(2), 358-368

Cox,J. and C. Huang, 1989, “Optimal Consumption and Portfolio Policies When Asset Prices Follow A Diffusion Process”, *Journal of Economics Theory* 49, 33-83

De Jong, F., 2008, “Pension Fund Investments and The Valuation Of Liabilities Under Conditional Indexation”, *Insurance: Mathematics and Economics*, 42, 1-13

Detemple, J. and M. Rindisbacher, 2008, “Dynamic Asset Liability Management With Tolerance For Limited Shortfalls”, *Insurance: Mathematics and Economics*, 43(3), 281-294

Hovenaars, R., R. Molenaar, R. Schotman and T. Steenkamp, 2008, “Strategic Asset Allocation With Liabilities: Beyond Stocks and Bonds”, *Journal of Economic Dynamics and Control* 32, 2939-2970

Inkmann, J. and D. Blake 2008, “The Term Structure Of Funding Spreads”, The Pensions Institute Discussion Paper PI-0709

Jorgensen, P., 2007, “Traffic Light Options”, *Journal of Banking and Finance* 31(2), 3698-3719

Kleinow, T., 2010, “Pension Fund Management and Conditional Indexation”, working paper

Kraft, H. and M. Steffenssen, 2012, “A dynamic programming approach to constrained portfolios”

Koijen, R. and T. Nijman, 2006, “Valuation and Risk Management Of Inflation Sensitive Pension Rights”

Mamon, R., 2004, “Three Ways To Solve For Bond Prices In The Vasicek Model”, *Journal of Applied Mathematics and Decision Sciences*, 8(1), 1-14

Margrabe, W., 1978, “The Value Of An Option To Exchange One Asset For Another”, *Journal of Finance* 33, 177-186

Martellini, L. and V. Milhau, 2012, “Dynamic Allocation Decisions In The Presence Of Funding Ratio Constraints”, *Journal of Pension Economics and Finance*, 11(4), 549-580

Munk, C., C. Sorensen, and T. Vinther, 2004, “Dynamic Asset Allocation Under Mean-Reverting Returns, Stochastic Interest Rates and Inflation Uncertainty Are Popular Recommendations Consistent With Rational Behavior?”, *International Review of Economics and Finance* 13, 141-166

Ponds, E.H.M. and B. Van Riel, 2007, “The Recent Evolution Of Pension Funds In The Netherlands: The Trend To Hybrid DB-DC Plans and Beyond”, Boston Center for Retirement Research
Rudolf, M. and W. Ziemba 2004, “Intertemporal Surplus Management”, *Journal of Economic Dynamics and Control* 28(5), 975-990
Shi, Z. and B.J.M Werker, 2012, “Short-Horizon Regulation For Long-Term Investors”, *Journal of Banking and Finance*, 36(12), 3227-3238
Van Binsbergen, J.H. and M.W. Brandt, 2006, “Optimal Asset Allocation in Asset Liability Management”, Duke University working papers
Vasicek, O., 1977, “An Equilibrium Characterization Of The Term Structure”, *Journal of Financial Economics* 5, 177-188
Wachter, J.A., 2002, “Portfolio and Consumption Decisions Under Mean-Reverting Returns: An Exact Solution For Complete Markets”, *Journal of Financial and Quantitative Analysis* 37, 63-91
Whitehouse, E.R., 2009, “Pensions, Purchasing-Power Risk, Inflation and Indexation”, *OECD Social, Employment and Migration Working Papers* 77