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DP 10/2011-044 (revised version May 15, 2013)

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May 15, 2013

Abstract

On average, "young" people underestimate whereas "old" people overestimate their chances to survive into the future. We adopt a Bayesian learning model of ambiguous survival beliefs which replicates these patterns. The model is embedded within a non-expected utility model of life-cycle consumption and saving. Our analysis shows that agents with ambiguous survival beliefs (i) save less than originally planned, (ii) exhibit undersaving at younger ages, and (iii) hold longer on to their assets than their rational expectations counterparts who correctly assess survival probabilities. Our ambiguity-driven model therefore simultaneously accounts for three important empirical findings on household saving behavior.

JEL Classification: D91, D83, E21.

Keywords: Ambiguity, Dynamic inconsistency, Life-cycle hypothesis, Subjective survival rates

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*Financial support from the Network for Studies on Pensions, Aging and Retirement (Netspar) is gratefully acknowledged. Groneck and Ludwig also thank the State North Rhine-Westphalia for partial funding. We thank Hippolyte d'Albis, Ben Heijdra, Christian Hellwig and seminar participants at SED 2012, 2012 Netspar Pension Workshop, EEA 2012, SAET 2011, 2011 CESIfo Conference on Macroeconomics, 2011 Annual Congress of the Verein fuer Socialpolitik, 2011 CMR Macroeconomic Workshop, University of Cologne, UC Louvain and Pontificia Universidad Católica de Chile, for helpful comments.

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1 Introduction

One important element of life-cycle models of consumption and saving is the process of how individuals form and revise beliefs about their life expectancy when they grow older. In line with the rational expectations paradigm it has been common in the literature to consider expected utility (EU) maximizing agents whose updated subjective beliefs coincide with objective conditional survival probabilities. Only recently, researchers have focused on subjective assessments of survival probabilities which greatly deviate from projected life-table survival rates. According to the Health and Retirement Study (HRS)[1], younger people strongly underestimate their (relatively high) probability to survive to some target age. At the same time older people strongly overestimate their lower probability to survive to some target age.[2] Such patterns can neither be reconciled with the rational expectations paradigm nor with the standard notion of Bayesian learning.

In this paper we argue that these biases between subjective beliefs and objective probabilities can be explained by a model of belief formation that takes individuals' ambiguity attitudes into account. We demonstrate that such an ambiguity-driven model of belief formation has profound implications for life-cycle consumption and savings decisions. Our approach can accommodate relevant empirical findings on household saving behavior which proved to be puzzling for rational expectations life-cycle models. We proceed in several steps.

First, in Section 2, we adopt a Bayesian learning model of ambiguous survival beliefs as developed by Ludwig and Zimper (2013) which can explain the key stylized facts on survival beliefs observed in the HRS data. Subjective survival beliefs are modeled as conditional *neo-additive capacities* (Chateauneuf, Eichberger, and Grant 2007). These are a parsimonious class of non-additive probability measures that arise within *Choquet ex-*

[1] In the HRS (Health and Retirement Study) people are asked about their subjective probability assessment to survive from some interview age up to a specific target age. Target age is mostly 10 to 15 years in advance, see, e.g., Ludwig and Zimper (2013) for details.

[2] Similar differences between subjective beliefs and objective data have been reported in various other datasets, cf. Ludwig and Zimper (2013) for a discussion.

pected utility (CEU) theory (Schmeidler 1989, Gilboa 1987).[3] Neo-additive capacities stand for a linear transformation of (*n*on-*e*xtreme *o*nly) additive probabilities so that the probability weighting function for probabilities between zero and one is flatter than the 45-degree line. Biases of subjective beliefs from some additive probability measure are captured by two parameters only. One parameter measures the degree of ambiguity—or the lack of confidence—a decision maker has in the additive probability measure. A second parameter measures the degree of over-, respectively under-estimation through which ambiguity is resolved.[4]

In Section 3, we build on this biased Bayesian learning model by generalizing a stochastic life-cycle model to the case of CEU decision makers. Our CEU life-cycle model generally describes dynamically inconsistent behavior. Throughout the remainder of our analysis we focus on naive households who do not anticipate that their future selves will depart from their own current self's optimal consumption and saving plan.[5]

In Section 4 we present a simplified version of our model which provides guidance for our quantitative analysis. We show analytically that we can expect households (i) to save less than originally planned at young age if they are "moderately optimistic" about their future survival in old age, (ii) to save less than under rational expectations if they are "sufficiently pessimistic" at young age and (iii) to have higher asset holdings than their rational expectations counterparts if they are optimistic in old age for sufficiently many periods.

To investigate whether these conditions hold quantitatively we return to the full model in Section 5 and calibrate it to simulate household decisions. We estimate parameters of ambiguous survival beliefs and demonstrate—in the first part of Section 6—that calibrated beliefs can account for the empirical facts as elicited in the HRS. Our parameter estimates are in line with standard estimates of the curvature of probability weighting functions as reported in Wu and Gonzalez (1996).

[3]When restricted to gains this is equivalent to the celebrated concept of *cumulative prospect theory* (Tversky and Kahneman 1992).

[4]Neo-additive capacities are in line with empirical evidence in the decision theoretic literature suggesting inversely S-shaped probability weighting functions—either due to *ambiguity attitudes* (Wu and Gonzalez 1999) or *likelihood insensitivity* (Wakker 2010).

[5]We provide arguments for this focus in Subsection 3.5.

As our main result we show—in the second part of Section 6—that the misperception of lifespan risk indeed simultaneously adds to existing explanations for three stylized facts in the data on savings behavior: On average, CEU agents with subjective survival beliefs at working age have a saving rate of 21.9% compared to a rational expectations model with an average saving rate of 22.8%. In addition, the realized saving rate is 2.8 percentage points lower than what the CEU agent at age 20 actually planned to save. This corresponds to ample empirical evidence on dynamically inconsistent behavior, in particular undersaving. Survey evidence indicates that people save less for retirement than actually planned and they save less than they think they should.[6] In addition, according to the National Retirement Risk Index, 51% of working age households are at risk of being unable to maintain their pre-retirement standard of living in retirement, cf. Munnell et al. (2010).

Simultaneously, the extended CEU life-cycle model matches well the important stylized fact that people hold substantial assets late in life and dissave less than predicted by the standard life-cycle model.[7] Using HRS data, Hurd and Rohwedder (2010) report that the median household at age 85 (95) holds around 49% (21%) of assets of age 65.[8] Our model predicts average asset holdings at age 85 (95) of 46.4% (23.5%) of the assets at age 65 which is very close to the data. Asset holdings are also substantially higher than respective values for rational agents.

The extent towards which our model can quantitatively resolve one or the other puzzle on household saving crucially hinges on the inter-temporal elasticity of substitution (IES). The IES governs the relative strength of undersaving at young age and high asset holdings at old age. Our benchmark

[6] Bernheim and Rangel (2007), and Laibson et al. (1998) quote numerous studies indicating self-reported mistakes in terms of private saving decisions for retirement. See also Bernheim (1998) and Choi et al. (2006) for studies on undersaving.

[7] There are numerous extensions of the standard life-cycle model which aim at explaining this feature of the data. The two main explanations for large assets holdings late in life are bequest motives (Hurd 1989, Lockwood 2012), and precautionary saving due to possibly large health expenditures (De Nardi, French, and Jones 2010).

[8] Reported values are extrapolated median wealth paths for single households according to Figure 1a and 1b in Hurd and Rohwedder (2010), p. 28. Note that the results depend on how to measure wealth changes and whether to concentrate on singles or couples. See also De Nardi et al. (2010) for data on elderly asset holdings across income quintiles.

results are for a value of the IES of one third. A lower (higher) value of 0.25 (0.5) leads to higher (lower) asset holdings of the elderly and less (more) undersaving. Specifically, an IES of 0.25 generates simulated median asset paths that are consistent with the empirical evidence reported by Hurd and Rohwedder (2010).

The phenomena of undersaving and dynamically inconsistent behavior have also been studied within the context of hyperbolic discounting models, cf. Laibson et al. (1998) and Laibson (1998). In contrast, our ambiguity-driven approach is based on an axiomatic decision theoretic model. We thereby attempt to "opening the black box of decision makers instead of modifying functional forms" (Rubinstein 2003). In addition, our model simultaneously generates undersaving at young age and large old-age asset holdings. This is not possible within the standard hyperbolic time-discounting life-cycle model.[9]

Finally, Section 7 concludes our analysis with an outlook on future research. Appendix A recalls formal definitions from Choquet decision theory. Appendix B sketches the construction of ambiguous survival beliefs through a model of Bayesian learning by Ludwig and Zimper (2013). All formal proofs are relegated to Appendix C. Supplementary material is contained in Appendix D.

2 Ambiguous Survival Beliefs

2.1 Motivation: Biases in Survival Beliefs

This paper is concerned with, first, modeling age-dependent updating of subjective beliefs of an agent about her chances to survive into the future and, second, with merging such a model of subjective beliefs into a standard life-cycle model of consumption and savings. Point of departure of our analysis is HRS data on subjective survival beliefs. Figure 1 summarizes this data by plotting average age-specific biases in survival beliefs—the difference between the respective average subjective belief and the average objective data—for three waves of the HRS between 2000 and 2004.

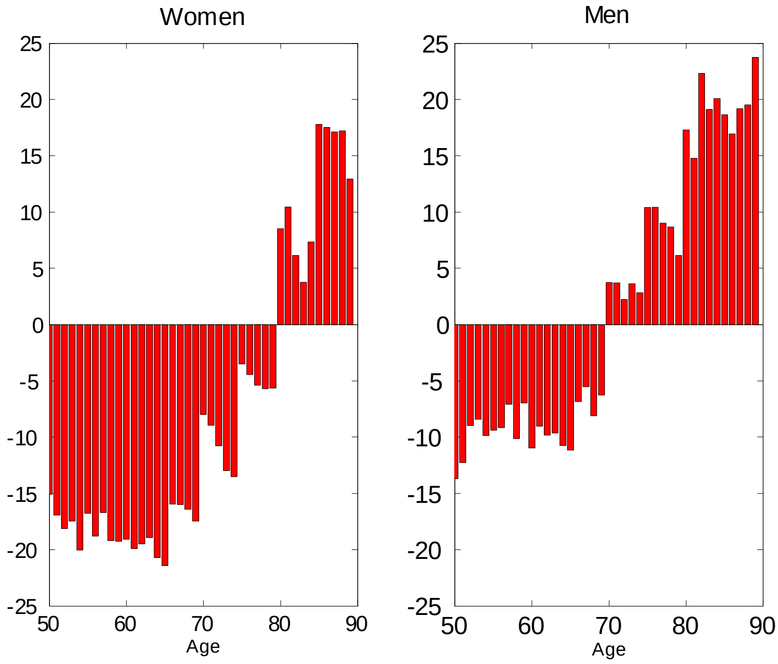
[9] In a companion paper we study the difference between the CEU model and the quasi-hyperbolic discounting model, cf. Groneck et al. (2013).

We observe that relatively "young"—younger than age 65-70—respondents underestimate whereas relatively "old"—above age 70—respondents overestimate their chances to survive into the future. For example, a 65 year old women underestimates her objective probability to become 80 years by about 20 percentage points. Respondents between ages 85 and 89 in the sample exhibit an average overestimation by about 15 to 20 percentage points.

To model such subjective survival beliefs, fix some state space Ω and some σ-algebra $\mathcal{F}$ on Ω with the interpretation that $\mathcal{F}$ contains all survival events that are relevant to an agent's decision on an optimal life-time consumption plan. We denote by $Z_{k,t} \in \mathcal{F}$ the event that the agent survives from period k to the end of period t. The standard expected utility approach would consider an additive probability space $(\Omega, \mathcal{F}, \pi)$ such that the agent maximizes at each age $h \geq 0$ the expected utility of her future consumption streams with respect to the conditional probability measure $\pi\left(\cdot \mid \tilde{I}_h\right)$. $\tilde{I}_h \in \mathcal{F}$ denotes the age-dependent random information that the agent receives about her future survival chances. This information process is typically described as a *filtration process* over the agent's age so that the agent learns more relevant (statistical) information about her future survival chances while she grows older. Standard consistency results for Bayesian updating (e.g., Doob 1949) imply that $\pi\left(Z_{k,t} \mid \tilde{I}_h\right)$ converges, for all $Z_{k,t}$, with probability one to the agent's objective survival probability, denoted $\psi_{k,t}$, if she receives more statistical information. If this expected utility approach was a good description of reality, we would, e.g., expect that the subjective belief of a representative 85 year old agent to survive from age 90 to 91 is closer to the true survival probability $\psi_{90,91}$ than the corresponding subjective survival belief of a representative 75 year old agent.

As the data in Figure 1 show, this convergence property of subjective additive conditional measures is violated for subjective survival beliefs in the HRS (as well as in other datasets, cf. the discussion in Ludwig and Zimper (2013)). To explain these patterns we therefore follow Ludwig and Zimper (2013) who develop a closed-form model of Bayesian learning under ambiguity which gives rise to a parsimonious notion of *ambiguous survival beliefs*. Our approach uses this model of ambiguous survival beliefs to

Figure 1: Relative difference of subjective survival probabilities and cohort data



Notes: This graph shows deviations in percentage points of subjective survival probabilities from objective data. Objective survival rates are based on cohort life table data. Future objective data is predicted with the Lee and Carter (1992) procedure.
Source: Own calculations based on HRS, Human Mortality Database and Social Security Administration data.

construct a non-additive probability space $(\Omega, \mathcal{F}, \nu(\cdot \mid \cdot))$ such that $\nu(\cdot \mid \cdot)$ constitutes a conditional neo-additive capacity in the sense of Chateauneuf et al. (2007) which is updated in accordance with the Generalized Bayesian update rule.[10] In contrast to additive conditional probabilities $\pi\left(Z_{k,t} \mid \tilde{I}_h\right)$, neo-additive conditional probabilities $\nu\left(Z_{k,t} \mid \tilde{I}_h\right)$ can replicate the patterns of Figure 1 because they do not necessarily converge through Bayesian updating to the objective probabilities $\psi_{k,t}$. The neo-additive probability space constructed in this paper can thus provide a more realistic model of survival beliefs than any additive probability space.

[10]For the formal definitions of these decision theoretic terms see Appendix A and references therein.

2.2 A Parsimonious Model

In a nutshell, Ludwig and Zimper (2013) model ambiguous survival beliefs as a weighted average of a standard additive probability measure and the degree of optimism, respectively pessimism, by which the decision maker resolves her ambiguity towards objective information. The respective weight represents the degree of ambiguity—or the lack of confidence—that the decision maker has in the additive probability measure. The model is information driven so that ambiguity—and thus the weight on relative optimism—increases in the amount of information. To simplify the analysis, Ludwig and Zimper (2013) associate information with age to the effect that ambiguity increases in the agent's actual age (i.e., when the objective survival probability decreases). For the reader's benefit we present a rigorous review of the Ludwig and Zimper (2013) learning model in Appendix B. In what follows, we only restate the model's parsimonious characterization of ambiguous survival beliefs.

Fix some $T \geq h \geq 0$ with the interpretation that the agent perceives it as possible to live until the end of period T whereas she perceives it as impossible to live longer than T. Denote by $\delta \in [0, 1]$ an initial degree of ambiguity (or degree of likelihood insensitivity). $\lambda \in [0, 1]$ denotes a psychological bias parameter which measures whether the agent resolves her ambiguity through over- or rather through under-estimation of the true probability. In this framework, Ludwig and Zimper (2013) derive the following result which we build on:

Observation 1 *(Ludwig and Zimper 2013). The h-old agent's age-dependent ambiguous belief to survive from age k with $h \leq k < T$ to target age t with $k < t \leq T$ is given by*

$$\nu_{k,t}^{h} = \delta_h \cdot \lambda + (1 - \delta_h) \cdot \psi_{k,t} \tag{1}$$

with

$$\delta_h = \frac{\delta}{\delta + (1 - \delta) \cdot \frac{1}{1+\sqrt{h}}} \tag{2}$$

for age-independent parameters $\delta, \lambda \in [0, 1]$.

The h-old agent's belief to survive from age k to some target age t

is thus formally described as an age-dependent weighted average of the objective survival probability with weight $1-\delta_h$ and the psychological bias parameter λ with weight δ_h. For $\delta = 0$ we have for all h that $\nu_{k,t}^{h} = \psi_{k,t}$ so that all ambiguous survival beliefs reduce to objective survival probabilities and the standard rational expectations approach is nested as a special case.

For any $\delta > 0$, the dynamics of the model imply that agents exhibit more pronounced ambiguity attitudes with increasing age. This feature captures the intuitive notion that, as the objective risk of survival becomes less likely, agents attach less and less certainty to this objective probability. According to our estimates of $\{\delta, \lambda\}$ presented in Section 5, objective survival probabilities $\psi_{k,t}$ decrease with age to values lower than λ. The model's convergence property hence implies that survival rates are overestimated eventually even when the initial degree of likelihood insensitivity, δ, is low.[11]

2.3 Neo-additive Probability Space

It remains to translate the above notion of ambiguous survival beliefs into the construction of the relevant conditional neo-additive probability space $(\Omega, \mathcal{F}, \nu(\cdot \mid \cdot))$. To this purpose define the finite state space $\Omega = \{0, 1, ..., T\}$ and let the σ-algebra $\mathcal{F}$ be the powerset of Ω. We interpret $D_t = \{t\}, t \in \Omega$ as the event in $\mathcal{F}$ that the agent dies at the end of period t. Define age h of the agent as the following event in $\mathcal{F}$: $h = D_h \cup ... \cup D_T$. Further, formally define $Z_{k,t} = D_t \cup ... \cup D_T$ as the event in $\mathcal{F}$ that the agent survives from period k to the beginning of period t.

The relevant information filtration of our model is simply given by $\mathcal{F}_1 \subset ... \subset \mathcal{F}_T = \mathcal{F}$ such that, for each age h, $\mathcal{F}_h$ is generated by the following partition of Ω: $\{\{0\}, .., \{h-1\}, \{h, ..., T\}\}$. That is, if the agent turns age h she (trivially) observes that she has not died in any previous period but will die at the end of either period h or $h+1$ or ... or T.

Finally, we assume that $\nu(\cdot \mid \cdot)$ satisfies (i) $\nu(\emptyset \mid \cdot) = 0$, $\nu(\Omega \mid \cdot) = 1$, (ii)

[11] If we were to assume that households do not have any memory to the effect that $\delta_h = \delta$ for all $h = 0, \ldots, T$, we would get qualitatively similar results. Quantitatively this would however imply that the degree of ambiguity would be substantially higher for many ages than with increasing δ_h. We will come back to this aspect in the interpretation of our main results below.

$\nu(\cdot \mid \cdot)$ is a conditional neo-additive capacity in the sense of Chateauneuf et al. (2007) which is updated in accordance with the Generalized Bayesian update rule where (iii) for all h, $Z_{k,t} \neq \emptyset$ and $Z_{k,t} \neq \Omega$ with $h \leq k < T$ and $k < t \leq T$, $\nu(Z_{k,t} \mid h) \equiv \nu_{k,t}^{h}$ with $\nu_{k,t}^{h}$ given by (1).

We make explicit use of this conditional neo-additive probability space $(\Omega, \mathcal{F}, \nu(\cdot \mid \cdot))$ in Section 3.4 where we introduce Choquet expected utility (=CEU) decision makers within the life-cycle model of consumption and savings.

3 Life-Cycle Model

This section generalizes a life-cycle model to the case of CEU decision makers with ambiguous survival beliefs. The non-additive survival beliefs are given as neo-additive capacities, cf. equation (1).

3.1 Demographics

We consider a large number of ex-ante identical agents (=households). Households become economically active at age (or period) 0 and live at most until age T. The number of households of age t is denoted by N_t. Population is stationary and we normalize total population to unity, i.e., $\sum_{t=0}^{T} N_t = 1$. Households work full time during periods $1, \ldots, t_r - 1$ and are retired thereafter. The working population is $\sum_{t=0}^{t_r-1} N_t$ and the retired population is $\sum_{t=t_r}^{T} N_t$.

We refer to age $h \leq t$ as the planning age of the household, i.e., the age when households make their consumption and saving plans for the future. At ages $h = 1, \ldots, T$, households face objective risk to survive to some future period t. We denote corresponding objective survival probabilities for all in-between periods k, $h \leq k < t$, by $\psi_{k,t}$ where $\psi_{k,t} \leq 1$ for all $t \leq T$ and $\psi_{k,t} = 0$ for $t = T + 1$. Population dynamics are correspondingly given by $N_{t+1} = \psi_{t,t+1} N_t$, for N_0 given.

3.2 Endowments

There are discrete shocks to labor productivity in every period $t = 0, 1, ..., , t_r - 1$ denoted by $\eta_t \in E$, E finite, which are i.i.d. across households of the same age. The reason for stochastic labor productivity in our model is to impose discipline on calibration. For sake of comparability, our fully rational model should feature standard elements as used in numerous structural empirical studies, as, e.g., by Laibson et al. (1998), Gourinchas and Parker (2002) and several others. By $\eta^t = (\eta_1, \ldots, \eta_t)$ we denote a history of shocks and $\eta^t \mid \eta^h$ with $h \leq t$ is the history $(\eta_1, \ldots, \eta_h, ..., \eta_t)$. Let E be the powerset of the finite set E and E^∞ be the σ-algebra generated by $\mathsf{E}, \mathsf{E}, ...$ We assume that there is an objective probability space $(\times_{t=0}^{\infty} E, \mathsf{E}^\infty, \pi)$ such that $\pi_t(\eta^t \mid \eta^h)$ denotes the probability of η^t conditional on η^h.

After retirement at age t_r households receive a lump-sum pension income, b. Retirement income is modeled in order to achieve a realistic calibration. Without a pension system, the old-age saving motive would lead to unrealistic saving behavior. Pension contributions are levied at contribution rate τ.

Collecting elements, income of a household of age t is given by

$$y_t = \begin{cases} \eta_t \phi_t w \, (1 - \tau) & \text{for } t < t_r \\ b & \text{for } t \geq t_r. \end{cases}$$

The interest rate is fixed at r. There are no annuity markets, an assumption which can be justified by the observed small size of private annuity markets.[12] We assume a fixed zero borrowing constraint. We define cash-on-hand as $x_t \equiv a_t \, (1 + r) + y_t$ so the budget constraint writes as

$$x_{t+1} = (x_t - c_t) \, (1 + r) + y_{t+1} \geq 0 \tag{3}$$

Define total income as $y_t^{tot} = y_t + r a_t$, saving as $s_t = y_t^{tot} - c_t$ and gross savings as assets tomorrow, a_{t+1}.

3.3 Government

We assume a pure PAYG public social security system. We denote by χ the net pension benefit level, i.e., the ratio of pensions to net wages. The

[12] See Friedman and Warshawsky (1990). Observe that pessimistic survival beliefs extenuate the annuity puzzle.

government budget is assumed to be balanced each period and is given by

$$\tau w \sum_{t=0}^{t_r-1} \phi_t N_t = b \sum_{t=t_r}^{T} N_t = \chi (1-\tau) w \sum_{t=t_r}^{T} N_t. \tag{4}$$

Accidental bequests—arising because of missing annuity markets—are taxed away at a confiscatory rate of 100%.[13]

3.4 CEU Preferences

Households face two dimensions of uncertainty, respectively risk, about period t consumption. First, due to our assumption of productivity shocks, agents face a risky labor income. Second, agents are uncertain with respect to their life expectancy. While we model income risk in the standard objective EU way, we model uncertainty about life-expectancy in terms of a CEU agent who holds neo-additive survival beliefs as stated in Observation 1.

Denote by $u(c_t)$ the agent's utility from consumption at age t. We assume that utility is strictly increasing in consumption and that the agent is strictly risk-averse, i.e., $u'(c_t) > 0$, $u''(c_t) < 0$. Given the productivity shock history η^h, denote by $\mathbf{c} \equiv (c_h, c_{h+1}, c_{h+2}...)$ a shock-contingent consumption plan such that the functions c_t, for $t = h, h+1, ...$, assign to every history of shocks $\eta^t|\eta^h$ some amount of period t consumption.

Expected utility of an h-old agent from consumption in period $t > h$ contingent on the observed history of productivity shocks η^h is given as

$$E_t [u(c_t)] \equiv E_t \left[u(c_t), \pi\left(\eta^t|\eta^h\right)\right] = \sum_{\eta^t|\eta^h} u(c_t)\, \pi\left(\eta^t|\eta^h\right)$$

where we introduce $E_t[\cdot]$ as a shortcut notation for the expectation operator with respect to productivity shock η^t in period t, conditional on period h.

We assume additive separability and discounting at rate β. For any $s \in \{h, h+1, ..., T\}$ and survival until period s, the agent's von Neumann

[13] Revenue from this source is used for government consumption which is otherwise neutral.

Morgenstern utility (vNM) from a consumption plan $\mathbf{c}$ is then defined as

$$U(\mathbf{c}(s)) = u(c_h) + \sum_{t=h+1}^{s} \beta^{t-h} E_t [u(c_t)].$$

To model survival uncertainty of an agent of age h we use the conditional neo-additive probability space $(\Omega, \mathcal{F}, \nu(\cdot \mid \cdot))$ of Section 2.3. Denote by $\nu^h \equiv \nu(\cdot \mid h)$ the agent's age-conditional neo-additive capacity. In order to formalize utility maximization over life-time consumption with respect to neo-additive probability measures, we henceforth describe an h-old agent as a CEU decision maker who maximizes her Choquet expected utility from life-time consumption with respect to ν^h. By Observation 8 in Appendix A, this agent's CEU from consumption plan $\mathbf{c}$ with respect to ν^h is given as

$$\begin{aligned} E\left[U(\mathbf{c}), \nu^h\right] &= \delta_h \left[\lambda \sup_{s \in \{h,h+1,\ldots\}} U(\mathbf{c}(s)) + (1-\lambda) \inf_{s \in \{h,h+1,\ldots\}} U(\mathbf{c}(s))\right] \\ &\quad + (1-\delta_h) \cdot \sum_{s=h}^{T} [U(\mathbf{c}(s)), \psi(D_s)] \end{aligned}$$

where we have for any $\mathbf{c}$ that

$$\begin{aligned} \sup_{s \in \{h,h+1,\ldots\}} U(\mathbf{c}(s)) &= u(c_h) + \sum_{t=h+1}^{T} \beta^{t-h} E_t [u(c_t)], \\ \inf_{s \in \{h,h+1,\ldots\}} U(\mathbf{c}(s)) &= u(c_h). \end{aligned}$$

Observe that we require

$$\sum_{t=h+1}^{T} \beta^{t-h} E_h [u(c_t)] > 0, \qquad \forall\, h = 0, \ldots \tag{5}$$

a condition we insure to hold via calibration, see below.

Proposition 1 *Consider an agent of age h. The agent's Choquet expected utility from consumption plan $\mathbf{c}$ is given by*

$$E\left[U(\mathbf{c}), \nu^h\right] = u(c_h) + \sum_{t=h+1}^{T} \nu_{h,t}^{h} \cdot \beta^{t-h} \cdot E_h [u(c_t)] \tag{6}$$

where the subjective belief to survive from age h to $t \geq h$ is given by

$$\nu_{h,t}^{h} = \begin{cases} \delta_h \cdot \lambda + (1-\delta_h) \cdot \psi_{h,t} & \textit{for } t > h \\ 1 & \textit{for } t = h \end{cases} \tag{7}$$

with

$$\delta_h = \frac{\delta}{\delta + (1-\delta) \cdot \frac{1}{1+\sqrt{h}}},$$

cf. Observation 1.

Proof. *Relegated to Appendix C.* ■

Like the survival beliefs model of Section 2 lifetime utility of CEU agents in equation (6) reduces to the standard rational expectations EU case if and only if there is no initial ambiguity, i.e., if $\delta = 0$, implying for every age h that $\nu_{h,t}^{h} = \psi_{h,t}$. In contrast to the standard EU model, which is dynamically consistent, the CEU model (6) describes a dynamically inconsistent decision maker whenever the agent has ambiguous survival beliefs.

3.5 Naivety versus Sequential Sophistication

In models with time inconsistency one has to take a stance on what information the agent has concerning the future. The literature dealing with time inconsistency distinguishes between naive and sophisticated agents, cf. Strotz (1955) or inter alia O'Donoghue and Rabin (1999) for procrastination models.

Naifs are not aware of their time inconsistency and believe that their future "selves" will be acting rational, i.e., in their interest. Naive agents construct consumption and saving plans that maximize lifetime utility at age h. Self h then implements the first action of that sequence expecting future selves to implement the remaining plan. Coming to the next period, self $h+1$ conducts her own maximization problem and implements actions that do not necessarily coincide with the plan of self h.

In contrast, sophisticates are fully aware of their time inconsistent behavior, cf., e.g., Angeletos et al. (2001). Sophisticates correctly predict that their own future selves will not be acting according to the preference of the current self. Thus, they take actions that seek to constrain future selves behavior (commitment devices).

The CEU framework differs from the aforementioned models in being information driven. Within this model, the naive agent is a much more logical construction. It assumes that the agent cannot exactly anticipate information she will gain in the future. In contrast, the sophisticated agent anticipates receiving new information. This is an extremely stylized theoretical construction. Furthermore, there exists a large literature which, from an empirical perspective, rather supports the assumption of naive agents, cf. O'Donoghue and Rabin (1999) and the literature cited therein. For these reasons and in order to focus our analysis we concentrate on naive agents. The comparison to sophisticated agents is done in Groneck et al. (2013).

Naive CEU Agent

In order to characterize optimal behavior, it is convenient to work with the recursive representation of the planning problem. We assume that income risk is first-order Markov such that $\pi(\eta^t \mid \eta^{t-1}) = \pi(\eta^t \mid \eta_t)$. It is then straightforward to set up the recursive formulation of model (6) for the naive agent. The value function of age $t \geq h$ viewed from planning age h is given by

$$V_t^h (x_t, \eta_t) = \max_{c_t, x_{t+1}} \left\{ u(c_t) + \beta \frac{\nu_{h,t+1}^h}{\nu_{h,t}^h} E_t \left[V_{t+1}^h \left(x_{t+1}, \eta_{t+1} \right) \right] \right\}.$$

The naive CEU agent's first order condition is given by the standard Euler equation:

Proposition 2 *The consumption plan* $\mathbf{c} = (c_h, c_{h+1}, ...)$ *of a naive CEU agent must satisfy, for all* $t \geq h$,

$$\frac{du}{dc_t} \geq \beta (1+r) \cdot \frac{\nu_{h,t+1}^h}{\nu_{h,t}^h} \cdot E_t \left[\frac{du}{dc_{t+1}} \right] \tag{8}$$

which holds with equality if $a_{t+1} > 0$.

The first-order condition has important implications for expected growth of marginal utility (and hence for consumption and savings decisions):

Observation 2 *Suppose that $a_{t+1} > 0$. For a naive household of age h the expected growth of marginal utility*

1. *from h to $h+1$ is higher than under rational expectations if the household is pessimistic with regard to survival to the next period, i.e., if $\nu^h_{h,h+1} < \psi_{h,h+1}$, and vice versa for an optimistic household;*

2. *from t to $t+1$, $t > h$ is always lower than under rational expectations.*

Proof. Relegated to Appendix C. ■

Consider a household that is (at least initially) pessimistic with regard to survival into the future as in the data shown in Figure 1. The first part of Observation 2 then implies that the household *tends* to spend too much today relative to tomorrow in comparison to rational expectations. The second part of the observation entails that the household plans in period h to overcorrect this consumption-savings decision in future periods. Notice that this is a difference to the hyperbolic time discounting model in which the naive household thinks that he will just revert back to the plan of a rational agent in future periods. One may view it as more realistic that households plan to correct savings choices in the future rather than just reverting to a path considered as rational. On the contrary, an optimistic household in our sense always saves more than his rational expectations counterpart and plans to continue doing so in the future.

These two parts of the observation exemplify two potentially offsetting effects on today's saving behavior of pessimistic agents. Analyzing marginal propensities to consume of a simplified version of our model, cf. Section 4, shows that the extent towards which CEU households indeed overspend depends on whether current survival beliefs are *sufficiently pessimistic* relative to future optimistic survival beliefs. Hence, first-order conditions derived here only provide partial insights.

That such a plan also implies time inconsistent behavior follows from inspection of the marginal rates of substitution (MRS) between any two subsequent periods from the perspective of different planning periods. Under time consistency, these objects would be identical. We have for any planning period h and periods $h < k < t$ that

$$MRS^{h}_{c_k,c_t} = \left(\beta^{t-k}\right)^{-1} \frac{\upsilon^{h}_{h,k}}{\upsilon^{h}_{h,t}} \frac{E_h\left[\frac{du}{dc_k}\right]}{E_h\left[\frac{du}{dc_t}\right]}.$$

Comparing the MRS of age h with the MRS of any age $h+i < k$ we find that the decisive term is the ratio of subjective beliefs which obeys the relationship

$$\frac{\nu^{h}_{h,k}}{\nu^{h}_{h,t}} = \frac{\delta_h \cdot \lambda + (1-\delta_h) \cdot \psi_{h,k}}{\delta_h \cdot \lambda + (1-\delta_h) \cdot \psi_{h,t}} \neq \frac{\delta_{h+i} \cdot \lambda + (1-\delta_{h+i}) \cdot \psi_{h+i,k}}{\delta_{h+i} \cdot \lambda + (1-\delta_{h+i}) \cdot \psi_{h+i,t}} = \frac{\nu^{h+i}_{h+i,k}}{\nu^{h+i}_{h+i,t}}.$$

Therefore, $MRS^{h}_{c_k,c_t} \neq MRS^{h+i}_{c_k,c_t}$.

Aggregation over Households

Wealth dispersion within each age bin is only driven by productivity shocks. We denote the cross-sectional measure of agents with characteristics (a_t, η_t) by $\Phi_t(a_t, \eta_t)$. Denote by $\mathcal{A} = [0, \infty]$ the set of all possible asset holdings and let $\mathcal{E}$ be the set of all possible income realizations. Define by $\mathcal{P}(\mathcal{E})$ the power set of $\mathcal{E}$ and by $\mathcal{B}(\mathcal{A})$ the Borel σ-algebra of $\mathcal{A}$. Let $\mathcal{Y}$ be the Cartesian product $\mathcal{Y} = \mathcal{A} \times \mathcal{E}$ and $\mathcal{M} = (\mathcal{B}(\mathcal{A}))$. The measures $\Phi_t(\cdot)$ are elements of $\mathcal{M}$. We denote the Markov transition function—telling us how people with characteristics (t, a_t, η_t) move to period $t+1$ with characteristics $t+1, a_{t+1}, \eta_{t+1}$—by $Q_t(a_t, \eta_t)$. The cross-sectional measure evolves according to

$$\Phi_{t+1}(\mathcal{A} \times \mathcal{E}) = \int Q_t\left((a_t, \eta_t), \mathcal{A} \times \mathcal{E}\right) \cdot \Phi_t\left(da_t \times d\eta_t\right)$$

and for newborns

$$\Phi_1(\mathcal{A} \times \mathcal{E}) = N_1 \cdot \begin{cases} \Pi(\mathcal{E}) & \text{if } 0 \in \mathcal{A} \\ 0 & \text{else.} \end{cases}$$

The Markov transition function $Q_t(\cdot)$ is given by

$$Q_t\left(\left(a_t, \eta_t\right), \mathcal{A} \times \mathcal{E}\right) = \begin{cases} \sum_{\eta_{t+1} \in \mathcal{E}} \pi\left(\eta_{t+1} | \eta_t\right) \cdot \psi_{t,t+1} & \text{if } a_{t+1}\left(a_t, \eta_t\right) \in \mathcal{A} \\ 0 & \text{else} \end{cases}$$

for all $(a_t, \eta_t) \in Y$ and all $(\mathcal{A} \times \mathcal{E}) \in \mathcal{Y}$. Observe that the transition from t to $t+1$ is governed by the objective survival probabilities $\psi_{t,t+1}$.

Aggregation gives average (or aggregate)

$$\begin{aligned}
&\text{consumption:} && \bar{c}_t = \int c_t(a_t, \eta_t)\Phi_t(da_t \times d\eta_t), \\
&\text{assets:} && \bar{a}_t = \int a_t(a_t, \eta_t)\Phi_t(da_t \times d\eta_t), \\
&\text{income:} && \bar{y}_t = (1-\tau)w\left(\sum_{t=0}^{t_r-1} \phi_t N_t + \chi \sum_{t=t_r}^{T} N_t\right), \\
&\text{total income:} && \bar{y}_t^{tot} = \bar{y}_t + r\bar{a}_t, \\
&\text{savings:} && \bar{s}_t = \bar{y}_t^{tot} - \bar{c}_t.
\end{aligned}$$

4 Simple Model

In order to provide insights for the numerical analysis below, we now develop a simple three-period model ($T = 2$) without productivity risk ($\eta_t = 1$ for all t) which can be solved analytically. We abstract from borrowing constraints, hence $a_{t+1} < 0, t < T$ is possible. The no-Ponzi condition $a_{T+1} \geq 0$ is of course assumed. Let

$$U^0 = u(c_0) + \beta\nu_{0,1}^0 u(c_1) + \beta^2 \nu_{0,2}^0 u(c_2) = u(c_0) + \beta\nu_{0,1}^0 \left(u(c_1) + \beta\frac{\nu_{0,2}^0}{\nu_{0,1}^0} u(c_2)\right)$$

and $U^1 = u(c_1) + \beta\nu_{1,2}^1 u(c_2)$, where superscripts again denote the respective planning age. We assume a *CRRA* per-period utility function with $\theta \neq 1$ given by

$$u\left(c_t\right) = \Gamma + \frac{c_t^{1-\theta}}{1-\theta}, \tag{9}$$

for all t with preference shifter $\Gamma \geq 0$ such that condition (5) holds.

In light of the data on subjective beliefs displayed in Figure 1 we interpret period 0 of the simple model as the period when survival beliefs express relative pessimism with respect to survival, i.e., up to actual age of about $65 - 70$. Period 1 then reflects the period when there is relative optimism in the data. Correspondingly, we make the following assumption:

Assumption 1 *We assume for some $\delta > 0$ that*

$$\psi_{0,1} > \nu_{0,1}^0 = \delta_0 \lambda + (1 - \delta_0)\psi_{0,1} \tag{10}$$

i.e., that $\lambda < \psi_{0,1}$ (pessimistic beliefs), as well as

$$\psi_{1,2} < \nu_{1,2}^1 = \delta_1 \lambda + (1 - \delta_1)\psi_{1,2} \tag{11}$$

i.e., that $\lambda > \psi_{1,2}$ (optimistic beliefs).[14]

4.1 Rational Expectations

The reference model is the standard solution to the rational expectations model (where $\delta_0 = \delta_1 = 0$):

Observation 3 *Policy functions of the rational expectations solution to the simple model are linear in total wealth, $w_t \equiv x_t + h_t$ (where $x_t \equiv a_t R + y_t$ is cash on hand and $h_t \equiv \sum_{s=t+1}^{T} \left(\frac{1}{1+r}\right)^{s-t} y_s$ is human wealth): $c_t = m_t w_t$ where $w_{t+1} = (w_t - c_t)R$ and*

$$m_t = \begin{cases} \frac{b\psi_t^{-\frac{1}{\theta}} m_{t+1}}{1 + b\psi_t^{-\frac{1}{\theta}} m_{t+1}} = \frac{1}{1 + \frac{1}{b\psi_t^{-\frac{1}{\theta}} m_{t+1}}} & \text{for } t < T \\ 1 & \text{for } t = T \end{cases}$$

for $b \equiv \left(\beta R^{1-\theta}\right)^{-\frac{1}{\theta}}$.

Proof. See, e.g., Deaton (1992). ■

4.2 Naive CEU Household

To draw a distinction between rational expectations and CEU households, we use superscript n to denote policy functions (in terms of marginal

[14] Notice that, despite equation (11), we may have that the household in period 0 is pessimistic with respect to survival from period 1 to 2, hence we may have that

$$\psi_{1,2} > \nu_{1,2}^0 = \delta_0 \lambda + (1 - \delta_0)\psi_{1,2}.$$

This is so because $\delta_0 < \delta_1$ and therefore less weight is put on the optimism parameter λ.

propensities to consume) of $n$aive CEU households. Given that the household consumes all outstanding wealth in the final period 2 (i.e. $m_2^n = 1$) the solution of the household's problem for all other periods are as follows:

Proposition 3 *The solution for the naive CEU household is as follows:*

- *The solution to the problem in period 1 is:*

$$c_1^{1,n} = m_1^{1,n} w_1 \quad \textit{where} \quad m_1^{1,n} = \frac{1}{1 + \frac{1}{b\left(\nu_{1,2}^1\right)^{-\frac{1}{\theta}}}}$$

- *The plan in period 0 for period 1 is:*

$$c_1^{0,n} = m_1^{0,n} w_1 \quad \textit{where} \quad m_1^{0,n} = \frac{1}{1 + \frac{1}{b\left(\frac{\nu_{0,2}^0}{\nu_{0,1}^0}\right)^{-\frac{1}{\theta}}}}$$

- *The solution in period 0 is:*

$$c_0^{0,n} = m_0^{0,n} w_0 \quad \textit{where} \quad m_0^{0,n} = \frac{1}{1 + \frac{1}{b(\nu_{0,1}^0)^{-\frac{1}{\theta}} m_1^{0,n}}}$$

Proof. See Appendix C. ■

Interpreting this proposition yields the following observation:

Observation 4 *Under Assumption 1, equation (11), we get $m_1^{1,n} < m_1$ so that the naive household saves more out of accumulated wealth in period 1 than the household with rational expectations.*

Accumulated wealth in turn is an endogenous object. We shall see below that a *sufficiently pessimistic* naive CEU household will save less out of initial wealth in period 0. While it is therefore clear that accumulated wealth of the naive CEU household in period 1 is lower than for rational expectations, relative wealth positions across the two households in period 2 depend on the relative strength of *sufficient pessimism* in period 0 vis-a-vis *optimism* in period 1. It is therefore ultimately a quantitative question whether accumulated wealth in period 2 of CEU households exceeds wealth of households with rational expectations.[15]

To understand savings behavior in period 0 we next provide a definition which bounds optimism in period 1 by the ratio of subjective beliefs from the perspective of period 0:

Definition 1 *A household is* moderately optimistic *if* $\nu_{1,2}^{1} < \frac{\nu_{0,2}^{0}}{\nu_{0,1}^{0}}$.

Observation 5 *From the expressions in Proposition 3 it immediately follows that* $m_1^{0,n} < m_1^{1,n}$ *under* moderate optimism, *i.e., a* moderately optimistic *naive CEU household plans in period 0 to save more out of accumulated wealth in period 1 than he actually does.*

That is, only if optimism expressed in the data on subjective beliefs is not too large there is hope for our quantitative analysis to match the stylized fact that households, in the course of the life-cycle, save less than originally planned.

Observation 6 *We have that* $m_1^{0,n} \leq m_1$*, i.e., the naive CEU household plans in period 0 to save more out of wealth in period 1 than the rational expectations household.*

Proof. Relegated to Appendix C. ■

This observation directly follows from our earlier Observation 2 in Section 3. It implies that, to generate undersaving in our model, pessimism in the first period must be sufficiently large in order to dominate the effect of Observation 6:

Definition 2 (Sufficient pessimism) *We label the period 0 naive household as* sufficiently pessimistic *if* $\frac{\nu_{0,1}^{0}}{\psi_{0,1}} < \left(\frac{m_1^{1,n}}{m_1}\right)^{\theta} < 1$.

The second inequality in the above definition follows from Observation 5. To satisfy sufficient pessimism for a given fraction $\frac{m_1^{1,n}}{m_1}$, $\nu_{0,1}^{0}$ must be decreased more strongly relative to $\psi_{0,1}$ when risk aversion, θ, is increased.

[15] To provide a full characterization we could of course express consumption in all periods as a function of initial wealth. Terms however get messy and interpretation is easier with marginal propensities to consume out of current wealth.

Observation 7 *Under* sufficient pessimism *we get that*

$$m_0^{0,n} > m_0,$$

i.e., the CEU household consumes more than the RE household in period 0.

Proof. See Appendix C. ■

Only if the household is sufficiently pessimistic in the sense of Definition 2, the model generates undersaving (relative to the solution to the rational expectations model).

The analysis of the simple model clarifies that it is a quantitative question whether the full-blown stochastic model can generate the three empirical regularities on saving behavior we discussed in Section 1: (i) time inconsistent behavior to the effect that people save less than originally planned (under "moderate optimism"); (ii) undersaving at young age (under "sufficient pessimism"); (iii) high old age asset holdings (if optimism eventually outweighs initially low asset accumulation due to pessimism). We now return to this quantitative question by calibrating the full-blown stochastic model.

5 Calibration

5.1 Household Age

Households enter the model at age 20 (model age 0). The last working year is age 64, hence $t_r = 45$. We set the horizon to some maximum biological human lifespan at age 125, hence $T = 105$. This choice is motivated by Weon and Je (2009) who estimate a maximum human lifespan of around 125 years using Swedish female life-table data between $1950 - 2005$.

5.2 Objective Cohort Data

For objective survival rates we estimate cohort specific survival rates for US cohorts alive in 2007. Objective cross-sectional data is taken from the Social Security Administration (SSA) for $1890 - 1933$ and the Human Mortality Database (HMD) for the years $1934 - 2007$. To obtain complete cohort tables, future survival rates are predicted by the Lee and Carter (1992) procedure. Details are described in Ludwig and Zimper (2013).

Since data on survival rates is unreliable for ages past 100 we estimate survival rates assuming the Gompertz-Makeham law.[16] Accordingly, the mortality rate μ_t at age t is assumed to follow

$$\mu_t = \alpha_1 + \alpha_2 \cdot \exp(\alpha_3 \cdot t) + \epsilon_t, \qquad \epsilon_t \sim \mathcal{N}(0, \sigma^2).$$

We estimate parameters $\{\alpha_i\}_{i=1}^3$ to get an out of sample prediction for ages past 100. The resulting predicted mortality rate function fits actual data very well and is used as objective cohort data in the simulation. According to our estimates, the average mortality rate approaches 1 at ages around age 110 ($t = 90$). For all ages $t = 91, \ldots, 105$, we set the objective survival rate to $\psi_{t,t+1} = \varepsilon = 0.01$.

5.3 Estimated Subjective Survival Beliefs

We follow Ludwig and Zimper (2013) and estimate parameters $\delta \equiv \delta_{h=0}$ and λ by pooling a sample of HRS data formed of HRS waves $\{2000, 2002, 2004\}$. Except for heterogeneity in sex and age, we ignore all other heterogeneity across individuals. This gives $\delta = 0.118$ and $\lambda = 0.406$.[17]

5.4 Preferences

As in the simple model of the previous section, per period utility is assumed to be CRRA, $u(c_t) = \Gamma + \frac{c_t^{1-\theta}-1}{1-\theta}$, at all ages t. As a benchmark, we choose $\theta = 3.0$—corresponding to an inter-temporal elasticity of substitution (IES) of one third—and consider as range for sensitivity analysis $\theta \in \{2, 4\}$.

Given $\theta > 1$, per period utility is negative and therefore the preference shifter Γ must be calibrated such that condition (5) holds for all t, η^t. We set $\Gamma = 76.7$ for the naive CEU agent which turns out to be sufficiently high.[18] We further set the discount rate ρ to 5%.

[16] See, e.g., Preston et al. (2001), p. 192.

[17] Estimation results are separately for men and women. We take an equally weighted average of the estimated parameters to get an approximation for λ and δ in the population.

[18] This relates to Hall and Jones (2007) who calibrate—in a different model setup—

Table 1: Calibrated parameters

Technology and Prices	
$w = 1$	Gross wage
$r = 0.042$	Interest rate
$\tau = 0.124$	Social security contribution rate
$\chi = 0.322$	Net pension benefit level
Income Process	
$\kappa = 0.97$	Persistence of income
$\epsilon = 0.68$	Variance of income
$\{\phi_t\}$	Age specific productivity estimated from PSID
Preferences	
$\theta \in \{2, 3, 4\}$	Coefficient of relative risk aversion
$\rho = 0.05$	Subjective discount rate
$\Gamma^{CEU} = 76.65$	Preference shifter for naive CEU agent
Subjective Survival Beliefs	
$\delta = 0.118$	Initial degree of ambiguity
$\lambda = 0.406$	Degree of optimism
Age Limits and Survival Data	
0	Initial model age (age 20)
$t_r = 45$	retirement (age 65)
$T = 105$	Maximum human lifespan (age 125)
$\{\psi_{k,t}\}$	Objective survival rates from SSA and HMD

5.5 Prices and Endowments

Wages are normalized to $w = 1$. We consider a symmetric two-state first-order Markov chain for the income process in periods $t = 0, \ldots, t_r$ with state vector $E = [1 + \epsilon, 1 - \epsilon]$ and symmetric transition matrix $\Pi = [\kappa, 1 - \kappa; 1 - \kappa, \kappa]$. We take as initial probability vector of the Markov chain $\pi_0 = [0.5, 0.5]'$. Values of persistence and conditional variance of the income shock process are based on the estimates of Storesletten et al. (2004) yielding $\kappa = 0.97$ and $\epsilon = 0.68$.

Age specific productivity $\{\psi_t\}$ of wages is estimated based on data from the Panel Study of Income Dynamics (PSID) applying the method developed in Huggett et al. (2007). The interest rate is set to $r = 0.042$ based on Siegel (2002). For the social security contribution rate we take the US contribution rate of $\tau = 0.124$. The pension benefit level then follows from the social security budget constraint, cf. equation (4).

All parameters are summarized in Table 1.

6 Results

6.1 Subjective Survival Beliefs

Predicted and Actual Subjective Survival Beliefs

Figure 2 compares predicted subjective survival rates resulting from the decision theoretic model with their empirical counterparts and corresponding objective survival rates. Jumps in the figure are due to changes in interview age and respective target age in the survey, cf. Ludwig and Zimper (2013). Predicted subjective beliefs fit data on subjective survival probabilities well. In particular, the model replicates underestimation of survival rates at younger ages and overestimation at older ages.[19]

a preference shifter in the range of $[22.1; 131.9]$. Notice that this is just an arbitrary monotone transformation. Any choice of $\Gamma > 76.7$ ensures that the value of life is always higher than the value of death.

[19]Ludwig and Zimper (2013) perform sensitivity analyses with regard to choice of initial age and the specific form of the experience function. They show that results do not hinge on these aspects.

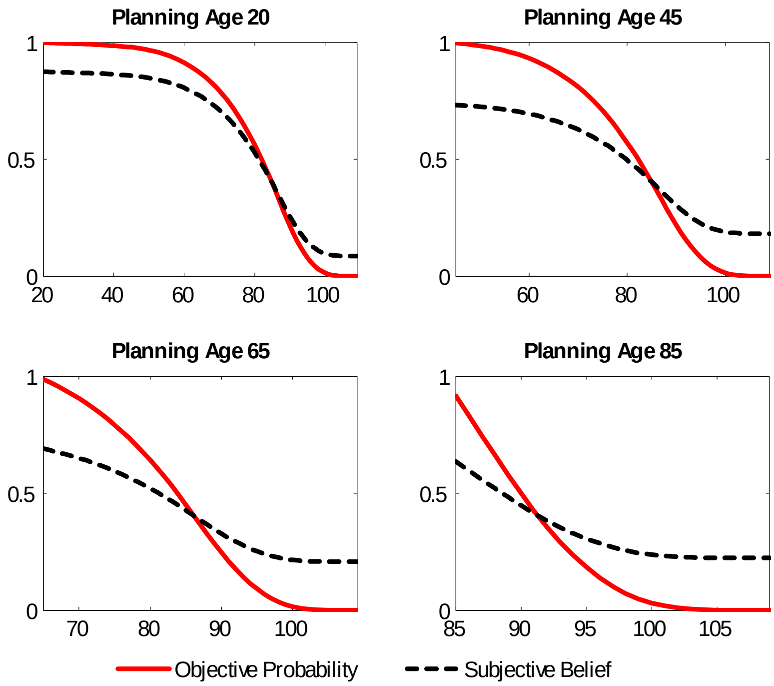
Figure 2: Objective, subjective and predicted survival rates

Notes: In the HRS interviewees are asked about their survival belief to a specific target age depending on the age at interview: Respondents between ages 50-69 are asked their probability to survive to 80, while agents between 70-74 (and 75-79, 80-84, 85-89) are asked about their belief to survive until 85 (and 90, 95, 100). The figure shows these subjective survival beliefs for different target ages (solid blue line), the corresponding objective survival rates (dashed-dotted red line) and the simulated subjective survival beliefs from the estimated CEU model (dashed green line).

Survival Belief Functions

Figure 3 compares subjective survival belief functions of CEU agents to objective cohort data. The panels in the figure show unconditional survival rates viewed from different planning ages where target age t is depicted on the abscissa. In each of the panels experience and thus likelihood insensitivity does not change. In line with Hammermesh (1985) and several others, the subjective survival function is generally flatter than its objective counterpart. The figure confirms that for younger ages underestimation of objective probabilities dominates while for old agents overestimation becomes more pronounced. Observe the initial drop of subjective beliefs which might be viewed as extreme. It is a consequence of our parsimonious specification. In our companion paper, cf. Groneck et al. (2013), we document that these survival belief functions are similar to quasi-hyperbolic time discounting functions which also feature such an initial drop. Just as quasi-hyperbolic discount functions approximate hyperbolic discount functions (which do not feature an initial drop), our specification is an approximation to a more complex structure of non-additive beliefs where the initial drop would wash out.[20] One can therefore view our model as a micro-foundation of the (quasi-)hyperbolic time discounting approach.

Figure 3: Unconditional probabilities



Notes: Unconditional objective and subjective survival probabilities viewed from different planning ages h.

Relationship to Experimental Findings on Probability Weighting

We compare estimates of our model and the implied linear probability weighting functions to inversely S-shaped probability weighting arising in cumulative prospect theory (CPT), cf. Tversky and Kahneman (1992). With this exercise we check whether the estimated parameters $\{\delta, \lambda\}$ are in the range of what is found in CPT.

In CPT, it is standard to assume a single-parameter functional form for the probability weighting function. Applied to (age dependent) survival beliefs such a functional form, as, e.g., used by Wu and Gonzalez (1996),

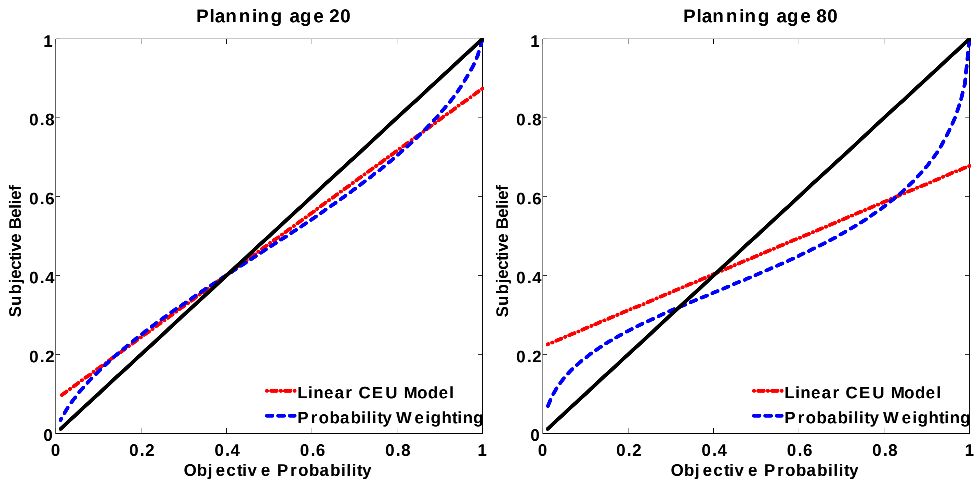
[20] For example, feeding into our model the corresponding continuous probability weighting functions, see below, results in similar quantitative predictions regarding the phenomenon of undersaving. However, such an ad-hoc model does not generate higher asset holdings at older ages. Results are available upon request.

is given by

$$\varpi_h\left(\psi_{h,t}, \xi_h\right) = \frac{\left(\psi_{h,t}\right)^{\xi_h}}{\left[\left(\psi_{h,t}\right)^{\xi_h} + \left(1-\psi_{h,t}\right)^{\xi_h}\right]^{\frac{1}{\xi_h}}}. \tag{12}$$

For $\xi_h = 1$, we have $\varpi_h\left(\psi_{h,t}, \xi_h\right) = \psi_{h,t}$. Decreasing ξ_h means increasing curvature of the probability weighting function. Standard estimates of ξ_h—not featuring age dependency—reported by Wu and Gonzalez (1996) are in the range of $[0.5, 0.9]$.

Figure 4: Probability weighting vs. linear CEU model



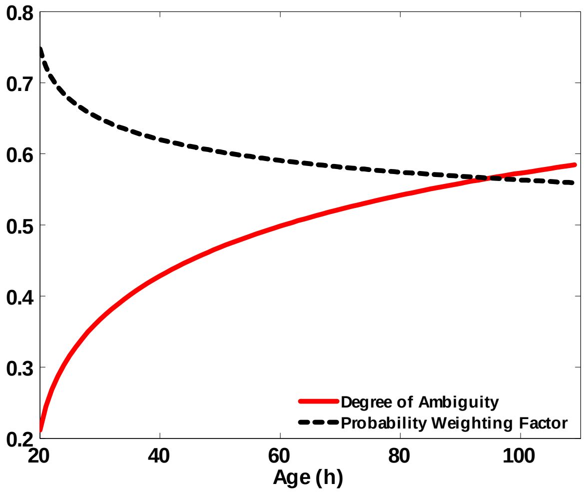
Notes: The figure shows the linear CEU subjective probabilities and the probability weighting functions compared to the 45-degree line. The estimates used for the probability weighting function are $\xi_1 = 0.75$ (for age 20) and $\xi_{61} = 0.57$ (for age 80)

We illustrate how our model relates to this concept. To this end we use that the neo-additive capacity of Observation 1 can be seen as a linear approximation to the probability weighting function (12), $\tilde{\varpi}_h = \beta_0 + \beta_1 \psi_{k,t}$ with $\beta_0 = \lambda\delta_h$ and $\beta_1 = (1-\delta_h)$, cf. Wakker (2004). Minimizing the Euclidean distance we back out the implied ξ_h for every planning age h that best matches the respective neo-additive capacity.

Figure 4 displays neo-additive beliefs and the probability weighting function for young agents (age 20) and older agents (age 80). In both models the functions exhibit an overestimation of low probabilities and an underestimation of high probabilities. Recall that δ_h is increasing with age. Consequently, the neo-additive line gets flatter and curvature of the implied inversely S-shaped probability weighting function increases, i.e.,

ξ_h decreases. At younger ages (left panel) likelihood insensitivity (or ambiguity) is low so that the probability weighting function is close to the 45-degree line. For older agents, as likelihood insensitivity is increased, both functions flatten out. Notice that younger agents form beliefs about an objective survival rate that is closer to one so that the underestimation part of the function is relevant. On the other hand, the average survival belief of 80 year old agents is further away from one making the overestimation part of the function relevant.

Figure 5: CEU and probability weighting



Notes: δ_h is calculated with equation (2), the corresponding probability weighting factor ξ is determined by minimizing the quadratic distance $\frac{1}{2}\left[\nu_{k,t}^{h}-\varpi\left(\psi_{k,t},\xi\right)\right]^{2}$ for all h.

Estimates of ξ_h, δ_h for each planning age h are depicted in Figure 5. Parameters of the probability weighting function corresponding to the CEU subjective survival rates for every planning period are in a range of $\xi_h = [0.56, 0.75]$ which is well within the bounds of conventional estimates reported by Wu and Gonzalez (1996). We are thus content with our own estimates applied to the specific context of subjective survival beliefs.[21]

[21] Relating back to our discussion in footnote 11, assuming no updating gives an estimate of about $\delta = 0.56$ and a degree of optimism of about $\lambda = 0.42$. This implies a curvature parameter of the probability weighting function of about 0.6. Accordingly, the downward drop in conditional probabilities displayed in figure 3 is even stronger for young ages. Otherwise, results are not affected much by this ad hoc—but arguably even more parsimonious—specification.

6.2 Life-Cycle Profiles with Ambiguous Beliefs

Plan vs. Realization of Naive CEU Agents

This section compares average plans and realized actions of naive CEU agents. These agents update their plans in each period. As a way to compare any gap between plans made at age h and realizations in $t \geq h$ for CEU agents we denote planned average consumption with superscripts and compute

$$\tilde{c}_t^h = \int c_t^h(a_t, \eta_t)\Phi_t^h(da_t \times d\eta_t) \tag{13}$$

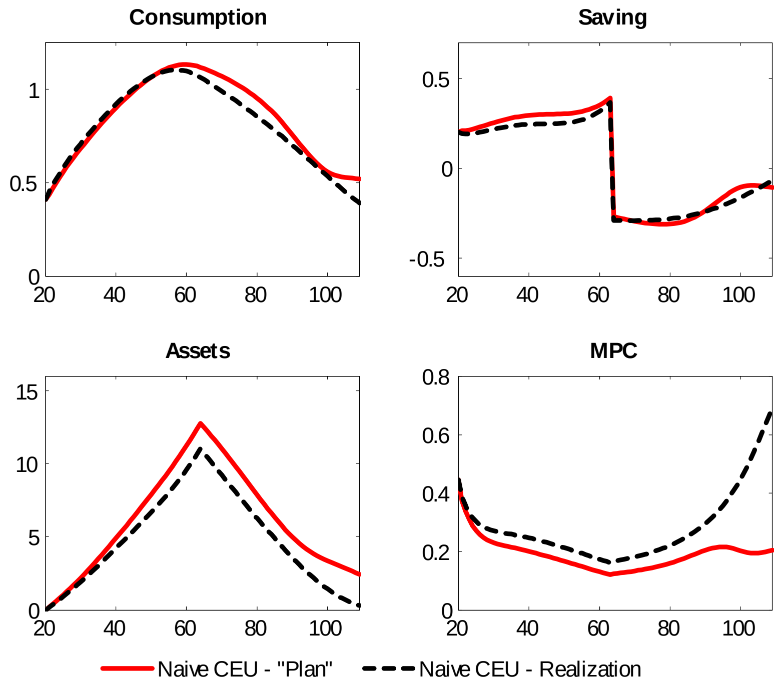
for all t. This gives us hypothetical average consumption profiles in the population if households would stick to their respective period-h plans in all periods $t = h, \dots, T$. Observe that $\Phi_t^h(\cdot)$ is an artificial distribution generated by respective plans of households. We refer to equation (13) as (average) "planned" consumption (asset, ...) profile in the figures that follow. By dynamic consistency, we have for RE agents that

$$c_t^h(a_t, \eta_t) = c_t^1(a_t, \eta_t) \qquad \text{hence} \qquad \tilde{c}_t^h = \tilde{c}_t$$

for all $h = 1, \dots, T$. These equalities do not hold for naive CEU agents.

Figure 6 compares these objects for naive CEU agents. We compute average consumption, $\tilde{c}_t^h$, net savings, $\tilde{s}_t^h$, assets, $\tilde{a}_t^h$ and total income, $\tilde{y}_t^{tot}$ as well as corresponding average realizations. Initially, CEU agents, on average, plan to save more and consume less during working life which would result in higher assets. The planned average saving rate of 20 year old CEU agents is 24.7 percent, whereas the realized saving rate is 21.9 percent. We can thus replicate empirical findings reviewed above that people save less than originally planned, cf. Choi et al. (2006). In the sense of the simple model of Section 4, cf. Observation 5, the naive CEU agent is "moderately optimistic": the marginal propensity to consume (MPC) is lower throughout when making their plan and $\frac{\nu_{h,h+2}^h}{\nu_{h,h+1}^h} > \nu_{h+1,h+2}^{h+1}$ at all ages, cf. Figure 8 in Appendix D for the latter.

Figure 6: Average "planned" and realized life-cycle profiles of CEU agents



Notes: Average planned life-cycle profiles of CEU agents at planning age 20 ($h = 1$) compared to average ex-post profiles. MPC denotes the marginal propensity to consume out of cash-on-hand which is approximated by computing averages of $\Delta c/\Delta x$ from the associated policy functions.

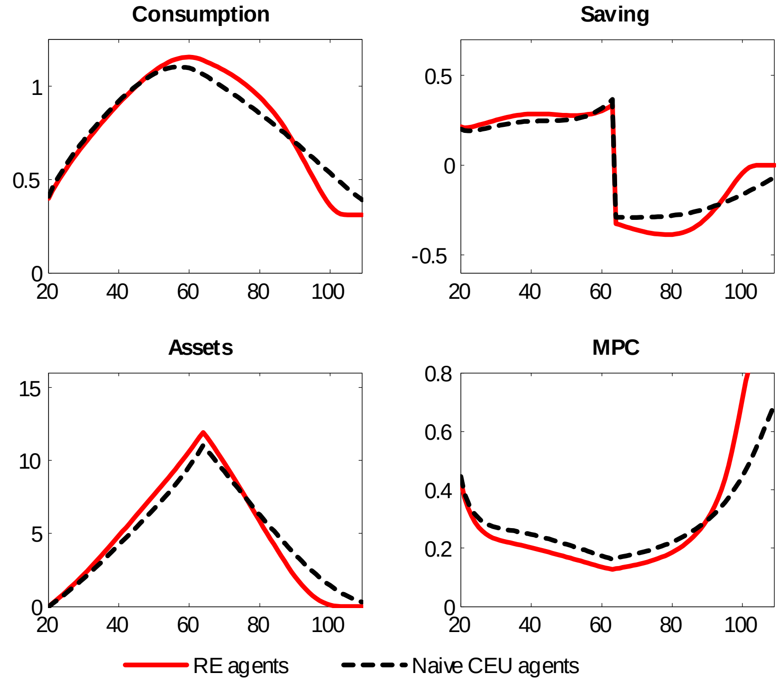
6.3 Ambiguous versus Rational Survival Beliefs

Life-Cycle Profiles in Baseline Calibration

To highlight the effects of modeling subjective survival beliefs on life-cycle profiles of consumption, saving and asset holdings we compare the CEU model (assuming naive agents) with ambiguous survival beliefs with a rational expectations (RE) model where agents use objective survival data.

Figure 7 compares average life-cycle profiles for CEU with RE agents. On average, CEU agents exhibit undersaving during early working life until age 57 relative to RE agents. Naive CEU agents first consume more than RE agents but start to consume less at age 46 leading to higher saving at age 58 and higher asset holdings later in life. The subjective survival belief model gives rise to undersaving at younger ages—due to an underestimation of future survival—and to higher asset holdings at older ages—due to an

Figure 7: Average life-cycle profiles: RE versus naive CEU



Notes: Average life-cycle profile of naive CEU agents compared to RE agents. MPC denotes the marginal propensity to consume out of cash-on-hand which is approximated by by computing averages of $\Delta c/\Delta x$ from the associated policy functions.

overestimation of the survival rate at older ages. In line with the insights of Section 4, naive CEU agents are sufficiently pessimistic, cf. Definition 2, to generate undersaving relative to RE agents. Thus, the MPC is higher at younger ages. With rising age, CEU agents become more and more optimistic with respect to their survival beliefs so that at older ages the marginal propensity is higher for the RE agents.

Table 2 comprises these results by reporting summary statistics. The average saving rate[22] of CEU agents during their working life is roughly one percentage point lower than the average saving rate of RE agents. More strikingly, average asset holdings of the elderly of ages 85+ are very different between the two types. For CEU agents assets of the elderly are roughly 107 percent of average assets. On the contrary, for RE agents, average asset holdings of the elderly are only 52 percent of average assets.

[22]The saving rate is defined as the ratio of average savings to average income.

Table 2: Summary Statistics

		RE	*CEU*
Ratio of max. consumption[1)]		2.07	1.91
Age at max. consumption		60	57
Saving rate[2)]		22.8%	21.9%
Assets at age 85 relative to 65[3)]	Average	35.4%	46.9%
	Median	31.0%	37.3%
Assets at age 95 relative to 65[3)]	Average	8.1%	23.9%
	Median	3.1%	11.6%
Assets of ages 85+ rel. to lifetime average[4)]		52.2%	106.9%

[1)] Maximal consumption relative to consumption at age 25

[2)] We define the "average" saving rate as the ratio of averages during working life. We hence compute $\sum s_t / \sum y_t$

[3)] Assets of age 85 (95) relative to assets at retirement entry.

[4)] Percentage difference of average assets during ages 85-110 relative to average assets through whole life.

Assets of an agent at age 85 (95) relative to her assets at retirement entry are still 46.9% (23.9%) while these values are much lower for RE agents, especially at very old ages. To make our results comparable to the empirical evidence reported by Hurd and Rohwedder (2010), cf. Section 1, Table 2 also reports numbers on median asset holdings.[23] These numbers indicate that the median decumulation speed of RE agents is $\frac{31.0}{3.1} \approx 10.0$ compared to $\frac{37.3}{11.6} \approx 3.2$ for CEU agents which is substantially closer to the empirical Hurd and Rohwedder (2010) benchmark of 2.3.

A Trade-off Between Matching Both Empirical Facts

The inter-temporal elasticity of substitution (IES)—the inverse of the coefficient of relative risk aversion θ—influences the willingness to smooth consumption over time. Increasing the IES leads to more consumption at younger ages and to a higher degree of undersaving by the CEU agent. This leads to less asset accumulation. In contrast, high old-age asset holdings of CEU agents is less pronounced when the intertemporal substitution elasticity is high. Thus, the choice of the IES determines whether undersaving

[23] Recall that median wealth paths for single households in the HRS indicate that households at age 85 (90) still hold around 49 (21) percent of their assets of age 65 Hurd and Rohwedder (2010). This corresponds to a decumulation speed of $\frac{49}{21} = 2.3$.

or high old-age asset holdings is predominant.

Table 3: Summary Statistics for different IES[1)]

		High IES		*Low IES*	
		RE	*CEU*	*RE*	*CEU*
Ratio of max. consumption		1.88	1.70	2.23	2.08
Age at max. consumption		58	55	60	59
Saving rate		19.9%	16.6%	25.0%	24.9%
Assets at 85 rel. to 65	Average	25.1%	25.7%	42.3%	57.7%
	Median	17.3%	9.4%	35.9%	51.7%
Assets at 95 rel. to 65	Average	2.6%	5.0%	13.5%	36.5%
	Median	0.7%	1.0%	9.1%	29.4%
Assets at 85+ rel. to average		31.4%	40.0%	68.7%	145.8%

[1)] High IES is $\theta = 2$, low IES is $\theta = 4$. For a description of how the statistics are constructed see Table 2.

Table 3 shows the saving rate and asset holdings for different values of the IES by setting $\theta \in \{2, 4\}$. In case of a high IES ($\theta = 2$), undersaving by CEU agents increases to 3.3 percentage points. At the same time the difference of average asset holdings of the elderly between CEU and RE are less pronounced. Nevertheless, CEU agents have on average roughly 8.6 percentage points higher relative average assets at old age than RE agents. A lower elasticity ($\theta = 4$) leads to very pronounced high asset holdings of elderly CEU agents which are 77.2 percentage points higher than for average RE agents. The undersaving effect almost vanishes, though. With $\theta = 4$, assets of a 85 (95) year CEU agent relative to the assets at age 65 are at 57.7 (36.5) percent. CEU median asset holdings at age 85 relative to age 65 are 51.7 percent and close to the empirical point estimate of 49 percent, cf. Hurd and Rohwedder (2010). The median CEU decumulation speed is $\frac{51.7}{29.4} \approx 1.75$ which is in fact lower than the empirical benchmark of 2.3.[24] Therefore, our model would exactly replicate this fact with an IES somewhere in the reasonable range of $[0.25, 0.33]$.

[24] The corresponding median RE decumulation speed is $\frac{35.9}{9.1} \approx 3.94$ which again is far off the empirical benchmark.

7 Conclusion

This paper studies implications of ambiguous survival beliefs for consumption and saving behavior. Point of departure of our analysis is the observation that "young" people tend to underestimate whereas "old" people tend to overestimate their survival probabilities. In a first step, we develop and parameterize a model of Bayesian learning of ambiguous survival beliefs which replicates these patterns. In a second step, we merge the resulting conditional neo-additive survival beliefs into a stochastic life-cycle model with CEU (=Choquet expected utility) agents.

We show that agents in our model behave dynamically inconsistent. As a result (naive) CEU agents save less at younger ages than they actually planned to save. Due to pessimism at young age, CEU agents also save less than an agent with rational expectations. Despite this tendency to undersave, CEU agents eventually have higher asset holdings after retirement because of the overestimation of survival probabilities in old age. Overall, our model adds to explanations for three empirical findings at once: (i) time inconsistency of agents, (ii) undersaving at younger ages and (iii) high asset holdings at old age. Hence, our model hits at—but does not kill—"three birds with one stone".

Our approach relates to recent models in behavioral economics which shed light on several phenomena of economic decision making that cannot be explained by standard rational expectations models. This line of research highlights features such as bounded rationality and bounded self-control, see Bernheim and Rangel (2007) for a review. Studies report large gaps between self-reported behavior and self-reported plans and/or preferences. A problem of calibrating these studies is that additional preference parameters reflecting the degree of present-bias are hard to observe and there is not much consensus concerning their values. As opposed to this stream of literature, parameters necessary to calculate ambiguous survival beliefs can be directly estimated from data on subjective survival beliefs. Thus, our approach seems quantitatively more reliable.

Although deviations from exponential discounting have been studied extensively, only few studies have looked at peoples' subjective beliefs about their own life expectancy and the consequences in life-cycle models. Initiated by Hammermesh (1985), empirical researchers have become interested in using data on subjective expectations in such and related studies.[25] A number of studies report significant effects of subjective survival beliefs on economic decision making which strongly supports our approach.[26] Presumably, constructing simple heuristic subjective survival rates from HRS survey data would lead to qualitatively similar life-cycle behavior as our model with ambiguous belief formation.[27] In contrast to such a heuristic construction of missing data points, our structural model provides an explanation of the data within a Bayesian learning model under ambiguity with a sound decision theoretic foundation.

As an avenue for future research we intend to study welfare consequences of social security in a life-cycle model with ambiguous survival beliefs. Laibson et al. (1998) document large welfare gains of a defined contribution plan as a commitment device for a sophisticated hyperbolic consumer. In contrast, Imrohoroglu et al. (2003) find no welfare gains in general equilibrium. A problem when calibrating these studies is that additional preference parameters reflecting the degree of present-bias are not observable. The size of welfare effects of a pay-as-you-go social security system crucially depends on the hyperbolic discount rate.[28] In contrast, parameters of our model can be directly estimated from data on subjective survival beliefs.

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[25]See, e.g., Hurd and Smith (1999), Attanasio and Hoynes (2000), and Puri and Robinson (2007).

[26]In particular, using HRS data, Bloom et al. (2006) find that an increased subjective survival probability leads to higher wealth accumulation for couples. This confirms results of Hurd et al. (1998) estimating the correlation between subjective survival beliefs and saving behavior.

[27]However, if embedded into an otherwise additive probabilistic framework, this would not give rise to dynamic inconsistency.

[28]A similar criticism applies to Gul and Pesendorfer (2001) preferences. In applied papers like Kumru and Thanopoulos (2008) who study welfare effects of social security with temptation preferences the results crucially depend on the choice for the degree of temptation.

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A Appendix: Choquet Decision Theory

A.1 Choquet Integration and Neo-additive Capacities

Consider a measurable space $(\Omega, \mathcal{F})$ with $\mathcal{F}$ denoting a σ-algebra on the state space Ω and a non-additive probability measure (=*capacity*) $\kappa : \mathcal{F} \rightarrow [0, 1]$ satisfying

(i) $\kappa(\emptyset) = 0$, $\kappa(\Omega) = 1$

(ii) $A \subset B \Rightarrow \kappa(A) \leq \kappa(B)$ for all $A, B \in \mathcal{F}$.

The Choquet integral of a bounded $\mathcal{F}$-measurable function $f : \Omega \rightarrow \mathbb{R}$ with respect to capacity κ is defined as the following Riemann integral extended to domain Ω (Schmeidler 1986):

$$E[f, \kappa] = \int_{-\infty}^{0} (\kappa(\{\omega \in \Omega \mid f(\omega) \geq z\}) - 1)\, dz + \int_{0}^{+\infty} \kappa(\{\omega \in \Omega \mid f(\omega) \geq z\})\, dz. \tag{14}$$

For example, assume that f takes on m different values such that $A_1, ..., A_m$ is the unique partition of Ω with $f(\omega_1) > ... > f(\omega_m)$ for $\omega_i \in A_i$. Then the Choquet expectation (14) becomes

$$E[f, \kappa] = \sum_{i=1}^{m} f(\omega_i) \cdot [\kappa(A_1 \cup ... \cup A_i) - \kappa(A_1 \cup ... \cup A_{i-1})].$$

This paper focuses on non-additive probability measures that are defined as *neo-additive capacities* in the sense of Chateauneuf et al. (2007). Recall that the set of *null events*, denoted $\mathcal{N}$, collects all events that the decision maker deems impossible.

Definition 3 *Fix some set of null-events $\mathcal{N} \subset \mathcal{F}$ for the measurable space $(\Omega, \mathcal{F})$. The neo-additive capacity, ν, is defined, for some $\delta, \lambda \in [0, 1]$ by*

$$\nu(A) = \delta \cdot \nu_\lambda(A) + (1 - \delta) \cdot \mu(A) \tag{15}$$

for all $A \in \mathcal{F}$ such that μ is some additive probability measure satisfying

$$\mu(A) = \begin{cases} 0 & \text{if } A \in \mathcal{N} \\ 1 & \text{if } \Omega \backslash A \in \mathcal{N} \end{cases}$$

and the non-additive probability measure ν_λ is defined as follows

$$\nu_\lambda(A) = \begin{cases} 0 & \textit{iff } A \in \mathcal{N} \\ \lambda & \textit{else} \\ 1 & \textit{iff } \Omega \backslash A \in \mathcal{N}. \end{cases}$$

In this paper, we are exclusively concerned with the empty set as the only null event, i.e., $\mathcal{N} = \{\emptyset\}$. In this case, the neo-additive capacity ν in (15) simplifies to

$$\nu(A) = \delta \cdot \lambda + (1 - \delta) \cdot \mu(A)$$

for all $A \neq \emptyset, \Omega$. The following observation extends a result (Lemma 3.1) of Chateauneuf et al. (2007) for finite random variables to the more general case of random variables with a bounded range (see Zimper 2012 for a formal proof).

Observation 8 *Let $f : \Omega \to \mathbb{R}$ be an $\mathcal{F}$-measurable function with bounded range. The Choquet expected value (14) of f with respect to a neo-additive capacity (15) is then given by*

$$E[f, \nu] = \delta(\lambda \sup f + (1 - \lambda) \inf f) + (1 - \delta) E[f, \mu]. \tag{16}$$

According to Observation 8, the Choquet expected value of a random variable f with respect to a neo-additive capacity is a convex combination of the expected value of f with respect to some additive probability measure μ and an ambiguity part. If there is no ambiguity, i.e., $\delta = 0$, then the Choquet expected value (16) reduces to the standard expected value of a random variable with respect to an additive probability measure. In case there is some ambiguity, however, the second parameter λ measures how much weight the decision maker puts on the least upper bound of the range of f. Conversely, $(1 - \lambda)$ is the weight he puts on the greatest lower bound.

A.2 The Generalized Bayesian Update Rule

CEU theory has been developed in order to accommodate paradoxes of the Ellsberg type which show that real-life decision-makers violate Savage's sure thing principle Savage (1954). Abandoning of the sure thing principle has two important implications for conditional CEU preferences. First, in contrast to Bayesian updating of additive probability measures, there exist several perceivable Bayesian update rules for non-additive probability measures (Gilboa and Schmeidler 1993; Pires 2002; Eichberger, Grant, and Kelsey 2007; Siniscalchi 2011). Second, if CEU preferences are updated in accordance with an updating rule that universally satisfies the principle of consequentialism, then these CEU preferences violate the principle of dynamic consistency (in a universal sense) whenever they do not reduce to EU preferences (cf. Zimper 2012 and references therein).

In the present paper we assume that the agents form conditional capacities in accordance with the Generalized Bayesian update rule such that, for all non-null $A, B \in \mathcal{F}$,

$$\kappa (A \mid B) = \frac{\kappa (A \cap B)}{\kappa (A \cap B) + 1 - \kappa (A \cup \neg B)}. \tag{17}$$

An application of (17) to a neo-additive capacity ν gives rise to the following observation.

Observation 9 *If the Generalized Bayesian update rule (17) is applied to a neo-additive capacity (15), we obtain, for all non-null $A, B \in \mathcal{F}$,*

$$\nu (A \mid B) = \delta_B \cdot \lambda + (1 - \delta_B) \cdot \mu (A \mid B)$$

such that

$$\delta_B = \frac{\delta}{\delta + (1 - \delta) \cdot \mu (B)}.$$

B Appendix: Bayesian Learning of Ambiguous Survival Beliefs

This appendix briefly recalls the learning model of ambiguous survival beliefs as introduced in Ludwig and Zimper (2013). We consider an h-old agent, with $0 \leq h \leq k$, who observes the random sample information $\tilde{I}_{n(h)}$ which counts how many individuals out of a sample of size $n(h)$ have survived from age k to t. By assumption, these individuals have the same i.i.d. objective survival probability as the agent.

B.1 The Benchmark Case of Additive Survival Beliefs

At first, consider a standard Bayesian decision maker whose additive estimator for the chance of surviving from k to t conditional on $\tilde{I}_{n(h)}$ is defined as the conditional expected value

$$E\left[\tilde{\theta}, \mu\left(\tilde{\theta} \mid \tilde{I}_{n(h)}\right)\right]$$

where the random variable $\tilde{\theta}$ stands for the agent's survival chance with support on $(0,1)$. By the i.i.d. assumption of individual survivals, $\tilde{I}_{n(h)}$ is, conditional on the true survival probability $\tilde{\theta} = \theta$, binomially distributed with probabilities

$$\mu\left(\tilde{I}_{n(h)} = j \mid \theta\right) = \binom{n(h)}{j} \theta^j (1-\theta)^{n-j} \text{ for } j \in \{0, ..., n(h)\}.$$

We further assume that the agent's prior over $\tilde{\theta}$ is given as a Beta distribution with parameters $\alpha, \beta > 0$, implying $E\left[\tilde{\theta}, \mu\left(\tilde{\theta}\right)\right] = \frac{\alpha}{\alpha+\beta}$. That is, we assume that

$$\mu\left(\tilde{\theta} = \theta\right) = K_{\alpha,\beta}\theta^{\alpha-1}(1-\theta)^{\beta-1}$$

where $K_{\alpha,\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$ is a normalizing constant.[29]

[29] The gamma function is defined as $\Gamma(y) = \int_0^\infty x^{y-1}e^{-x}dx$ for $y > 0$.

By Bayes' rule we obtain the following conditional distribution of $\tilde{\theta}$

$$\begin{aligned} \mu\left(\tilde{\theta}=\theta \mid \tilde{I}_{n(h)}=j\right) &= \frac{\mu\left(\tilde{I}_{n(h)}=j \mid \theta\right)\mu(\theta)}{\int_{(0,1)}\mu\left(\tilde{I}_{n(h)}=j \mid \theta\right)\mu(\theta)\,d\theta} \\ &= K^{\alpha+j-1}_{\alpha+j,\beta+n(h)-k}\theta^{\alpha+j-1}(1-\theta)^{\beta+n(h)-j-1} \text{ for } \theta\in(0,1) \end{aligned}$$

Note that $\mu\left(\tilde{\theta} \mid \tilde{I}_{n(h)}=j\right)$ is itself a Beta distribution with parameters $\alpha+j, \beta+n(h)-j$. The agent's subjective survival belief conditional on information $\tilde{I}_{n(h)}=j$ is thus given as

$$\begin{aligned} E\left[\tilde{\theta},\mu\left(\tilde{\theta}\mid j\right)\right] &= \frac{\alpha+j}{\alpha+\beta+n(h)} \\ &= \left(\frac{\alpha+\beta}{\alpha+\beta+n(h)}\right)E\left[\tilde{\theta},\mu\left(\tilde{\theta}\right)\right]+\left(\frac{n(h)}{\alpha+\beta+n(h)}\right)\frac{j}{n(h)}, \\ \text{for } j &\in \{0,...,n(h)\}. \end{aligned}$$

That is, the posterior estimator $E\left[\tilde{\theta},\mu\left(\tilde{\theta}\mid\tilde{I}_{n(h)}\right)\right]$ is a weighted average of her prior survival probability $E\left[\tilde{\theta},\mu\left(\tilde{\theta}\right)\right]$, not including any sample information, and the observed sample mean $\frac{j}{n(h)}$.

B.2 Ambiguous Survival Beliefs

Turn now to a Choquet decision maker with neo-additive capacity

$$\nu\left(\tilde{\theta}\right)=\delta\cdot\lambda+(1-\delta)\cdot\mu\left(\tilde{\theta}\right)$$

such that the conditional neo-additive capacity $\nu\left(\tilde{\theta}\mid\tilde{I}_{n(h)}\right)$ results from an application of the Generalized Bayesian update rule. Instead of the additive estimator $E\left[\tilde{\theta},\mu\left(\tilde{\theta}\mid\tilde{I}_{n(h)}\right)\right]$ we now suppose that the agent's estimator for her survival chance is given as the conditional Choquet expected value

$$E\left[\tilde{\theta},\nu\left(\tilde{\theta}\mid\tilde{I}_{n(h)}\right)\right]=\delta_{\tilde{I}_{n(h)}}\left(\lambda\sup\tilde{\theta}+(1-\lambda)\inf\tilde{\theta}\right)+\left(1-\delta_{\tilde{I}_{n(h)}}\right)E\left[\tilde{\theta},\mu\left(\tilde{\theta}\mid\tilde{I}_{n(h)}\right)\right].$$

For a Beta distribution $\mu\left(\tilde{\theta}\right)$, Ludwig and Zimper (2013) prove the following result:

Observation 10 *The Choquet decision maker's ambiguous survival belief is given as*

$$E\left[\tilde{\theta},\nu\left(\tilde{\theta}\mid\tilde{I}_{n(h)}\right)\right]=\delta_{\tilde{I}_{n(h)}}\cdot\lambda+\left(1-\delta_{\tilde{I}_{n(h)}}\right)\cdot E\left[\tilde{\theta},\mu\left(\tilde{\theta}\mid\tilde{I}_{n(h)}\right)\right],\qquad(18)$$

with

$$\delta_{\tilde{I}_{n(h)}}=\frac{\delta}{\delta+(1-\delta)\cdot\mu\left(\tilde{I}_{n(h)}\right)}$$

where the unconditional distribution of $\tilde{I}_{n(h)}$ *is given by*

$$\begin{aligned}\mu\left(\tilde{I}_{n(h)}=j\right)&=\binom{n\,(h)}{j}\frac{(\alpha+j-1)\cdot...\cdot\alpha\cdot(\beta+n\,(h)-j-1)\cdot...\cdot\beta}{(\alpha+\beta+n\,(h)-1)\cdot...\cdot(\alpha+\beta)}\qquad(19)\\ \text{for } j&\in\{0,...,n\,(h)\}\,.\end{aligned}$$

Finally, to derive from (18) the parsimonious characterization of ambiguous survival beliefs in Observation 1, we employ several simplifying assumptions:

Assumption 2 *The additive part* $E\left[\tilde{\theta},\nu\left(\tilde{\theta}\mid\tilde{I}_{n(h)}\right)\right]$ *is, for any information* $\tilde{I}_{n(h)}$*, given as the objective probability, denoted* $\psi_{k,t}$*, to survive from age* k *to* t*.*

Assumption 3 *The agent's additive prior over the parameter space is given as a uniform distribution, i.e., a Beta distribution with parameters* $\alpha=\beta=1$*, implying for (19) that* $\mu\left(\tilde{I}_{n(h)}=j\right)=\frac{1}{1+n(h)}$.

Assumption 4 *The age-dependent sample-size function is given as*

$$n\,(h)=\sqrt{h}\text{ for }h\leq T$$

which implies, by Assumption 3, that

$$\mu\left(\tilde{I}_{n(h)}=j\right)=\frac{1}{1+n\,(h)},\text{ for }j\in\{0,...,n\,(h)\}\,.$$

Assumption 2 is an extreme version of the rational Bayesian learning part of the model developed in Appendix B.1. It specifies a fully informed prior and hence simplifies upon Ludwig and Zimper (2013).[30] By this assumption any difference between subjective survival beliefs and objective survival probabilities are exclusively driven by the ambiguity part of the agent's belief. Assumption 3 allows for an explicit expression of the unconditional probability $\mu\left(\tilde{I}_{n(h)}\right)$ which only depends on age h, i.e., it is identical for every observed sample information $\tilde{I}_{n(h)}$ if h is fixed. By assumption 4, the agent observes a strictly increasing sample while growing older.

[30] Ludwig and Zimper (2013) are more explicit about the rational Bayesian learning part of the model and assume a proportional bias in prior additive beliefs.

C Appendix: Formal Proofs

Proof. [Proof of Proposition 1] The objective probability to survive until period t is given as

$$\psi_{h,t} = \prod_{s=h}^{t-1} \psi_{s,s+1}$$

implying

$$\psi_{h,t} = \sum_{s=t+1}^{T} \psi^h(D_s)$$

where D_t denotes the event that the agent dies at the end of period t. Consequently, (5) can be equivalently written in terms of survival beliefs as

$$\begin{aligned}
E\left[U(\mathbf{c}), \nu^h\right] &= \delta_h \left(\lambda \left(u(c_h) + \sum_{t=h+1}^{T} \beta^{t-h} E\left[u(c_t), \pi(\eta_t|\eta_h)\right] \right) + (1-\lambda) u(c_h) \right) \\
&\quad + (1-\delta_h) \left(u(c_h) + \sum_{t=h+1}^{T} \psi^h(D_t) \sum_{s=h+1}^{t} \beta^{s-h} E\left[u(c_s), \pi(\eta_s|\eta_h)\right] \right) \\
&= u(c_h) + \delta_h \lambda \sum_{t=h+1}^{T} \beta^{t-h} E\left[u(c_t), \pi(\eta_t|\eta_h)\right] \\
&\quad + (1-\delta_h) \sum_{t=h+1}^{T} \psi^h(D_t) \sum_{s=h+1}^{t} \beta^{s-h} E\left[u(c_s), \pi(\eta_s|\eta_h)\right] \\
&= u(c_h) + \delta_h \lambda \sum_{t=h+1}^{T} \beta^{t-h} E\left[u(c_t), \pi(\eta_t|\eta_h)\right] \\
&\quad + (1-\delta_h) \sum_{t=h+1}^{T} \psi_{h,t} \cdot \beta^{t-h} E\left[u(c_t), \pi(\eta_t|\eta_h)\right] \\
&= u(c_h) + \sum_{t=h+1}^{T} \left(\delta_h \lambda + (1-\delta_h)\psi_{h,t}\right) \beta^{t-h} E\left[u(c_t), \pi(\eta_t|\eta_h)\right] \\
&= u(c_h) + \sum_{t=h+1}^{T} \nu_{h,t}^h \beta^{t-h} E\left[u(c_t), \pi(\eta_t|\eta_h)\right].
\end{aligned}$$

■

Proof. [Proof of Observation 2] Rewrite equation (8) to

$$E_t\left[\frac{\frac{du}{dc_{t+1}}}{\frac{du}{dc_t}}\right]=\left(\beta\left(1+r\right)\cdot\frac{\nu^h_{h,t+1}}{\nu^h_{h,t}}\right)^{-1}$$

1. Proofing the first part of the observation is trivial because $\left(\beta\left(1+r\right)\cdot\nu^h_{h,h+1}\right)^{-1}>\left(\beta\left(1+r\right)\cdot\psi_{h,h+1}\right)^{-1}$ whenever $\nu^h_{h,h+1}<\psi_{h,h+1}$ and vice versa.

2. For the second part observe that expected marginal utility growth for the naive agent is lower for all ages $t>h$ than under rational expectations if $\frac{\nu^h_{h,t+1}}{\nu^h_{h,t}}>\frac{\psi_{h,t+1}}{\psi_{h,t}}=\psi_{t,t+1}$. For $\delta>0$ (giving $\delta_h>0$, all h) this always holds because

$$\begin{aligned}
& \frac{\nu^h_{h,t+1}}{\nu^h_{h,t}}>\psi_{t,t+1}\\
\Leftrightarrow\quad & \frac{\delta_h\lambda+(1-\delta_h)\psi_{h,t+1}}{\delta_h\lambda+(1-\delta_h)\psi_{h,t}}>\psi_{t,t+1}\\
\Leftrightarrow\quad & \delta_h\lambda+(1-\delta_h)\psi_{h,t+1}>\psi_{t,t+1}\left(\delta_h\lambda+(1-\delta_h)\psi_{h,t}\right)\\
\Leftrightarrow\quad & \delta_h\lambda+(1-\delta_h)\psi_{h,t+1}>\delta_h\lambda\psi_{t,t+1}+(1-\delta_h)\psi_{h,t+1}\\
\Leftrightarrow\quad & \delta_h\lambda(1-\psi_{t,t+1})>0.
\end{aligned}$$

■

Proof. [Proof of Proposition 3]

- The first-order condition in period 1 is:

$$u_c(c_1)=\beta R\nu^1_{1,2}u_c(c_2)$$

which gives

$$c_1^{1,n}=m_1^{1,n}w_1\quad\text{where}\quad m_1^{1,n}=\frac{1}{1+\frac{1}{b\left(\nu^1_{1,2}\right)^{-\frac{1}{\theta}}}}.$$

- Period 0: The plan for period 1 gives the first-order condition:

$$u_c(c_1)=\beta R\frac{\nu^0_{0,2}}{\nu^0_{0,1}}u_c(c_2)$$

which yields

$$c_1^{1,n} = m_1^{1,n} w_1 \quad \text{where} \quad m_1^{1,n} = \frac{1}{1 + \frac{1}{b\left(\frac{\nu_{0,2}^0}{\nu_{0,1}^0}\right)^{-\frac{1}{\theta}}}}$$

- The first-order condition in period 0 is:

$$u_c(c_0) = \beta R \nu_{0,1}^0 u_c(c_1)$$

yielding

$$c_0 = m_0^{0,n} w_0 = \frac{1}{1 + \frac{1}{b(\nu_{0,1}^0)^{-\frac{1}{\theta}} m_1^{0,n}}} w_0$$

■

Proof. [Proof of Observation 6] To get that $m_1^{0,n} \leq m_1$ we need

$$\frac{\nu_{0,2}^0}{\nu_{0,1}^0} \geq \psi_{1,2}.$$

This always holds, see Observation 2. ■

Proof. [Proof of Observation 7] We have already established above that

$$m_1^{0,n} < m_1,$$

i.e., the optimistic CEU household plans to save more in the second period than the RE household. In order to get that

$$m_0^{0,n} > m_0,$$

i.e., that the CEU household consumes more than the RE household in period 1 we require that

$$
\begin{aligned}
& \frac{1}{1+\frac{1}{b(\nu^0_{0,1})^{-\frac{1}{\theta}}m_1^{0,n}}} > \frac{1}{1+\frac{1}{b(\psi^0_{0,1})^{-\frac{1}{\theta}}m_1}} \\
\Leftrightarrow \quad & (\nu^0_{0,1})^{-\frac{1}{\theta}}m_1^{0,n} > (\psi^0_{0,1})^{-\frac{1}{\theta}}m_1 \\
\Leftrightarrow \quad & \frac{\nu^0_{0,1}}{\psi_{0,1}} < \left(\frac{m_1^{0,n}}{m_1}\right)^{\theta}.
\end{aligned}
$$

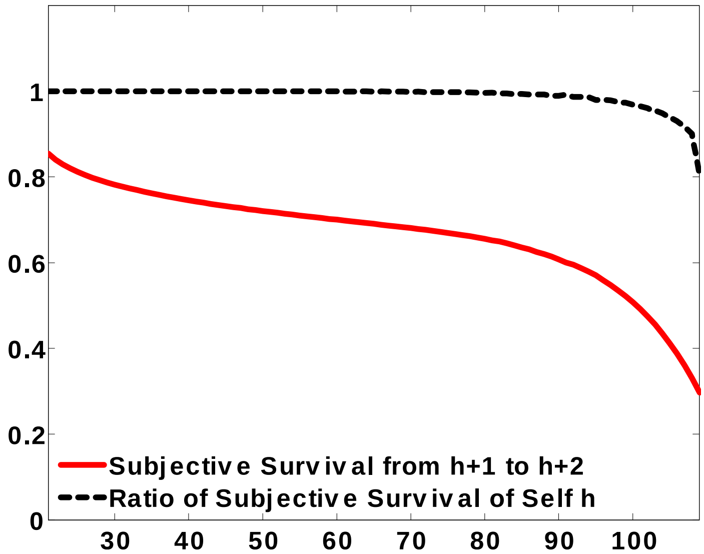
■

D Appendix: Supplementary Results

Moderate Optimism

Moderate optimism is defined in Observation 5, Section 4. It implies that the ratio of subjective beliefs of planning age h is always higher than the subjective belief to survive to the next period of planning age $h+1$, i.e. $\frac{\nu^h_{h,h+2}}{\nu^h_{h,h+1}} > \nu^{h+1}_{h+1,h+2}$. Figure 8 confirms this for the numerical exercise.

Figure 8: Moderate Optimism and Subjective Survival Belief



Notes: The figure shows that the ratio of subjective beliefs from planning age h, $\frac{\nu^h_{h,h+2}}{\nu^h_{h,h+1}}$ (black dotted line) is always higher than the belief to survive to the next period from planning age *h+1*, $\nu^{h+1}_{h+1,h+2}$ (red solid line).