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Estimation of Premiums for Heavy-Tailed Loss

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Estimation of Premiums for Heavy-tailed Loss

by

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Abstract

Pricing of an Insurance product is a necessary but not easy issue for Insurance company. To meet the no-arbitrary pricing model in asset pricing theory, Wang (1995) proposed a recent premium principle named proportional hazards transform. Amount of premiums under this principle can be estimated by parametric, nonparametric and semi-parametric methods. We compare different estimators for heavy-tailed losses within a limited sample size. In particular, we investigate the asymptotic normality of extremes estimators. The detailed simulation study in this thesis clearly demonstrates the excellent performance of extreme estimators.

Contents

Chapter 1

Introduction

Insurance is a fundamental financial product that is particularly useful to hedge against risk of losses. The insured transfers the risk of losses to the insurer, usually insurance companies, by paying a certain amount of premium. After pooling risks in a large scale, the insurer reduces the average risk over a large number of the insured by diversification or other hedge methods. One possible option for the insurer (cedant) is to buy reinsurance to further transfer their risk to the reinsurer. Consequently both of the reinsurer and the insurer allocate a portion of the risk of losses as well as the premiums.

With no doubt, the determination of the amount of premium is one of most crucial topics for both insurers and reinsurers. If the premium is too high, the insurance companies lose their market. If the amount of premium is too small, the insurers or reinsurers expose themselves to risk of significant financial loss. This problem is particularly relevant to the insurance company when the insurers or reinsurers are pricing heavy-tailed losses, in other words, the losses of which extreme values occurs with high probability. Here, we give the formal definition of heavy-tailed loss in this thesis as following.

Definition 1. *A loss X is said to be heavy-tailed if for some $r > 0$, $E[X^r] = \infty$.*

In order to quantify the risk premium properly, a variety of premium principles is developed. Examples of traditional premiums principles are the expected value principle, the standard deviation principle, value at risk and etc. In this thesis, we focus on a recent principle proposed by Wang based on proportional hazards transform. We call the premium computed according to this principle as proportional hazards premium, or in short PH-premium. Wang's principle satisfies all the desired properties of premium principle including subadditivity and layer additivity, which conforms to the adjusted distribution methods advocated by Venter for no-arbitrage pricing model. We give the formulation of PH-premium for regular insurance as following.

Definition 2. *Given a loss distribution $F(u) = Pr\{X \leq u\}$, for some exogenous index $0 < \eta \leq 1$,* *the proportional hazards (PH) transform refers to a mapping* $S_w(u) = [1 - F(u)]^\eta$*, and the PH-premium refers to the expected value under the transformed distribution:*

$$H_\eta[X] = \int_0^\infty [1 - F(u)]^\eta du$$

which represent the risk-adjusted premium and $H_\eta[X] \geq \int_0^\infty [1 - F(u)]du = E[X]$

A standard product of reinsurance is excess-of-loss reinsurance, which means the reinsurer only offset the loss of cedent exceeding a certain amount of retention. By purchasing excess-of-loss reinsurance, the cedants limit their risk at a certain level. Because of the additivity property of proportional hazards premium for losses, the PH-premium of excess-of-loss reinsurance immediately follows as bellowing.

Definition 3. *Given a loss distribution* $F(u) = Pr\{X \leq u\}$*, for some exogenous index* $\eta (0 < \eta \leq 1)$*, the risk-adjusted premium of excess-of-loss reinsurance at retention a according to proportional hazards (PH) transform is*

$$H_{\eta,a}[X] = \int_a^\infty [1 - F(u)]^\eta du$$

In practice, lacking prior knowledge of losses, the insurers often adopt estimates of premiums because accurate computation is unfeasible. Both parametric and nonparametric estimation approaches are available. The insurers favor for parametric approaches estimate the relevant parameters by restricting themselves on certain loss distribution. Parametric approach usually gives most effective results if the choice on loss distribution is appropriate. However, on the other hand, these insurers are obviously running the risk of misidentification of the loss distribution. Such risk can be devastating to the insurance company because an artificial loss distribution may seriously underestimate the actual amount of premium. In this sense, the insurers often switch to nonparametric method for robust results.

Although the limiting behaviors of these estimators are thoroughly studied in the literature, their performances within a limited sample size are not immediately clear. In fact, for heavy-tailed losses such as fire and flood that cannot be observed frequently, both parametric and nonparametric estimations can perform poorly. This motivates the development of a semiparametric method based on Extreme value theory. Extreme value theory restricts the behavior of the distribution function in the tail basically to resemble a limited class of functions that can be fitted to the tail of distribution (de Haan, 2006). In general, the semiparametric method improves the nonparametric estimators by integrating corrections for tail behavior.

The aim of this thesis is to compare different estimation approaches for proportional hazards premium of heavy-tailed losses. We investigate both the empirical and theoretic results for regular insurance and excess-of-loss reinsurance. In chapter 2 to 4, we respectively discuss parametric, nonparametric and semi-parametric approaches. The main attention is on extreme estimations. In particular, we offer several new theorems on the asymptotic normality of extreme estimators. The proofs are included in Appendix. For analysis on different estimation procedures in case of finite size of data, in Chapter 5, we carry out a simulation study. The outcomes clearly demonstrate the excellent performance of extreme estimators. Finally, we conclude this thesis in Chapter 6.

Chapter 2

Parametric Approach: Maximum Likelihood Estimator

When we know the loss has a given parametric distribution function, it suffices to estimate its parameter for any further inference. In this case, parametric process is usually most effective due to factual prior knowledge. In this chapter, we study the maximum likelihood estimation (MLE) of proportional hazards premiums.

We can easily derive MLE of PH-premium from the MLE of parameters of loss distribution based on the invariance property of maximum likelihood estimators. The invariance property of MLE states that the MLE of any transformation of parameters is nothing but the function value at MLE of parameters. Indeed, suppose loss X has a cumulative distribution function $F_\theta(x) = Pr\{X \leq x\}$, which is determined by parameter(s) $\theta \in \Theta \subset \mathbb{R}^d$. Denote $\hat{\theta}$ as the MLE[1] of parameter(s) θ. Define function $G_{\eta,a}(\theta) = \int_a^\infty [1 - F_\theta(u)]^\eta du$ for all $\theta \in \Theta$. Then , for regular insurance, the MLE of PH-premium is

$$G_{\eta,0}(\hat{\theta}) = \int_0^\infty [1 - F_{\hat{\theta}}(u)]^\eta du$$

and, for excess-of-loss reinsurance at retention level a, the MLE of PH-premium is

$$G_{\eta,a}(\hat{\theta}) = \int_a^\infty [1 - F_{\hat{\theta}}(u)]^\eta du$$

Usually above two estimators can be computed analytically. Numerical integration can solve other cases and can be easily implemented by computers. This parametric estimation procedure assumes the data are drawn from a certain probability distribution. An obvious risk embedded in this approach is misidentification of loss distribution. If the probability distribution of loss is not chosen properly, this parametric approach usually fail to deliver satisfying estimations.

[1] Here we make a natural assumption on the existence of MLEs.

2.1 Asymptotic Normality and consistency of MLE estimators

MLE estimators are always consistent and asymptotic normal under regularity conditions. Here, we do not discuss any further in the proof of this statement since it is written in a lot of past literatures. First we assume the parameter space is one-dimensional. The multidimensional cases are entirely analogous. The asymptotic normality of maximum likelihood estimation, for all $a \geq 0$ under regularity conditions, says

$$\sqrt{n}(G_{\eta,a}(\hat{\theta}) - H_{\eta,a}[X]) = \sqrt{n}(G_{\eta,a}(\hat{\theta}) - G_{\eta,a}(\theta)) \xrightarrow{d} N\left(0, \frac{(g_{\eta,a}(\theta))^2}{I(\theta)}\right) \tag{2.1}$$

where, for any positive value a, $g_{\eta,a}(\cdot)$ is the derivative of $G_{\eta,a}(\cdot)$ and $I(\theta)$ is the fisher information for parameter θ. Note here, by Leibniz integral rule, for all $a \geq 0$,

$$g_{\eta,a} = \frac{d}{d\theta}\int_a^\infty [1 - F_\theta(x)]^\eta dx = \int_a^\infty \frac{d}{d\theta}[1 - F_\theta(x)]^\eta dx = -\int_a^\infty [1 - F_\theta(x)]^{\eta-1}\frac{dF_\theta(x)}{d\theta}dx$$

Then it immediately follows by Slutsky'theorem that , that is , denote $M_{\eta,\theta}(a) = \frac{g_{\eta,a}(\theta)}{G_{\eta,a}(\theta)}$

$$\sqrt{n}\left(\frac{G_{\eta,a}(\hat{\theta})}{H_\eta[X]} - 1\right) = \sqrt{n}\left(\frac{G_{\eta,a}(\hat{\theta})}{G_{\eta,a}(\theta)} - 1\right) \xrightarrow{d} N\left(0, \frac{[M_{\eta,\theta}(a)]^2}{I(\theta)}\right) \tag{2.2}$$

which implies that the relative error of our estimation also converge to 0 in probability with converge rate of $\mathcal{O}_p(n^{-2})$. Here, note that $\lim_{a\to\infty} g_{\eta,a}(\theta) = 0$ and $\lim_{a\to\infty} G_{\eta,a}(\theta) = 0$, by L'Hôpital's rule

$$\begin{aligned}
&\lim_{a\to\infty} M_{\eta,\theta}(a) = \lim_{a\to\infty} \frac{g_{\eta,a}(\theta)}{G_{\eta,a}(\theta)} = \lim_{a\to\infty} \frac{dg_{\eta,a}(\theta)/da}{dG_{\eta,a}(\theta)/da} \\
&= \lim_{a\to\infty}\left\{-[1 - F_\theta(a)]^{\eta-1}\frac{dF_\theta(a)}{d\theta}\right\} / \left\{[1 - F_\theta(a)]^\eta\right\} \\
&= \lim_{a\to\infty}\left\{\frac{1}{1 - F_\theta(a)}\frac{d[1 - F_\theta(a)]}{d\theta}\right\} \\
&= \lim_{a\to\infty}\frac{d\log[1 - F_\theta(a)]}{d\theta}
\end{aligned}$$

Consequently, when $\lim_{a\to\infty}\frac{d\log[1-F_\theta(a)]}{d\theta} = \infty$, such as for Fréchet and Burr distribution, our MLE may perform very bad for large a within a finite size of sample because of the large asymptotic variance . The simulation study in Chapter 5 offers empirical evidence for this statement.

Chapter 3

Nonparametric Approach: Empirical Estimator

Another alternative estimation method for PH-premium is nonparametric empirical estimation. In this procedure, we estimate loss distribution function by the empirical distribution function. Then we construct the empirical estimators for PH-premium accordingly. Here, we do not need any assumption on loss distribution nor on any relevant parameter. It means this estimation procedure is entirely free of the risk of misidentification of loss distribution. However, an obvious drawback of the empirical estimators is its relatively low quality compared to MLEs if the specification of loss distribution is, in fact, appropriate.

Hence, given i.i.d. claims data $X_1, ..., X_n$, the empirical estimators of PH-premium for regular insurance is

$$\hat{\mu}_\eta^{EMP} = \int_0^\infty [1 - F_n(x)]^\eta dx = \sum_{i=1}^{n} \left[\left(\frac{n+1-i}{n} \right)^\eta - \left(\frac{n-i}{n} \right)^\eta \right] X_{i:n}$$

and for excess-of-loss reinsurance with retention a, when $a \leq X_{n:n}$, is

$$\begin{aligned}\hat{\mu}_{\eta,a}^{EMP} &= \int_a^\infty [1 - F_n(x)]^\eta dx = \int_{X_{k^*:n}}^\infty [1 - F_n(x)]^\eta dx + \int_a^{X_{k^*:n}} [1 - F_n(x)]^\eta dx \\ &= \sum_{i=k^*}^{n} \left[\left(\frac{n+1-i}{n} \right)^\eta - \left(\frac{n-i}{n} \right)^\eta \right] X_{i:n} - \left(1 - \frac{k^*-1}{n} \right)^\eta a\end{aligned}$$

where $k^* = \min\{k | X_{k:n} >= a\}$ and F_n is empirical distribution function of losses.

When $a > X_{n:n}$, we have no observations exceeding a.[1] In this case, empirical estimation may not be appropriate since empirical distribution function contains no information beyond the range of data. Nevertheless, one possible solution is simply to take estimator with value 0.

[1] Fortunately, this situation does not occur in our simulation outcomes.

3.1 Asymptotic Normality of Empirical Estimators

The empirical estimator of PH-premium for regular insurance proposed above is an L-statistic. Many authors have explored the limiting behavior under various conditions. However, to the best of this author's knowledge, no general statement is suggested in literatures. A recent discovery by Jones and Zitikis (2003) says following

Fact 1. *(Jones and Zitikis, 2003, Theorem 3.2). For any $1 > \eta > 1/2$, we have*

$$\sqrt{n}(\hat{\mu}_\eta^{EMP} - H_\eta[X]) \xrightarrow{d} N(0, \sigma_\eta^2)$$

where

$$\sigma_\eta^2 = \eta^2 \int_0^1 \int_0^1 [\min(s,t) - st] s^{\eta-1} t^{\eta-1} dQ(1-s) dQ(1-t) < \infty$$

provided that $E(|X|^r) < \infty$ for some $r > 2/(2\eta - 1)$.

Here, $Q(\cdot)$ is the generalized quantile function of loss distribution function $F(\cdot)$ defined by $Q(s) := \inf\{x \geqslant 0 : F(x) \geqslant s\}$ for all $0 \leq s \leq 1$.

The consistency of the extreme estimators then immediately follows from their asymptotic normality under corresponding conditions.

Note that the cases that $\eta \leq 1/2$ is not covered in the above proposition. Jones and Zitikis (2003) conclude that it is still an open question for these cases whether the asymptotic normality in the above theorem holds. In this thesis, we give a partial answer to this question. Please refer to Appendix for details.

We do not have findings about the asymptotic normality of the empirical estimator for excess-of-loss reinsurance. This may be a future adventure for future study. For the performance of empirical estimator of PH-premium for excess-of-loss reinsurance, please refer to the simulation study in Chapter 5.

Chapter 4

Semiparametric Approach: Extremes Estimator

To develop the estimation procedure for the semiparametric extreme estimator, we first give an introduction to Extreme Value Theory. We start from the asymptotic distribution of the maximum of claims data. If the maximum of claims data drawn from a particular distribution function F have a continuous asymptotic distribution, then it can be taken from a class of distributions determined by a single parameter γ. The γ is uniquely determined by F and named extreme value index. [1] Generally speaking, The extreme value index measures the tail heaviness: the larger γ, the heavier tail F has. When $\gamma > 0$, then loss X is heavy-tailed. In this Chapter we only consider the losses with positive extreme value index. Below lemma give an convenient way to compute $\gamma > 0$ of a given loss distribution F.

Lemma 1. *(de Haan and Ferreira 2006)Given a loss distribution $F(u) = Pr\{X \leq u\}$, it has an extreme value index $\gamma > 0$ if and only if*

$$\lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\frac{1}{\gamma}}, x > 0$$

However, if we do not know the loss distribution F, we cannot apply above lemma. Instead, we estimate $\gamma > 0$ by Hill estimator, that is, given i.i.d claims data $X_1, ..., X_n$ and a positive integer k,

$$\hat{\gamma}_k = \frac{1}{k} \sum_{i=0}^{k-1} \log(X_{n-i:n}) - \log(X_{n-k:n})$$

4.1 Extreme Estimators

In this section, we describe how we construct the extreme estimators.

From Lemma 1, we know $1 - F(tx) \approx (1 - F(t))x^{-\frac{1}{\gamma}}$ for large t.

[1] However,the reverse statement is not true, i.e., different distribution functions may give exactly the same extreme value index, for example, both Normal and Gamma distributions correspond to $\gamma = 0$.

Take $a = tx$ and $t = F^{-1}(1-\frac{k}{n})$, for large a

$$1-F(a) \approx \left(1-\frac{n-k}{n}\right)\left(\frac{a}{F^{-1}(1-\frac{k}{n})}\right)^{-\frac{1}{\gamma}} = \frac{k}{n}\left(\frac{F^{-1}(1-\frac{k}{n})}{a}\right)^{\frac{1}{\gamma}} \approx \frac{k}{n}\left(\frac{X_{n-k:n}}{a}\right)^{\frac{1}{\gamma}} \approx \frac{k}{n}\left(\frac{X_{n-k:n}}{a}\right)^{\frac{1}{\hat{\gamma}_k}}$$

Then, if $1 \geq \eta > \gamma$, we have

$$\begin{aligned}
&\textstyle\int_0^\infty [1-F(x)]^\eta dx \\
&= \textstyle\sum_{i=0}^{n-k-1}\int_{X_{i:n}}^{X_{i+1:n}}[1-F(x)]^\eta dx + \int_{X_{n-k:n}}^\infty [1-F(x)]^\eta dx \\
&\approx \textstyle\sum_{i=0}^{n-k-1}\int_{X_{i:n}}^{X_{i+1:n}}[1-F_n(x)]^\eta dx + \int_{X_{n-k:n}}^\infty \left[\frac{k}{n}\left(\frac{X_{n-k:n}}{x}\right)^{\frac{1}{\gamma}}\right]^\eta dx \\
&= \textstyle\sum_{i=0}^{n-k-1}\left(1-\frac{i}{n}\right)^\eta (X_{i+1:n}-X_{i:n}) + \left(\frac{k}{n}\right)^\eta X_{n-k:n}{}^{\frac{\eta}{\gamma}}\int_{X_{n-k:n}}^\infty x^{-\frac{\eta}{\gamma}}dx \\
&= \textstyle\frac{1}{n^\eta}\sum_{i=1}^{n-k-1}X_{i:n}[(n-i+1)]^\eta-(n-i)]^\eta] + \frac{1}{n^\eta}X_{n-k:n}(k+1)^\eta + \left(\frac{k}{n}\right)^\eta X_{n-k:n}{}^{\frac{\eta}{\gamma}}X_{n-k:n}{}^{1-\frac{\eta}{\gamma}}\frac{\gamma}{\eta-\gamma} \\
&= \textstyle\frac{1}{n^\eta}\sum_{i=1}^{n-k-1}X_{i:n}[(n-i+1)]^\eta-(n-i)]^\eta] + \frac{1}{n^\eta}X_{n-k:n}[(k+1)^\eta + k^\eta\frac{\gamma}{\eta-\gamma}] \\
&\approx \textstyle\frac{1}{n^\eta}\sum_{i=1}^{n-k-1}X_{i:n}[(n-i+1)]^\eta-(n-i)]^\eta] + \frac{1}{n^\eta}X_{n-k:n}[(k+1)^\eta + k^\eta\frac{\hat{\gamma}_k}{\eta-\hat{\gamma}_k}] \\
&= \textstyle\frac{1}{n^\eta}\sum\limits_{i=1}^{n-k}X_{i:n}[(n-i+1)^\eta-(n-i)^\eta] + \left(\frac{k}{n}\right)^\eta X_{n-k:n}\frac{\eta}{\eta-\hat{\gamma}_k}
\end{aligned}$$

Similarly,

$$\int_a^\infty [1-F(x)]^\eta dx \approx \int_a^\infty \left[\frac{k}{n}\left(\frac{X_{n-k:n}}{x}\right)^{\frac{1}{\gamma}}\right]^\eta dx = \left(\frac{k}{n}\right)^\eta X_{n-k:n}^{\frac{\eta}{\gamma}}\int_a^\infty x^{-\frac{\eta}{\gamma}}dx$$
$$= \left(\frac{k}{n}\right)^\eta X_{n-k:n}^{\frac{\eta}{\gamma}}\left[0-\left(\frac{a^{-\frac{\eta}{\gamma}+1}}{-\frac{\eta}{\gamma}+1}\right)\right] = \left(\frac{k}{n}\right)^\eta X_{n-k:n}^{\frac{\eta}{\gamma}}a^{-\frac{\eta}{\gamma}+1}\frac{\gamma}{\eta-\gamma} \approx \left(\frac{k}{n}\right)^\eta X_{n-k:n}^{\frac{\eta}{\hat{\gamma}_k}}a^{-\frac{\eta}{\hat{\gamma}_k}+1}\frac{\hat{\gamma}_k}{\eta-\hat{\gamma}_k}$$

We summarize the extreme estimators as following.

Definition 4. *Given i.i.d. claims data $X_1, ..., X_n$ and a positive integer k, suppose its distribution function is F with extreme value index $\gamma > 0$ and $\hat{\gamma}_k < \eta$, for all η such that $1 \geq \eta > \gamma$ we define our EVT estimator [2] for $\int_0^\infty [1-F(x)]^\eta dx$ as*

$$\hat{\mu}_\eta^{EVT} = \frac{1}{n^\eta}\sum_{i=1}^{n-k}X_{i:n}[(n-i+1)^\eta-(n-i)^\eta] + \left(\frac{k}{n}\right)^\eta X_{n-k:n}\frac{\eta}{\eta-\hat{\gamma}_k}$$

The EVT estimator for $\int_a^\infty [1-F(x)]^\eta dx$ is

$$\hat{\mu}_{\eta,a}^{EVT} = \left(\frac{k}{n}\right)^\eta X_{n-k:n}^{\frac{\eta}{\hat{\gamma}_k}}a^{-\frac{\eta}{\hat{\gamma}_k}+1}\frac{\hat{\gamma}_k}{\eta-\hat{\gamma}_k}$$

where $X_{1:n}, ..., X_{n:n}$ is the increasing order statistics of $X_1, ..., X_n$ and $\hat{\gamma}_k$ is Hill estimator of X with integer k.

4.2 First-order and Second-order Conditions

We have to impose some conditions in order to apply Extreme Value Theory. It will become clear that the so-called extreme value condition is on the one hand quite general (it is not easy to find

[2] $\int_0^\infty [1-F(x)]^\eta dx = \infty$ if $\eta < \gamma$

distribution that do not satisfy them) but on the other hand is sufficiently precise to serve as a basis for extrapolation(de Haan and Ferreira 2006).

In the beginning of this chapter, we already see loss distribution F, with extreme value index $\gamma > 0$, has to satisfy the condition in Lemma 1. This is the so called 'first-order condition' and it is widely used to determine the value of extreme value index. Besides, a 'second-order' condition is also often imposed. The 'second-order' condition is satisfied if there exists a positive or negative function $a(\cdot)$ with $\lim_{t\to\infty} a(t) = 0$, such that

$$\lim_{t\to\infty} (a(t))^{-1} \left(\frac{1-F(tx)}{1-F(t)} - x^{-1/\gamma} \right) = x^{-1/\gamma} \frac{x^{\rho/\gamma}-1}{\rho/\gamma} \tag{4.1}$$

or equivalent

$$\lim_{s\downarrow 0}(A(s)^{-1}) \left(\frac{Q(1-sx)}{Q(1-s)} - x^{-\gamma} \right) = x^{-\gamma} \frac{x^{\rho}-1}{\rho} \tag{4.2}$$

for all $x > 0$ with second-order index $\rho \le 0$. Here, $Q(\cdot)$ is the quantile function defined in Chapter 3 and $A(s) := \gamma^2 a(Q(1-s))$. When $\rho = 0$, interpret both $\frac{x^{\rho/\gamma}-1}{\rho/\gamma}$ and $\frac{x^{\rho}-1}{\rho}$ as $log(x)$. This condition is often used to evaluate the converge rate of optimal choice of k for Hill estimator. Generally speaking, the smaller the ρ, the smaller the k should be chosen for Hill estimator. The asymptotic normality of Hill estimator follows from the first-order and second-order conditions.

Fact 2. *(de Haan and Ferreira 2006, Theorem 3.2.5) Given i.i.d. claims data $X_1, ..., X_n$ and a positive integer k, suppose its distribution function is F with extreme value index $\gamma > 0$ and F satisfies the second-order condition* (4.1) *(and* (4.2)*) with second order parameter $\rho \le 0$, i.e., Then*

$$\sqrt{k}(\hat{\gamma}_k - \gamma) \xrightarrow{d} N(0, \gamma^2)$$

with N being normal distribution, provided $k = k(n) \to \infty$,$k/n \to 0$,$n \to \infty$, and $\sqrt{k} A\left(\frac{k}{n}\right) \to 0$.

The consistency of Hill estimator immediately follows from its asymptotic normality.[3]

4.3 EVT-transform

Peng (2001) has shown the asymptotic normality of the extreme estimator for regular insurance when $\eta = 1$, i.e. the PH-premium equals to the expected value of loss. To generalize the Peng's result for all $1 \ge \eta > \gamma$, we first determine a transformation of loss X, say Y, with expectation equals the PH-premium with index η. We name the transform from X to Y as EVT-transform. Theorem 1 below states the exact expression of EVT-transforms and some of its properties .

[3] In fact, the consistency of Hill estimator still holds under a weaker condition. For details please check Theorem 3.2.4 in (de Haan and Ferreira 2006)

Theorem 1. *Given a nonnegative random variables X of distribution function F with extreme value index $\gamma > 0$ and an index $0 < \eta \leq 1$, EVT-transform with transform index $1 > \eta > 0$ for X is a a mapping $J(\cdot) : R \to R$ such that $J_\eta(x) = \eta[1 - F(x)]^{\eta-1}$. Note that J_η is well defined because here $F(x) < 1$ for all $x > 0$ (de Haan and Ferreira 2006). Write*

$$Y = J_\eta(X) = \frac{\eta X}{[1 - F(X)]^{1-\eta}}$$

EVT-transform satisfies following properties:

1. $J_\eta(x)$ is strictly increasing therefore is invertible. 2. $E(Y) = E(J_\eta(X)) = H_\eta[X] = \int_0^\infty [1 - F(x)]^\eta dx$

3. Y has a heavy tail with extreme value index $\gamma^Y = \gamma + 1 - \eta$

4. $E(Y) < \infty$ if $1 \geq \eta > \gamma$ and $E(Y) = \infty$ if $0 < \eta < \gamma$

5. $y_p = (1 - p)^{\eta-1}\eta x_p$ where y_p and x_p are the p-quantile of X and Y.

6. $\int_a^\infty [1 - F(x)]^\eta dx = E([J(X) - J_\eta(a)]^+) - (1 - \eta)a[1 - F(a)]^\eta$

Unfortunately, EVT-transform depends on the distribution function of loss X. This means we cannot perceive the value of Y from the data thereby cannot apply Peng's result on Y directly. A solution is that we use the estimations of Y instead of the true values. We can establish estimations of Y base on estimation of the distribution function. With a suitable choice of the estimation of the distribution function, the extreme estimator for expectation of Y is just the extreme estimators of PH-premium of X for regular insurance.

Furthermore, Peng (2001) points out that his extreme estimator shows distinct asymptotic behaviors between circumstances $\gamma > 1/2$ and $\gamma \leq 1/2$. Analogously, we also shows that the (generalized) extreme estimators also shows different asymptotic behaviors between cases $\gamma^Y > 1/2$ and $\gamma^Y \leq 1/2$, in Theorem 2.

4.4 Asymptotic Normality and Consistency

In previous sections, we have introduced the extreme value theory and EVT-transform. Although EVT-transform is not used in the rest of this thesis, it helps a lot to motivate following theorem and clarify the necessary conditions. The main results for this chapter are given by the following theorems in which we establish the asymptotic normality of extreme estimators. Theorem 2 is a generalization of the statements made by Peng (2001) and Necir and Meraghni (2009). In the previous section, we already point out that the findings from (Peng 2001) can only apply to a particular case $\eta = 1$. The work of Necir and Meraghni (2009) is much more general but still limited in the cases $\gamma > 1/2$. Here, we make a significant contribution to this field by providing a general theorem that apply to all η and γ such that $1 \geq \eta > \gamma$.

Theorem 2. *Suppose F is a distribution function of loss X satisfying both first-order condition and second-order condition ((4.1) and (4.2)) with extreme value index $1 > \gamma > 0$. Assume $Q(\cdot)$ is continuously differentiable on [0,1). Take $Y = J_\eta(X)$ with $\gamma^Y = \gamma + 1 - \eta$. If $k = k_n$ is such that $k \to \infty$,$k/n \to 0$ and $k^{\frac{1}{2}} A(k/n) \to 0$ as $n \to \infty$, then for any $1 \geq \eta > \gamma$ we have*

$$\frac{\sqrt{n}}{\sigma_\eta(k/n)} \left(\hat{\mu}_\eta^{EVT} - H_\eta[X]\right) \xrightarrow{d} N(0, \sigma^2_{\eta,\gamma})$$

where

$$\sigma^2_{\eta,\gamma} = \begin{cases} 1 & \text{if } \gamma^Y = \gamma + 1 - \eta \in (0, 1/2] \\ \frac{(\gamma+1-\eta)(\gamma+1/2-\eta)[(\eta-\gamma)^2+1]}{(\eta-\gamma)^4} + \frac{[4(\eta-\gamma)-1]}{2(\eta-\gamma)} & \text{if } \gamma^Y = \gamma + 1 - \eta \in [1/2, 1) \end{cases}$$

and

$$\sigma^2_\eta(x) = \eta^2 \int_x^1 \int_x^1 [\min(s,t) - st] s^{\eta-1} t^{\eta-1} dQ(1-s) dQ(1-t) \ , \ for \ 0 \leqslant x \leqslant 1$$

Remark 1. *If F satisfies, as $x \to \infty$,*

$$1 - F(x) = c_1 x^{-1/\gamma} + c_2 x^{-1/\gamma + \rho/\gamma}(1 + o(1))$$

for constant $c_1 > 0$,$c_2 \neq 0$,$\gamma > 0$ and $\rho < 0$.
Then the condition $\sqrt{k}A(k/n) \to 0$ is equivalent to $k(n) = o(n^{-2\rho/(1-2\rho)})$.[4]

Remark 2. *From the proof of theorem 3, we get, under the assumptions in theorem 2, moreover, if $\gamma^Y = \gamma + 1 - \eta < 1/2$,*

$$\sqrt{n}\left(\hat{\mu}_\eta^{EVT} - \hat{\mu}_\eta^{EMP}\right) \xrightarrow{d} 0$$

This implies that,for regular insurance, empirical estimator and extreme estimator of the PH-premium shows same asymptotic behavior when $\gamma^Y = \gamma + 1 - \eta < 1/2$. We do not have finding for cases of $\gamma^Y = \gamma + 1 - \eta \geq 1/2$. However, the simulation outcomes in Chapter 5 suggests that under this circumstance the extreme estimator often significantly outperform the empirical estimator.

Though Theorem 2 applies to most general cases, still several possible improvement can be made in the future study. For example, the condition about continuously differentiability of quantile function Q may be eliminated. Necir, Rassoul and Zitikis (2010) use Vervaat process to avoid this requirement for the case $\eta = 1$ and $1 > \gamma > 1/2$.

Next, we provide Theorem 3 for theorectical completeness on the asymptotic normality of extreme estimator for excess-of-loss reinsurance.

Theorem 3. *Suppose F is a distribution function of loss X satisfying (4.1) and (4.2) with extreme value index $1 > \gamma > 0$ and $Q(\cdot)$ is continuously differentiable on [0,1). Let $Y = J_\eta(X)$ with*

[4] See Page 76-77 in de Haan and Ferreira (2006)

$\gamma^Y = \gamma + 1 - \eta$. If $k = k_n$ is such that $k \to \infty, k/n \to 0$ and $k^{\frac{1}{2}} A(k/n) \to 0$ as $n \to \infty$. For any $1 \geq \eta > \gamma$ we assume function $G(\cdot) : [0, \infty) \mapsto (0, H_\eta[X]]$ such that $G(x) = \int_x^\infty [1 - F(t)]^\eta dt$ satisfies the second-order condition with second-order parameter ρ'.[5] Write $a_n = c_n Q(1 - k/n)$ and let $\lim_{n\to\infty} c_n = 1$. If provided $k^{\frac{1}{2}} A_0(Q(1 - k/n)) \to 0$, we have

$$\frac{\sqrt{k}}{(k/n)^\eta Q(1 - k/n)} \left\{ \hat{\mu}^{EVT}_{\eta, c_n X_{n-k:n}} - H_{\eta, a_n}[X] \right\} \xrightarrow{d} \mathrm{N}(0, \frac{\gamma^4}{(\eta - \gamma)^2})$$

where

$$A_0(t) = \begin{cases} \rho' \left[1 - \lim_{s\to\infty} s^{\eta/\gamma - 1} G(s) / (t^{\eta/\gamma - 1} G(t)) \right] & , \rho' < 0 \\ 1 - \int_0^t s^{\eta/\gamma - 1} G(s) ds / (t^{\eta/\gamma} G(t)) & , \rho' = 0 \end{cases}$$

To be consistent with the formulation of Theorem 2, we develop following corollary.

Corollary 1.

$$\frac{\sqrt{n}}{\sigma_\eta^2(k/n)} \left\{ \hat{\mu}^{EVT}_{\eta, c_n X_{n-k:n}} - H_{\eta, a_n}[X] \right\} \xrightarrow{d} \begin{cases} 0 & if\ \gamma^Y = \gamma + 1 - \eta \leqslant 1/2 \\ N(0, \tilde{\sigma}^2_{\eta,\gamma}) & if\ \gamma^Y = \gamma + 1 - \eta > 1/2 \end{cases}$$

where

$$\tilde{\sigma}^2_{\eta,\gamma} = \frac{\gamma^2 (\gamma + 1 - \eta)(\gamma + 1/2 - \eta)}{\eta^2 (\eta - \gamma)^2}$$

The consistency of the extreme estimators then immediately follows from their asymptotic normality under corresponding conditions.

[5] The first-order condition can be easily verified by applying Karamata's Theorem (de Haan and Ferreira 2006) and Lemma 1.

Chapter 5

Simulation Study

In this chapter, we carry out a simulation study to compare the performance of different estimation approaches for heavy-tailed losses. We take the distortion index $\eta = 0.8$ for computation of PH-premium. Our sample size is N=1000 and , for each distribution, we simulate in total S=100 scenarios. For the actual loss distribution, we have chosen Fréchet distribution and Burr distribution. For comparison with parametric estimators, we fit a wide range of distribution, including the true distribution and many other artificial distributions, to my observations. Table 5.1 shows the details of the setting of distribution used in the simulation study.

Table 5.1: Distributions used in simulation study

	Distribution	γ	ρ
$\gamma > 0$	• Fréchet $F(x) = exp(-x^{-1/\gamma})$, $x > 0$, with $\gamma \in$ {0.6,0.2}	γ	-1
	• Burr $F(x) = 1 - (x^{-\rho/\gamma} + 1)^{1/\rho}$, $x \geq 0$, with $(\gamma, \rho) \in$ {(0.6,-1),(0.2,-2)}	γ	ρ
	• GPD with shape γ and scale σ=1	γ	$-\infty$
$\gamma = 0$	• Exponential distribution	0	$-\infty$
	• Gamma distribution	0	
	• Weibull distribution	0	-1
	• Lognormal distribution	0	0

The simulation study consists of two parallel parts, one for regular insurance and the other for excess-of-loss reinsurance. In each part, we examine the optimal choice of k for Hill estimators and compare the extreme estimators with other estimators. For (high) retention levels of excess-of-loss reinsurance, we use the 0.9 and 0.99 quantiles of actual loss distribution.

5.1 Regular Insurance

5.1.1 The Choice of k

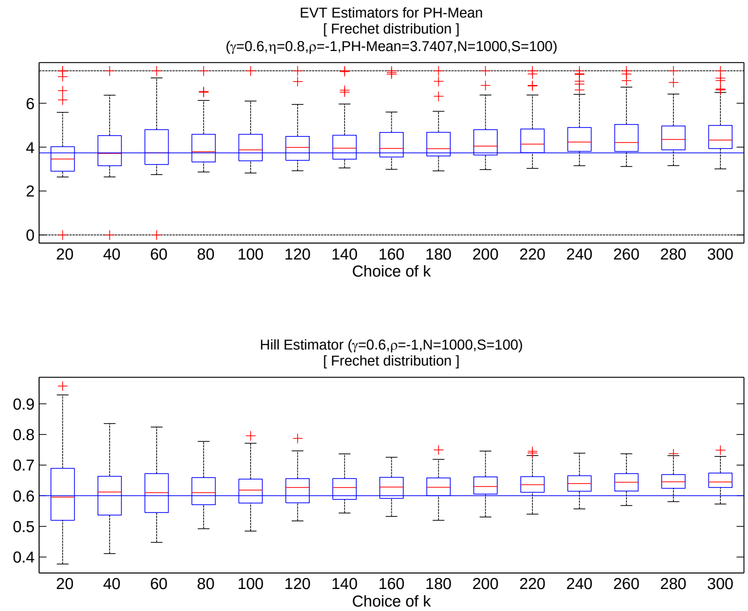


Figure 5.1: Fréchet Distribution,γ=0.6

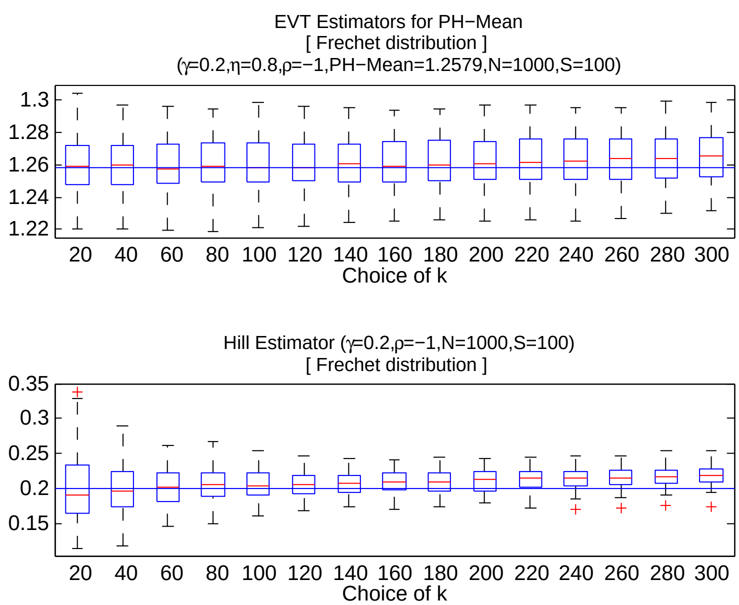


Figure 5.2: Fréchet Distribution,γ=0.2

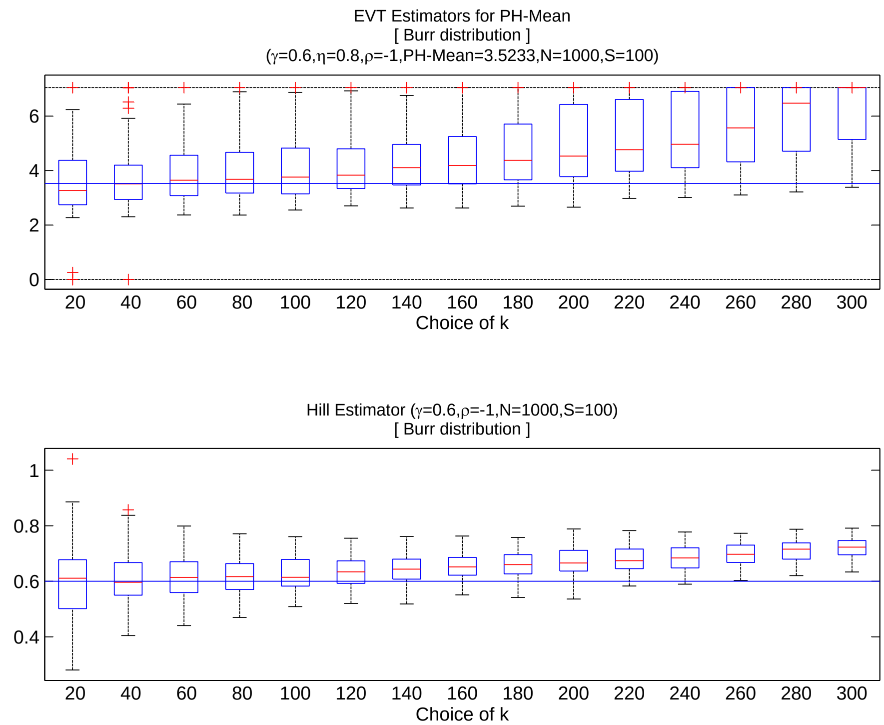


Figure 5.3: Burr Distribution,γ=0.6,$\rho = -1$

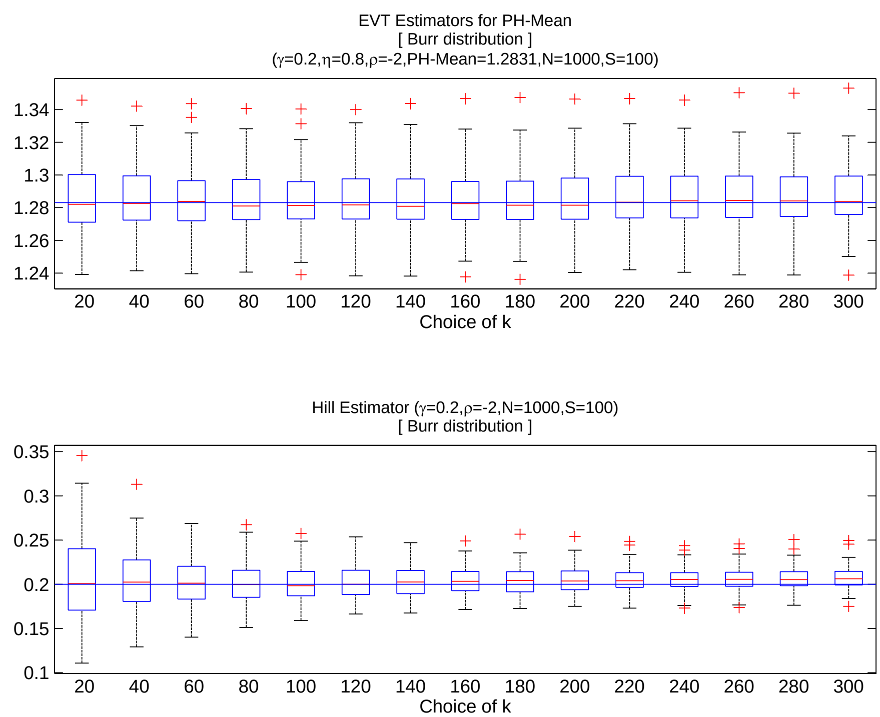


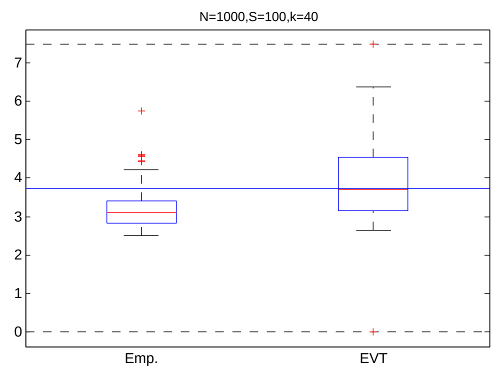
Figure 5.4: Burr Distribution,γ=0.2,$\rho = -2$

Generally speaking, the second-order coefficient $\rho \leq 0$ controls the convergent rate of k. When ρ is getting smaller, we can take a larger portion of sample size as k, which can be clear seem in figure

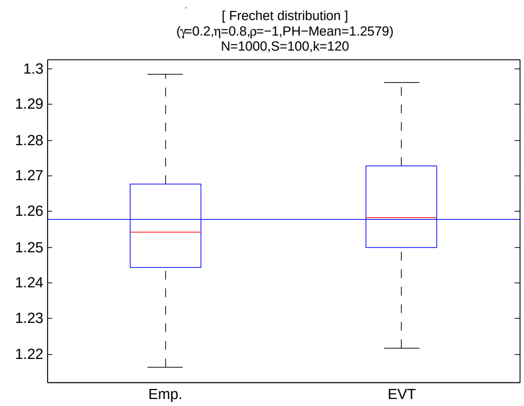
4.4. In figure 4.4 we have a relative high ρ and we observe that the performance of the extreme estimators are very stable over a relative large range of k. When ρ is relative large, the performance of our extremes estimator seems to depend on extreme value index γ of losses. For larger γ, the optimal choice of k tend to be smaller.

5.1.2 Empirical vs EVT Estimators

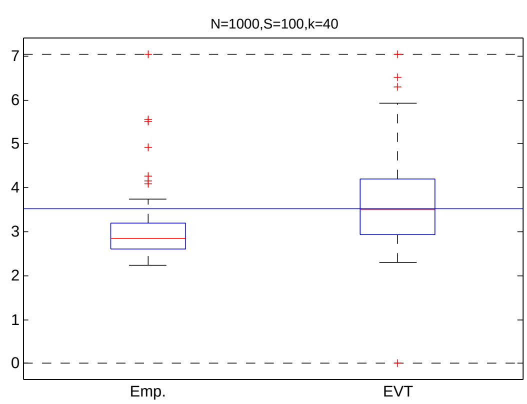
Figure 5.5: PH-Mean: Empirical vs EVT Estimators



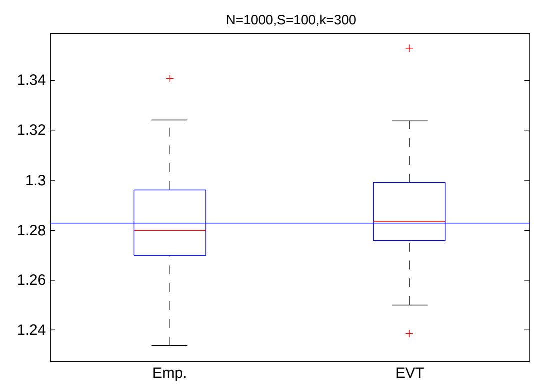
(a) Fréchet: γ=0.6



(b) Fréchet: γ=0.2



(c) Burr: (γ, ρ)=(0.6,-1)

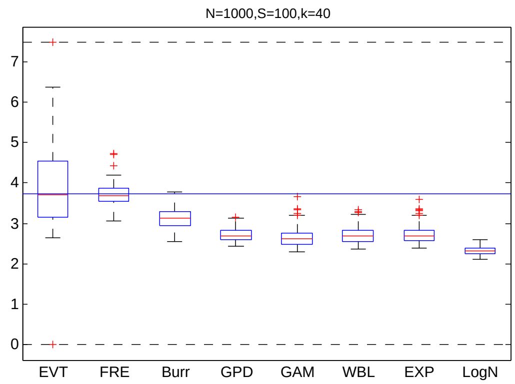


(d) Burr: (γ, ρ)=(0.2,-2)

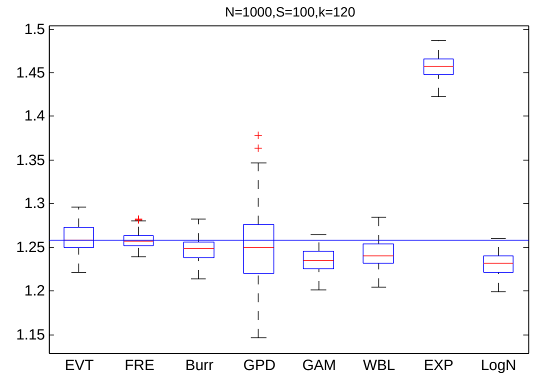
A clear message delivered by these four figures is that: empirical estimators may result in a terrifically poor estimation when we are dealing with heavy-tailed losses. On the other hand, the extremes estimator shows robust performance over different heaviness of losses. While the tail of losses is not so heavy, both empirical and extremes estimator has similar behavior, which is in line with Remark 2 in Theorem 2.

5.1.3 Parametric vs Semi-parametric Approach

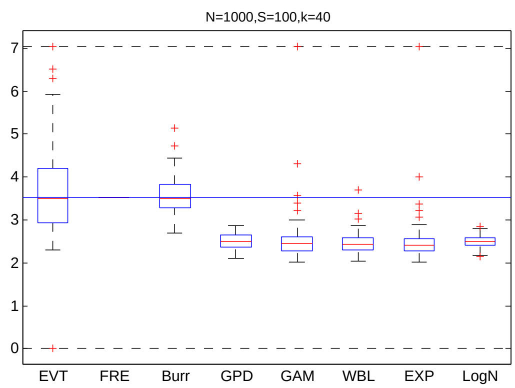
Figure 5.6: Parametric vs Semi-parametric Methods: PH-Mean



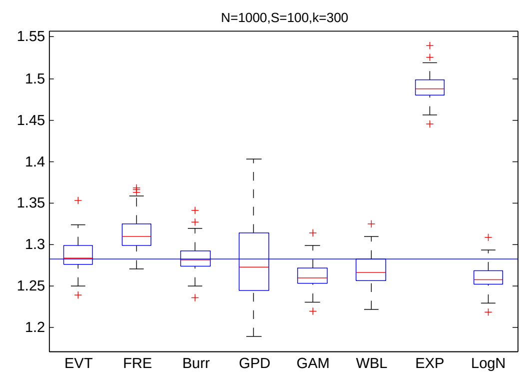
(a) Fréchet: γ=0.6



(b) Fréchet: γ=0.2



(c) Burr: (γ, ρ)=(0.6,-1)



(d) Burr: (γ, ρ)=(0.2,-2)

In this section, we compare the extremes estimators with parametric estimators based on a variety of models. Parametric always shows the best performance but only if we have the proper identification of the loss distribution. Losses with positive extreme value index can hardly be approximated by other distributions with zero or even positive extreme value index. Nevertheless, in certain circumstances it is still possible to approximate by another similar distribution. For example, in the graphs on the right side, generalize pareto distribution may also be a suitable model for estimating PH-premiums. The boxplot of Frechet distribution in left downside figure is missing because the maximum likelihood estimators cannot be obtained properly.

5.2 Excess-of-Loss Reinsurance

5.2.1 The Choice of k

Figure 5.7: Fréchet Distribution:γ=0.6

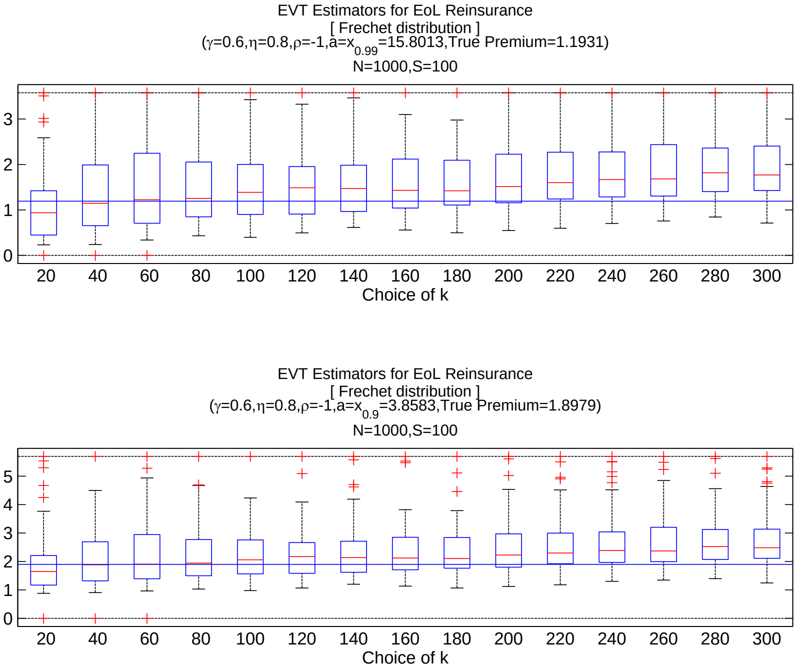


Figure 5.8: Fréchet Distribution:γ=0.2

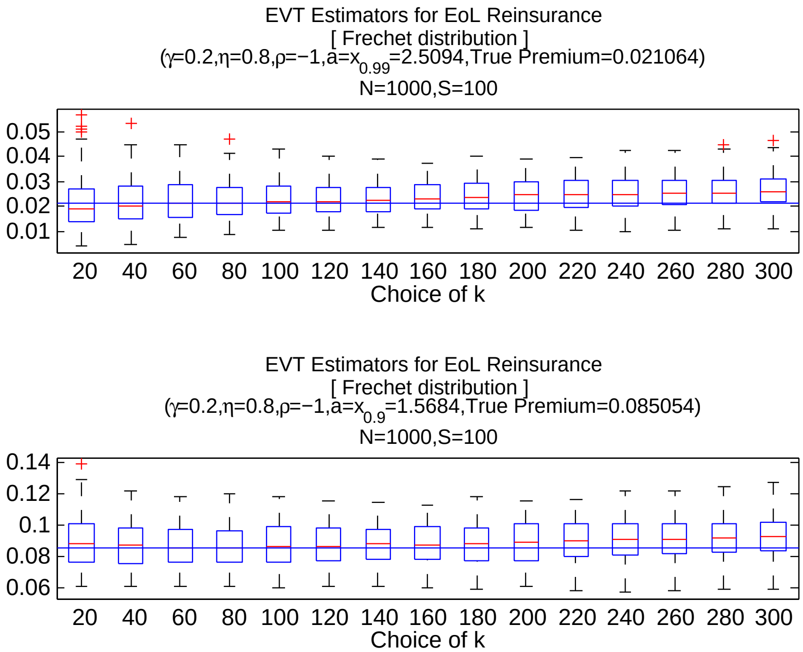


Figure 5.9: Burr Distribution:(γ, ρ)=(0.6,-1)

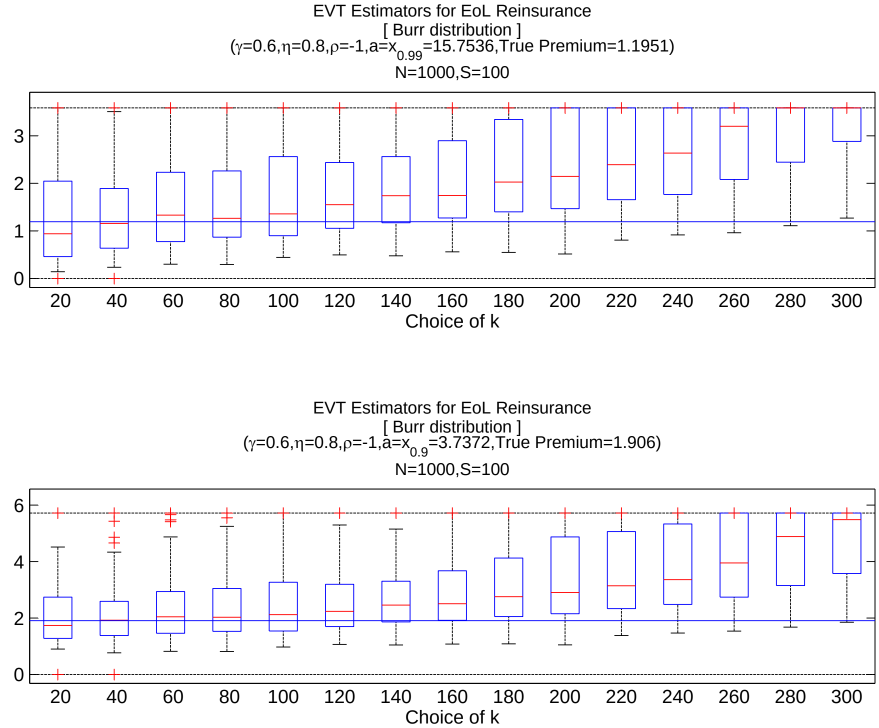
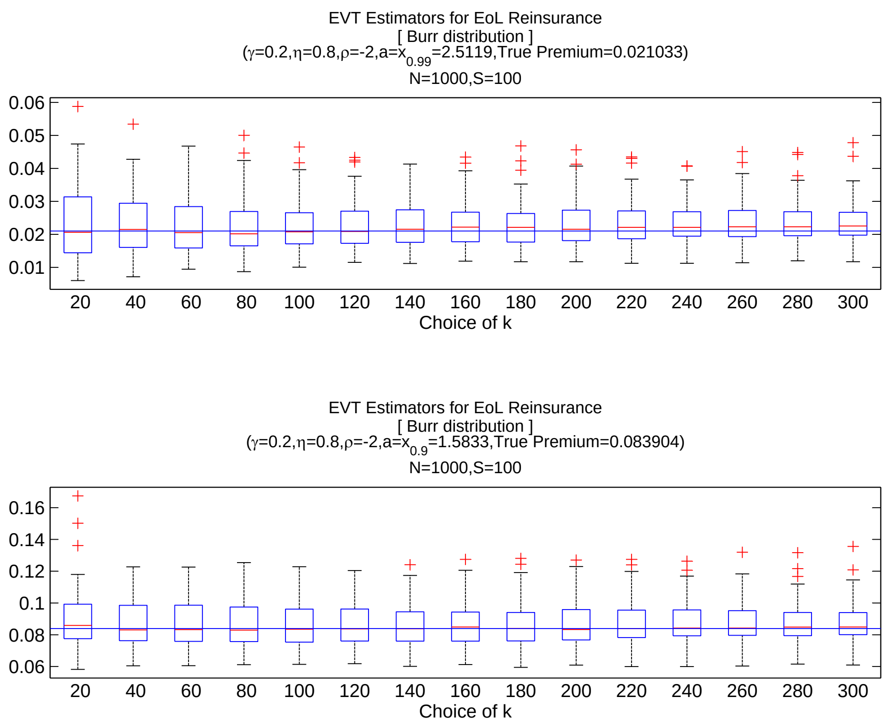


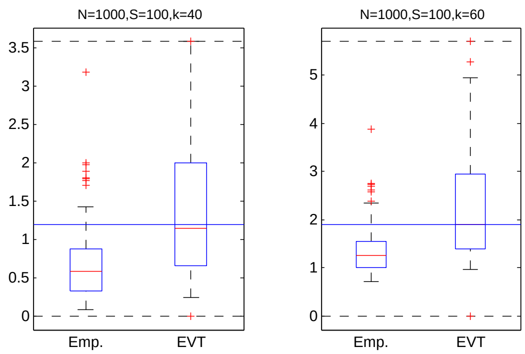
Figure 5.10: Burr Distribution:(γ, ρ)=(0.2,-2)



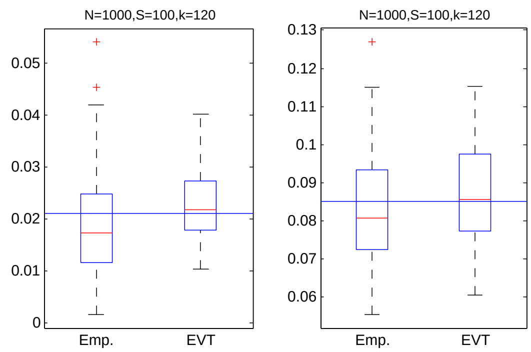
Similar observations as in section 5.2.1 are found. Accordingly we have a similar interpretation for the choice of k in case of pricing Excess-of-Loss Reinsurance.

5.2.2 Empirical vs EVT Estimators

Figure 5.11: Excess-of-Loss Reinsurance: Empirical vs EVT Estimators

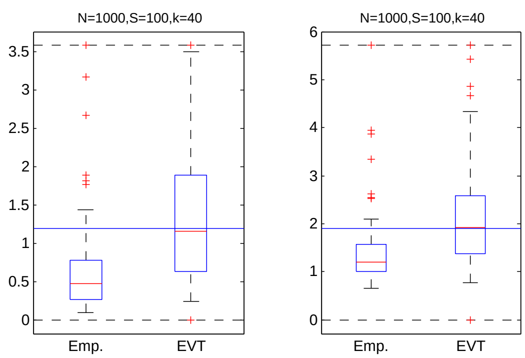


(a) Fréchet :γ=0.6

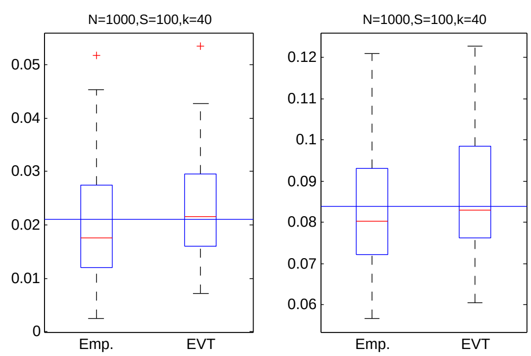


(b) Fréchet :γ=0.2

Figure 5.12: Excess-of-Loss Reinsurance: Empirical vs EVT Estimators



(a) Burr:(γ, ρ)=(0.6,-1)

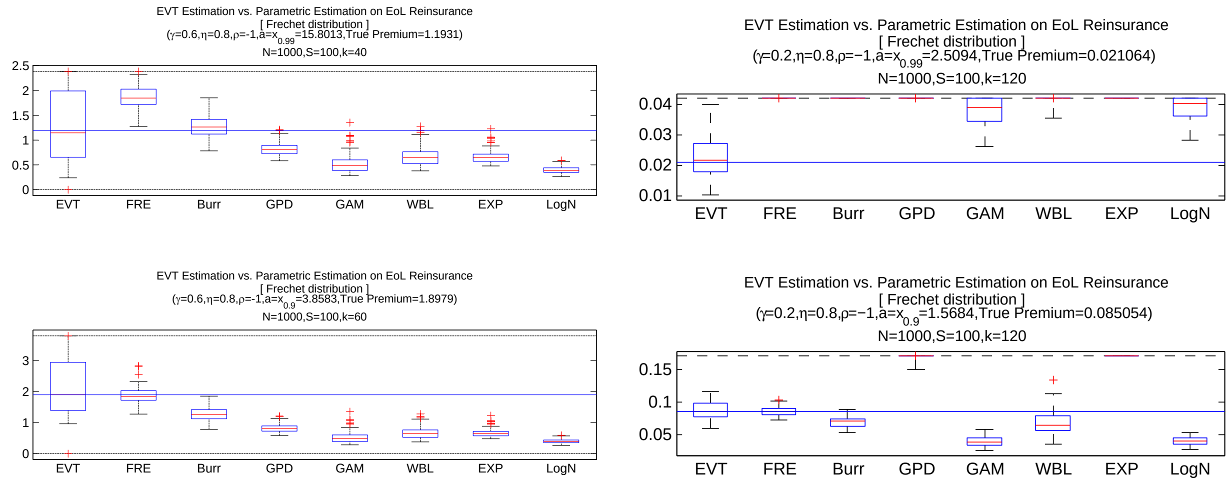


(b) Burr:(γ, ρ)=(0.2,-2)

Similar results as those in section 5.1.2 are observed. In case of losses with strong heaviness, our extremes estimator obviously outperforms to empirical ones in my simulation. It may due to that empirical estimators fail to recognize the asymptotic behavior of large losses. When Y lies in the sum domain of attraction of normal distribution, i.e. when $\gamma + 1 - \eta < 1/2$ both estimators show comparable behaviors.

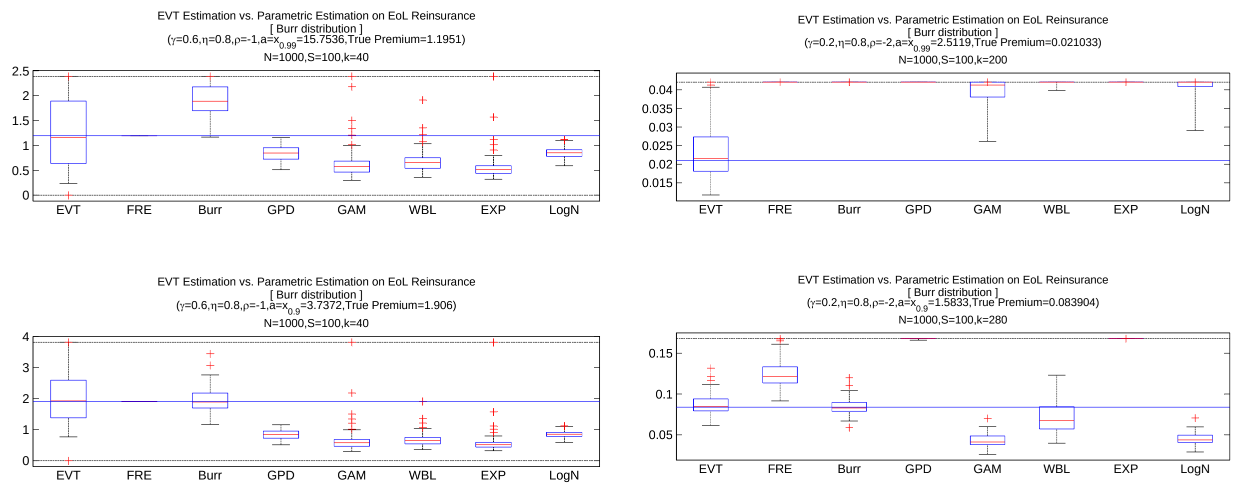
5.2.3 Parametric and Semi-parametric Approach

Figure 5.13: Excess-of-Loss Reinsurance: $a = x_{0.99}$ vs $a = x_{0.9}$



(a) Fréchet: γ=0.6 (b) Fréchet: γ=0.2

Figure 5.14: Excess-of-Loss Reinsurance: $a = x_{0.99}$ vs $a = x_{0.9}$



(a) Burr: (γ, ρ)=(0.6,-1) (b) Burr: (γ, ρ)=(0.2,-2)

Extremes estimators always provide reliable estimates in these scenarios. A critical remark is that even if we have correctly identify the loss distribution, the parametric estimators still shows poor performance when the retention level is high.high. This is because in above cases we have $\lim_{a\to\infty} \frac{d \log[1-F_\theta(a)]}{d\theta} = \infty$, which means the relative error asymptotic variance for the MLEs is very large if retention level is high.[1] This problem is particularly serious if the heaviness of losses is relatively low. The boxplots of Fréchet distribution are missing in the some figures of Burr

[1]Please refer to the end of Chapter 2 for detailed explanation.

distribution because the maximum likelihood estimator of parameters had not been properly found.

Chapter 6

Conclusion

In this thesis, we discuss three different types of estimators of the proportional hazards premium for regular insurance and excess-of-loss reinsurance: maximum likelihood, empirical and extreme estimators. Particularly, we investigate the asymptotic behavior and consistency of all these estimation procedures. A simulation study is carried out to compare the their performance within a limited sample size.

The simulation conclusion is consistent with the theoretic findings. It suggests that extremes estimators always shows promising performance under a variety of circumstances in simulation. The advantage of the extremes estimators is particularly significant if the heaviness of loss distribution is strong, or if we want to price an excess-of-loss reinsurance with high retention level. When $\gamma+1-\eta$ is large, the empirical estimators fail to produce satisfactory results. When $\gamma+1-\eta$ is small, the empirical estimators and extreme estimators are similar. No surprise that parametric methods usually give best estimations if we have suitable identification of loss distribution. However, if we are pricing excess-of-loss insurance with high retention level, the performance of parametric methods can still be poor and much worse than extreme estimators, even though we have correct identification of loss distribution. Moreover, the simulation analysis in Chapter 4 clearly suggests that, if misidentification error is present, we can hardly obtain satisfactory results by MLEs.

The asymptotic normality of the extreme estimators is the highlight of this thesis. We have made a strong generalization of the statements made by Peng (2001) and Necir and Meraghni (2009). The statement from Peng (2001) can only apply to a particular case $\eta = 1$. The work of Necir and Meraghni (2009) is much more general but still limited in the cases $\gamma > 1/2$. Here, we make a significant contribution to this field by providing a general theorem that apply to all η and γ such that $1 \geq \eta > \gamma$. Still, many topics are possible for future study. For example, the condition about continuously differentiability of quantile function Q in Theorem 2 may be eliminated. Moreover, the conditions in Theorem 3 about the convergence rate of k may be simplified since we do not detect transparent additional requirements for selection of k for excess-of-loss reinsurance, compared to regular insurance, in the simulation outcome. It is also an open question about the asymptotic behavior of the difference between the extremes estimators and empirical estimators if $\gamma^Y \geq 1/2$.

Chapter 7

Appendix

In this Appendix, we give the details of proofs for all the theorems mentioned above.

7.1 Proof of Theorem 1

Before we go the the proof of Theorem 1, we first introduce the definition of (first-order) regularly varying function and a lemma used in the proof.

Definition 5. *A function $h > 0$ is called regularly varying with index ρ, or called $\rho -$ varying, if*

$$\lim_{t\to\infty} \frac{h(tx)}{h(t)} = x^{\rho}, x > 0$$

Lemma 2. *(N.H.Bingham and J.L.Teugels 1975)Given a function $h > 0$ that is regularly varying with index $\rho > 0$,then its inverse h^{-1} ,if exists, is regularly varying with index ρ^{-1}.*

1. Let $x_1, x_2 \in [0, \infty)$ and $x_1 < x_2$. Then, if $1 \geq \eta > 0$

$$F(x_1) \leqslant F(x_2) < 1 \Rightarrow 1 - F(x_1) \geqslant 1 - F(x_2) > 0 \Rightarrow 0 < [1 - F(x_1)]^{\eta-1} \leqslant [1 - F(x_2)]^{\eta-1}$$

$$J_\eta(x_1) = \eta x_1[1 - F(x_1)]^{\eta-1} < \eta x_2[1 - F(x_1)]^{\eta-1} \leqslant \eta x_2[1 - F(x_2)]^{\eta-1} = J_\eta(x_2)$$

   This proves function $J_\eta(x)$ is strictly increasing therefore is invertible.

2.

$$E(Y) = E(J_\eta(X)) = \int_0^\infty J_\eta(x)dF(x) = \int_0^\infty x\eta[1 - F(x)]^{\eta-1}dF(x) = -\int_0^\infty xd[1 - F(x)]^\eta$$
$$= -\int_0^\infty \left(\int_0^x 1dt\right) d[1 - F(x)]^\eta = -\int_0^\infty \left(\int_t^\infty 1d[1 - F(x)]^\eta\right) dt = -\int_0^\infty (-[1 - F(t)]^\eta)dt$$
$$= \int_0^\infty ([1 - F(t)]^\eta)dt = H_\eta[X]$$

3.

$$\lim_{t\to\infty}\frac{J_\eta(tx)}{J_\eta(t)}=\lim_{t\to\infty}\left(\frac{\eta tx}{[1-F(tx)]^{1-\eta}}\cdot\frac{[1-F(t)]^{1-\eta}}{\eta t}\right)=\lim_{t\to\infty}\left(x\cdot\left(\frac{1-F(tx)}{1-F(t)}\right)^{\eta-1}\right)$$
$$=x\cdot\left(\lim_{t\to\infty}\frac{1-F(tx)}{1-F(t)}\right)^{\eta-1}=x^{1-\frac{\eta-1}{\gamma}}=x^{\frac{\gamma+1-\eta}{\gamma}}$$

It means function $J_\eta(x)$ is $\frac{\gamma+1-\eta}{\gamma}-varying$. From property 1 we know function $J_\eta(\cdot)$ is invertible. Denote its inverse be $K_\eta(x):[0,\infty)\to\mathbb{R}$. By lemma 2, we know function $K_\eta(x)$ is $\frac{\gamma}{\gamma+1-\eta}-varying$, i.e.,

$$\lim_{t\to\infty}\frac{K_\eta(tx)}{K_\eta(t)}=x^{\frac{\gamma}{\gamma+1-\eta}}$$

The distribution function of random variable Y is

$$F_Y(x)=\mathrm{P}[Y\leqslant x]=P[J_\eta(X)\leqslant x]=P[X\leqslant K_\eta(x)]=F(K_\eta(x))$$

Since function K_η is inverse J_η,

$$x=J_\eta(K_\eta(x))=\frac{\eta K_\eta(x)}{[1-F(K_\eta(x))]^{1-\eta}}\Rightarrow 1-F_Y(x)=1-F(K_\eta(x))=\left(\frac{\eta K_\eta(x)}{x}\right)^{\frac{1}{1-\eta}}$$

Subsequently,

$$\lim_{t\to\infty}\frac{1-F_Y(tx)}{1-F_Y(t)}=\lim_{t\to\infty}\left[\left(\frac{\eta K_\eta(tx)}{tx}\right)^{\frac{1}{1-\eta}}\left(\frac{t}{\eta K_\eta(t)}\right)^{\frac{1}{1-\eta}}\right]=\lim_{t\to\infty}\left(\frac{K_\eta(tx)}{K_\eta(t)}x^{-1}\right)^{\frac{1}{1-\eta}}$$
$$=\left(x^{-1}\cdot\lim_{t\to\infty}\frac{K_\eta(tx)}{K_\eta(t)}\right)^{\frac{1}{1-\eta}}=x^{\left(-1+\frac{\gamma}{\gamma+1-\eta}\right)\cdot\frac{1}{1-\eta}}=x^{-\frac{1}{\gamma+1-\eta}}$$

This proves Y is heavy-tailed with extreme value index $\gamma^Y=\gamma+1-\eta$ from lemma 1.

4. This property is immediately true by applying this fact (de Haan and Ferreira 2006): given a loss Y with extreme value index $\gamma+1-\eta>0$ then $E[Y^r]<\infty$ if $r>\frac{1}{\gamma+1-\eta}$ and $E[Y^r]=\infty$ if $r>\frac{1}{\gamma+1-\eta}$.

5. From above we know function J_η is strictly increasing,i.e. EVT-transform is monotonic, thereby obviously the p-quantile of random variable Y

$$y_p=J_\eta(x_p)=(1-p)^{\eta-1}\eta x_p$$

where x_p is the p-quantile of random variable X.

6.

$$
\begin{aligned}
&E[(J_\eta(X) - J_\eta(a))^+] \\
&= \int_a^\infty \left(\frac{\eta x}{[1-F(x)]^{1-\eta}} - \frac{\eta a}{[1-F(a)]^{1-\eta}} \right) dF(x) \\
&= -\int_a^\infty \left(x - \frac{a[1-F(x)]^{1-\eta}}{[1-F(a)]^{1-\eta}} \right) d[1-F(x)]^\eta \\
&= -\int_a^\infty \left(x - a - \frac{a}{[1-F(a)]^{1-\eta}}([1-F(x)]^{1-\eta} - [1-F(a)]^{1-\eta}) \right) d[1-F(x)]^\eta \\
&= -\int_a^\infty \left(\int_a^x \left\{ 1 - \frac{a}{[1-F(a)]^{1-\eta}} \frac{d[1-F(t)]^{1-\eta}}{dt} \right\} dt \right) d[1-F(x)]^\eta \\
&= -\int_a^\infty \left(\int_t^\infty \left\{ 1 - \frac{a}{[1-F(a)]^{1-\eta}} \frac{d[1-F(t)]^{1-\eta}}{dt} \right\} d[1-F(x)]^\eta \right) dt \\
&= -\int_a^\infty \left(\left\{ 1 - \frac{a}{[1-F(a)]^{1-\eta}} \frac{d[1-F(t)]^{1-\eta}}{dt} \right\} (-[1-F(t)]^\eta) \right) dt \\
&= \int_a^\infty [1-F(t)]^\eta dt - \frac{a(1-\eta)}{[1-F(a)]^{1-\eta}} \int_a^\infty \frac{[1-F(t)]^\eta}{1-\eta} d[1-F(t)]^{1-\eta} \\
&= \int_a^\infty [1-F(t)]^\eta dt - \frac{a(1-\eta)}{[1-F(a)]^{1-\eta}} \int_a^\infty d[1-F(t)] \\
&= \int_a^\infty [1-F(t)]^\eta dt + a(1-\eta)[1-F(a)]^\eta
\end{aligned}
$$

■

7.2 Proof of Theorem 2

We divide the proof of Theorem 2 into three distinct cases: $\gamma^Y < 1/2$, $\gamma^Y = 1/2$ and $\gamma^Y > 1/2$. Before the main body of the proof, we first give a lemma that is used in the latter two cases.

Lemma 3. *Let F be a distribution function satisfying the assumptions in Theorem 2 with $\gamma \geqslant \eta - \frac{1}{2}$, i.e., $\gamma^Y = \gamma + 1 - \eta \geqslant \frac{1}{2}$. We have*

$$
\lim_{s \downarrow 0} \frac{s^{2\eta-1} Q^2(1-s)}{\sigma_\eta^2(s)} = \frac{(\gamma+1-\eta)(\gamma+1/2-\eta)}{\eta^2\gamma^2} = \frac{\gamma^Y(\gamma^Y - 1/2)}{\eta^2\gamma^2}
$$

Proof: See the proof of Lemma 2 in Necir and Meraghni (2009). Please note that we have weaker assumptions than A.Necir et al. (2009)(Necir and Meraghni 2009). However, it's clear that both Lemma 1 and Lemma 2 in Necir and Meraghni (2009) can easily extended to prove our lemma.

7.2.1 Case 1: $\gamma^Y < \frac{1}{2}$

Let $\gamma^Y = \gamma + 1 - \eta < \frac{1}{2}$. Then it immediately follows that $\eta > \frac{1}{2} > \gamma$ and $\frac{1}{\gamma^X} > \frac{2}{2\eta - 1}$ because $\gamma > 0$ and $0 < \eta \leq 1$. We first want to prove

$$
\sqrt{n}(\hat{\mu}_\eta^{EVT} - H_\eta[X]) \xrightarrow{d} \mathrm{N}(0, \sigma_\eta^2) \tag{7.1}
$$

where

$$\sigma_\eta^2 = \eta^2 \int_0^1 \int_0^1 (\min(s,t) - st) s^{\eta-1} t^{\eta-1} dQ(1-s) dQ(1-t) < \infty$$

Recall

$$\hat{\mu}_\eta^{EMP} = \sum_{i=1}^{n} \left[\left(\frac{n+1-i}{n} \right)^\eta - \left(\frac{n-i}{n} \right)^\eta \right] X_{i:n}$$

By Theorem 3.2 in Jones and Zitikis (2003), we have

$$\sqrt{n}(\hat{\mu}_\eta^{EMP} - H_\eta[X]) \xrightarrow{d} \mathrm{N}(0, \sigma_\eta^2)$$

To prove (7.1),it suffice to show

$$\sqrt{n}(\hat{\mu}_\eta^{EMP} - \hat{\mu}_\eta^{EVT}) \xrightarrow{d} 0$$

Notice that

$$\hat{\mu}_\eta^{EMP} - \hat{\mu}_\eta^{EVT} = \sum_{i=1}^{k} \left(\frac{i}{n} \right)^\eta (X_{n-i+1} - X_{n-i}) - \left(\frac{k}{n} \right)^\eta X_{n-k:n} \frac{\hat{\gamma}_k}{\eta - \hat{\gamma}_k} =: I_1 - I_2$$

And by Theorem 1 and Theorem 2 in Necir, Meraghni and Meddi (2007), we have

$$\left(\frac{k}{n} \right)^{-\eta} \frac{\sqrt{k}}{X_{n-k:n}} \left\{ I_1 - \int_{Q\left(1-\frac{k}{n}\right)}^{\infty} [1 - F(x)]^\eta dx \right\} \xrightarrow{d} N(0, \sigma_1^2)$$

$$\left(\frac{k}{n} \right)^{-\eta} \frac{\sqrt{k}}{X_{n-k:n}} \left\{ I_2 - \int_{Q\left(1-\frac{k}{n}\right)}^{\infty} [1 - F(x)]^\eta dx \right\} \xrightarrow{d} N(0, \sigma_2^2)$$

where $\sigma_1^2 < \infty$ and $\sigma_2^2 < \infty$ Write $Q(1 - \frac{1}{u}) = u^{\gamma^X} L(u)$ with L a slowly varying function. Then $s^{\gamma^Y - \frac{1}{2}} L(s)$ regular varying with exponent $\gamma^Y - \frac{1}{2} < 0$. It implies

$$\lim_{s \to \infty} s^{\gamma^Y - \frac{1}{2}} L(s) = \lim_{n \to \infty} \left(\frac{n}{k} \right)^{\gamma^Y - \frac{1}{2}} L\left(\frac{n}{k} \right) = 0$$

by Proposition B.1.9 in de Haan and Ferreira (2006). On the other hand, from the proof of corollary in A.Necir (2009),we have

$$\frac{Q\left(1 - k/n\right)}{X_{n-k:n}} \xrightarrow{p} 1$$

Then, by Slutsky,

$$\frac{X_{n-k:n}}{Q\left(1-k/n\right)}\left(\frac{n}{k}\right)^{\gamma^Y-\frac{1}{2}}L(\frac{n}{k})\left(\frac{k}{n}\right)^{-\eta}\frac{\sqrt{k}}{X_{n-k:n}}\left\{I_1-\int\limits_{Q\left(1-\frac{k}{n}\right)}^{\infty}[1-F(x)]^{\eta}dx\right\}$$
$$=\frac{1}{(n/k)^{\gamma^X}L(n/k)}\left(\frac{n}{k}\right)^{\gamma^X-\eta+\frac{1}{2}}\left(\frac{k}{n}\right)^{-\eta+\frac{1}{2}}\sqrt{n}\left\{I_1-\int\limits_{Q\left(1-\frac{k}{n}\right)}^{\infty}[1-F(x)]^{\eta}dx\right\}$$
$$=\sqrt{n}\left\{I_1-\int\limits_{Q\left(1-\frac{k}{n}\right)}^{\infty}[1-F(x)]^{\eta}dx\right\}$$
$$\xrightarrow{d}1\cdot 0\cdot N(0,\sigma_1^2)=0$$

Similarly, we also have

$$\sqrt{n}\left\{I_2-\int\limits_{Q\left(1-\frac{k}{n}\right)}^{\infty}[1-F(x)]^{\eta}dx\right\}\xrightarrow{d}1\cdot 0\cdot N(0,\sigma_2^2)=0$$

Then we have

$$\sqrt{n}\left(\hat{\Pi}_{\eta,n}-\hat{\mu}_{\eta}^{EVT}\right)=\sqrt{n}\left(I_1-I_2\right)=\sqrt{n}\left\{I_1-\int\limits_{Q\left(1-\frac{k}{n}\right)}^{\infty}[1-F(x)]^{\eta}dx\right\}-\sqrt{n}\left\{I_2-\int\limits_{Q\left(1-\frac{k}{n}\right)}^{\infty}[1-F(x)]^{\eta}dx\right\}$$
$$\xrightarrow{d}0-0=0$$

This proves that (7.1) is true, that is

$$\frac{\sqrt{n}}{\sigma_{\eta}(k/n)}(\hat{\mu}_{\eta}^{EVT}-H_{\eta}[X])\xrightarrow{d}\frac{1}{\sigma_{\eta}}\mathrm{N}(0,\sigma_{\eta}^2)=\mathrm{N}(0,1)$$

since $\sigma_{\eta}(k/n)\to\sigma_{\eta}(0)=\sigma_{\eta}<\infty$

7.2.2 Case 2: $\gamma^Y>\frac{1}{2}$

Notice that

$$\hat{\mu}_{\eta}^{EVT}=\frac{1}{n^{\eta}}\sum_{i=1}^{n-k}X_{i:n}[(n-i+1)^{\eta}-(n-i)^{\eta}]+\left(\frac{k}{n}\right)^{\eta}X_{n-k:n}\frac{\eta}{\eta-\hat{\gamma}_k}$$
$$=\frac{1}{n^{\eta}}\sum_{i=1}^{n-k}X_{i:n}[(n-i+1)^{\eta}-(n-i)^{\eta}]+\left(\frac{k}{n}\right)^{\eta}X_{n-k:n}\frac{\hat{\gamma}_k}{\eta-\hat{\gamma}_k}+\left(\frac{k}{n}\right)^{\eta}X_{n-k:n}:=\Pi_3^{(n)}+\Pi_2^{(n)}+\Pi_1^{(n)}$$

and

$$H_{\eta}[X]=\int_0^{\infty}[1-F(s)]^{\eta}ds=\int_1^0 s^{\eta}d(Q(1-s))=-\int_0^{k/n}s^{\eta}d(Q(1-s))-\int_{k/n}^1 s^{\eta}d(Q(1-s))$$

Integrating the second term by parts yields

$$H_\eta[X] = \eta \int_{k/n}^{1} s^{\eta-1} Q(1-s)ds - \int_{0}^{k/n} s^\eta d(Q(1-s)) + \left(\frac{k}{n}\right)^\eta Q(1-k/n) := \Pi_3 + \Pi_2 + \Pi_1$$

Necir et al. (2007) have shown that under the assumptions of Theorem 3, there exists a sequence of Brownian bridges $\{B_n(s), 0 \leq 0 \leq 1\}_{n\in\mathbb{N}}$ such that, for all sufficiently large n,

$$\frac{\sqrt{n}}{(k/n)^{\eta-1/2}Q(1-k/n)}\left(\Pi_1^{(n)} - \Pi_1\right) = \gamma\left(\frac{k}{n}\right)^{-1/2} B_n\left(1-\frac{k}{n}\right) + o_p(1) \tag{7.2}$$

and

$$\begin{aligned}&\frac{\sqrt{n}}{(k/n)^{\eta-1/2}Q(1-k/n)}\left(\Pi_2^{(n)} - \Pi_2\right)\\ &= \frac{\gamma}{(\eta-\gamma)^2}(\eta\gamma - \gamma^2 + \eta)\left(\frac{k}{n}\right)^{-1/2} B_n\left(1-\frac{k}{n}\right) - \frac{\eta\gamma}{(\eta-\gamma)^2}\left(\frac{k}{n}\right)^{1/2}\int_{1-k/n}^{1}\frac{B_n(s)}{1-s}ds\end{aligned} \tag{7.3}$$

Then we need to evaluate, for all large n,

$$\frac{\sqrt{n}}{(k/n)^{\eta-1/2}Q(1-k/n)}\left(\Pi_3^{(n)} - \Pi_3\right)$$

Necir and Meraghni (2009) shows that

$$\Pi_3^{(n)} - \Pi_3 = S_{n1} + S_{n2}$$

with

$$S_{n1} = \eta\int_{k/n}^{1-1/n} s^{\eta-1}(Q_n(1-s) - Q(1-s))ds$$

$$S_{n2} = (1-(1-1/n)^\eta)X_{1:n} - \eta\int_{1-1/n}^{1} s^{\eta-1}Q(1-s)ds$$

First, we show that, for all large n,

$$\frac{\sqrt{n}}{(k/n)^{\eta-1/2}Q(1-k/n)}S_{n2} = o_p(1) \tag{7.4}$$

We have

$$(k/n)^{\eta-1/2}Q(1-k/n) = (k/n)^{\eta-1/2}(k/n)^{-\gamma}L(n/k) = (n/k)^{\gamma-\eta+1/2}L(n/k) = (n/k)^{\gamma^Y-1/2}L(n/k) \to \infty$$

by Proposition B.1.9 in de Haan and Ferreira (2006) since function $s^{\gamma^Y-1/2}L(s)$ is regularly varying with index $\gamma^Y - 1/2 > 0$. To prove (7.4),it suffice to show $\sqrt{n}S_{n2} \xrightarrow{p} 0$.

Note that $\sqrt{n}X_{1:n}$ has same distribution as $\sqrt{n}Q(\xi_{1:n})$ where $\xi_{1:n}$ is the minimum of n samples drawn from uniform distribution on [0,1). Since $\sqrt{n}\xi_{1:n} \xrightarrow{d} 0$ and Q is continuously differentiable with Q(0)=0, by delta method, we have $\sqrt{n}(Q(\xi_{1:n})-Q(0)) \xrightarrow{d} 0\cdot Q'(0) = 0$. Thereby, for $1 \geq \eta > 0$

$$\sqrt{n}(1-(1-1/n)^{\eta})X_{1:n} = [(1-(1-1/n)^{\eta})]\sqrt{n}X_{1:n} \xrightarrow{d} [1-(1-0)^{\eta}]\cdot 0 = 0 \tag{7.5}$$

On the other hand,

$$0 \leqslant \sqrt{n}\eta\int_{1-1/n}^{1} s^{\eta-1}Q(1-s)ds \leqslant \sqrt{n}\eta\int_{1-1/n}^{1}\left(1-\frac{1}{n}\right)^{\eta-1}Q\left(\frac{1}{n}\right)ds \leqslant \frac{1}{\sqrt{n}}\eta\left(1-\frac{1}{n}\right)^{\eta-1}Q\left(\frac{1}{n}\right)$$

since function $s^{\eta-1}Q(1-s)$ is decreasing on (0,1]. We have

$$\frac{1}{\sqrt{n}}\eta\left(1-\frac{1}{n}\right)^{\eta-1}Q\left(\frac{1}{n}\right) \to 0\cdot\eta(1-0)^{\eta-1}Q(0) = 0$$

consequently,

$$\sqrt{n}\eta\int_{1-1/n}^{1} s^{\eta-1}Q(1-s)ds \to 0 \tag{7.6}$$

From (7.5),(7.6), we know $\sqrt{n}S_{n2} \xrightarrow{p} 0$. This complete the proof of (7.4). Next, we derive the asymptotic behavior of

$$\frac{\sqrt{n}}{(k/n)^{\eta-1/2}Q(1-k/n)}S_{n1}$$

Necir and Meraghni (2009) have shown that

$$\frac{\sqrt{n}}{(k/n)^{\eta-1/2}Q(1-k/n)}S_{n1} = -\frac{\eta\int_{k/n}^{1-1/n} s^{\eta-1}B_n(1-S)Q'(1-s)ds}{(k/n)^{\eta-1/2}Q(1-k/n)} + o_p(1) \tag{7.7}$$

Recall that, $E[(B_n(s))^2] = [(1-s)s] \leqslant \frac{1}{4}$, for any $0<s<1$. By Jensen's inequality, for each $n\in\mathbb{N}$ we get, for any $0<s<1$,

$$E(|B_n(1-s)|) \leqslant \sqrt{E[(B_n(1-s))^2]} \leqslant \frac{1}{2}$$

then for all large n we have

$$\begin{aligned}
&E\left(\left|\int_{1-1/n}^{1} s^{\eta-1}B_n(1-S)Q'(1-s)ds\right|\right) \leqslant \int_{1-1/n}^{1} s^{\eta-1}E[|B_n(1-S)|]\left|Q'(1-s)\right|ds \\
&\leqslant \int_{1-1/n}^{1} s^{\eta-1}\frac{1}{2}\left|Q'(1-s)\right|ds \leqslant \int_{1-1/n}^{1}\left(1-\frac{1}{n}\right)^{\eta-1}\frac{1}{2}\left|Q'(1-s)\right|ds \\
&= \left(1-\frac{1}{n}\right)^{\eta-1}\frac{1}{2}\int_{1-1/n}^{1}\left|Q'(1-s)\right|ds = -\left(1-\frac{1}{n}\right)^{\eta-1}\frac{1}{2}\int_{1-1/n}^{1}Q'(1-s)ds \\
&= \left(1-\frac{1}{n}\right)^{\eta-1}\frac{1}{2}\left[Q\left(\frac{1}{n}\right)-Q(0)\right]
\end{aligned}$$

The function Q is continuous on [0,1), then $Q(\frac{1}{n}) \to Q(0) = 0$ as $n\to\infty$. Hence

$$\left(1-\frac{1}{n}\right)^{\eta-1}\frac{1}{2}\left[Q\left(\frac{1}{n}\right)-Q(0)\right] \to (1-0)^{\eta-1}\frac{1}{2}[Q(0)-Q(0)] = 0$$

consequently,

$$\int_{1-1/n}^{1} s^{\eta-1} B_n(1-S) Q'(1-s) ds = o_p(1) \tag{7.8}$$

From (7.7) and (7.8)

$$-\frac{\eta \int_{k/n}^{1} s^{\eta-1} B_n(1-S) Q'(1-s) ds}{(k/n)^{\eta-1/2} Q(1-k/n)} + o_p(1)$$

together with (7.4), it implies that

$$\frac{\sqrt{n}}{(k/n)^{\eta-1/2} Q(1-k/n)} \left(\Pi_3^{(n)} - \Pi_3\right) = -\frac{\eta \int_{k/n}^{1} s^{\eta-1} B_n(1-S) Q'(1-s) ds}{(k/n)^{\eta-1/2} Q(1-k/n)} + o_p(1) \tag{7.9}$$

Eventually, (7.2),(7.3) and (7.9) yield that, for all large n,

$$\frac{\sqrt{n}}{(k/n)^{\eta-1/2} Q(1-k/n)} \left(\hat{\mu}_\eta^{EVT} - H_\eta[X]\right) = \frac{\sqrt{n}}{(k/n)^{\eta-1/2} Q(1-k/n)} \left(\sum_{i=1}^{3} \Pi_i^{(n)} - \sum_{i=1}^{3} \Pi_i\right) = \Delta_n(\eta,\gamma) + o_p(1)$$

where

$$\begin{aligned}\Delta_n(\eta,\gamma) = & \frac{\eta\gamma}{(\eta-\gamma)^2}(\eta-\gamma+1)\left(\frac{k}{n}\right)^{-1/2} B_n\left(1-\frac{k}{n}\right) \\ & -\frac{\eta\gamma}{(\eta-\gamma)^2}\left(\frac{k}{n}\right)^{1/2} \int_{1-k/n}^{1} \frac{B_n(s)}{1-s} ds \\ & -\frac{\eta \int_{k/n}^{1} s^{\eta-1} B_n(1-s) Q'(1-s) ds}{(k/n)^{\eta-1/2} Q(1-k/n)}\end{aligned}$$

Necir and Meraghni (2009) have shown that $\Delta_n(\eta,\gamma)$ is a Gaussian random variable with mean zero and variance

$$\sigma_\Delta^2 + o(1) = \frac{\eta^2\gamma^2[(\eta-\gamma)^2+1]}{(\eta-\gamma)^4} + \frac{[4(\eta-\gamma)-1]\eta^2\gamma^2}{(\gamma+1-\eta)[2\gamma+1-2\eta](\eta-\gamma)} + o(1)$$

This implies that, under assumptions of theorem 3,

$$\frac{\sqrt{n}}{(k/n)^{\eta-1/2} Q(1-k/n)} \left(\hat{\mu}_\eta^{EVT} - H_\eta[X]\right) \xrightarrow{d} N(0,\sigma_\Delta^2) \tag{7.10}$$

By Lemma 3, we know

$$\frac{\sqrt{n}}{\sigma_\eta(k/n)} \left(\hat{\mu}_\eta^{EVT} - H_\eta[X]\right) = \frac{(k/n)^{\eta-1/2} Q(1-k/n)}{\sigma_\eta(k/n)} \frac{\sqrt{n}}{(k/n)^{\eta-1/2} Q(1-k/n)} \left(\hat{\mu}_\eta^{EVT} - H_\eta[X]\right) \xrightarrow{d} N(0,\sigma_{\eta,\gamma}^2)$$

where

$$\begin{aligned}\sigma_{\eta,\gamma}^2 &= \frac{(\gamma+1-\eta)(\gamma+1/2-\eta)}{\eta^2\gamma^2}\left[\frac{\eta^2\gamma^2[(\eta-\gamma)^2+1]}{(\eta-\gamma)^4} + \frac{[4(\eta-\gamma)-1]\eta^2\gamma^2}{(\gamma+1-\eta)[2\gamma+1-2\eta](\eta-\gamma)}\right] \\ &= \frac{(\gamma+1-\eta)(\gamma+1/2-\eta)[(\eta-\gamma)^2+1]}{(\eta-\gamma)^4} + \frac{[4(\eta-\gamma)-1]}{2(\eta-\gamma)}\end{aligned}$$

7.2.3 Case 3: $\gamma^Y = 1/2$

Suppose now $\gamma^Y = 1/2$. Then from Lemma 3, we have, as $n \to \infty$

$$\frac{(k/n)^{2\eta-1} Q^2(1-k/n)}{\sigma_\eta^2(k/n)} \to 0$$

consequently, from (7.2) and (7.3), by Slutsky's Theorem,we get

$$\frac{\sqrt{n}}{\sigma_\eta(k/n)}\left(\Pi_1^{(n)} - \Pi_1\right) = \frac{(k/n)^{\eta-1/2}Q(1-k/n)}{\sigma_\eta(k/n)} \frac{\sqrt{n}}{(k/n)^{\eta-1/2}Q(1-k/n)}\left(\Pi_1^{(n)} - \Pi_1\right) \xrightarrow{d} 0 \quad (7.11)$$

and

$$\frac{\sqrt{n}}{\sigma_\eta(k/n)}\left(\Pi_2^{(n)} - \Pi_2\right) = \frac{(k/n)^{\eta-1/2}Q(1-k/n)}{\sigma_\eta(k/n)} \frac{\sqrt{n}}{(k/n)^{\eta-1/2}Q(1-k/n)}\left(\Pi_2^{(n)} - \Pi_2\right) \xrightarrow{d} 0 \quad (7.12)$$

Moreover, from above, we also have, for all large n,

$$\begin{aligned} &\frac{\sqrt{n}}{\sigma_\eta(k/n)}\left(\Pi_3^{(n)} - \Pi_3\right) \\ &= \frac{(k/n)^{\eta-1/2}Q(1-k/n)}{\sigma_\eta(k/n)} \frac{\sqrt{n}}{(k/n)^{\eta-1/2}Q(1-k/n)}\left(\Pi_3^{(n)} - \Pi_3\right) \\ &= \frac{\sqrt{n}}{\sigma_\eta(k/n)} S_{n2} - \frac{(k/n)^{\eta-1/2}Q(1-k/n)}{\sigma_\eta(k/n)} \frac{\eta \int_{k/n}^{1-1/n} s^{\eta-1} B_n(1-s) Q'(1-s) ds}{(k/n)^{\eta-1/2}Q(1-k/n)} + o(1)\cdot o_p(1) \\ &= -\frac{\eta \int_{k/n}^{1} s^{\eta-1} B_n(1-s) Q'(1-s) ds}{\sigma_\eta(k/n)} + o_p(1) \end{aligned} \quad (7.13)$$

since $\sqrt{n} S_{n2} = o_p(1)$, $\int_{1-1/n}^{1} s^{\eta-1} B_n(1-s) Q'(1-s) ds = o_p(1)$ and $\sigma_\eta(k/n) \to \sigma_\eta \in (0, \infty]$. Finally, (7.11), (7.12) and (7.13) yields

$$\frac{\sqrt{n}}{\sigma_\eta(k/n)}\left(\hat{\mu}_\eta^{EVT} - H_\eta[X]\right) = \frac{\sqrt{n}}{\sigma_\eta(k/n)}\left(\sum_{i=1}^{3} \Pi_i^{(n)} - \sum_{i=1}^{3} \Pi_i\right) = \Delta_n^*(\eta,\gamma) + o_p(1)$$

where

$$\Delta_n^*(\eta,\gamma) = -\frac{\eta \int_{k/n}^{1} s^{\eta-1} B_n(1-s) Q'(1-s) ds}{\sigma_\eta(k/n)} = \frac{\eta \int_{k/n}^{1} s^{\eta-1} B_n(1-s) dQ(1-s)}{\sigma_\eta(k/n)}$$

It is clear that $\Delta_n^*(\eta,\gamma)$ is a Gaussian random variable with mean zero and variance

$$\begin{aligned} E[\Delta_n^*(\eta,\gamma)]^2 &= \left\{\eta^2 \int_{k/n}^{1}\int_{k/n}^{1} [\min(s,t) - st] s^{\eta-1} t^{\eta-1} dQ(1-s) dQ(1-t) / \sigma_\eta^2(k/n)\right\} \\ &= \sigma_\eta^2(k/n)/\sigma_\eta^2(k/n) + o(1) = 1 \end{aligned}$$

This proves that

$$\frac{\sqrt{n}}{\sigma_\eta(k/n)}\left(\hat{\mu}_\eta^{EVT} - H_\eta[X]\right) \xrightarrow{d} N(0,1)$$

■

7.3 Proof of Theorem 4

Write

$$\hat{\mu}^{EVT}_{\eta,c_nX_{n-k:n}} - H_{\eta,c_nQ(1-k/n)}[X] = c_n^{-\frac{\eta}{\hat{\gamma}_k}+1}\left(\frac{k}{n}\right)^{\eta} X_{n-k:n}\frac{\hat{\gamma}_k}{\eta-\hat{\gamma}_k} - G(c_nQ(1-k/n))$$

Then

$$\begin{aligned}
&\frac{\sqrt{k}}{(k/n)^{\eta}Q(1-k/n)}\left\{\hat{\mu}^{EVT}_{\eta,c_nX_{n-k:n}} - H_{\eta,c_nQ(1-k/n)}[X]\right\}\\
&= \frac{\sqrt{k}}{(k/n)^{\eta}Q(1-k/n)}\left\{c_n^{-\frac{\eta}{\hat{\gamma}_k}+1}\left(\frac{k}{n}\right)^{\eta} X_{n-k:n}\frac{\hat{\gamma}_k}{\eta-\hat{\gamma}_k} - c_n^{-\frac{\eta}{\hat{\gamma}_k}+1}G(Q(1-k/n))\right\}\\
&+\frac{\sqrt{k}}{(k/n)^{\eta}Q(1-k/n)}\left\{c_n^{-\frac{\eta}{\hat{\gamma}_k}+1}G(Q(1-k/n)) - c_n^{-\frac{\eta}{\gamma}+1}G(Q(1-k/n))\right\}\\
&+\frac{\sqrt{k}}{(k/n)^{\eta}Q(1-k/n)}\left\{c_n^{-\frac{\eta}{\gamma}+1}G(Q(1-k/n)) - G(c_nQ(1-k/n))\right\}\\
&:= \Xi_1+\Xi_2+\Xi_3
\end{aligned}$$

Note that

$$\begin{aligned}
&\Xi_1 = c_n^{-\frac{\eta}{\hat{\gamma}_k}+1}\frac{\sqrt{k}}{(k/n)^{\eta}Q(1-k/n)}\left\{\left(\frac{k}{n}\right)^{\eta} X_{n-k:n}\frac{\hat{\gamma}_k}{\eta-\hat{\gamma}_k} - \int_{Q(1-k/n)}^{\infty}[1-F(t)]^{\eta}dt\right\}\\
&\xrightarrow{d} 1^{-\frac{\eta}{\gamma}+1}\cdot N(0,\tilde{\sigma}^2_{\eta,\gamma}) = N(0,\tilde{\sigma}^2_{\eta,\gamma})
\end{aligned}$$

by applying the Theorem 2 in Necir et al. (2007). It's suffice now to prove

$$\Xi_2 \xrightarrow{d} 0 \ and \ \Xi_3 \xrightarrow{d} 0 \tag{7.14}$$

Write

$$\Xi_2 = \frac{\int_{Q(1-k/n)}^{\infty}[1-F(t)]^{\eta}dt}{(k/n)^{\eta}Q(1-k/n)}\sqrt{k}\left\{c_n^{-\frac{\eta}{\hat{\gamma}_k}+1} - c_n^{-\frac{\eta}{\gamma}+1}\right\}$$

and

$$\Xi_3 = \frac{\int_{Q(1-k/n)}^{\infty}[1-F(t)]^{\eta}dt}{(k/n)^{\eta}Q(1-k/n)}\sqrt{k}\left\{c_n^{-\frac{\eta}{\gamma}+1} - \frac{G(c_nQ(1-k/n))}{G(Q(1-k/n))}\right\}$$

Recall, from Karamata's Theorem, that

$$\frac{\int_{Q(1-k/n)}^{\infty}[1-F(t)]^{\eta}dt}{(k/n)^{\eta}Q(1-k/n)} \to \frac{1}{\eta/\gamma-1} = \frac{\gamma}{\eta-\gamma} \ , as \ n\to\infty$$

since function$[1-F(t)]^{\eta}$ is regular varying with index $-\frac{\eta}{\gamma} < -1$ and $Q(1-k/n)\to\infty$ as $n\to\infty$. Now it's suffice to prove

$$\sqrt{k}\left\{c_n^{-\frac{\eta}{\hat{\gamma}_k}+1} - c_n^{-\frac{\eta}{\gamma}+1}\right\} \xrightarrow{d} 0 \ , as \ n\to\infty \tag{7.15}$$

$$\sqrt{k}\left\{c_n^{-\frac{\eta}{\gamma}+1} - \frac{G(c_nQ(1-k/n))}{G(Q(1-k/n))}\right\} \xrightarrow{d} 0 \ , as \ n\to\infty \tag{7.16}$$

We First prove (7.16).From Theorem 2.3.9 in de Haan and Ferreira (2006), we have, for any ε,δ ¿0, there exists M_0 such that for all $n > M_0$

$$\left| \frac{c_n^{-\frac{\eta}{\gamma}+1} - \frac{G(c_n Q(1-k/n))}{G(Q(1-k/n))}}{A_0(Q(1-k/n))} - c_n^{-\frac{\eta}{\gamma}+1} \frac{c_n^{\rho'} - 1}{\rho'} \right| \leqslant \varepsilon c_n^{-\frac{\eta}{\gamma}+1+\rho'} \max(c_n^{\delta}, c_n^{-\delta})$$

that is,

$$\begin{aligned} &\sqrt{k} A_0(Q(1-k/n)) \left\{ c_n^{-\frac{\eta}{\gamma}+1} \frac{c_n^{\rho'} - 1}{\rho'} - \varepsilon c_n^{-\frac{\eta}{\gamma}+1+\rho'} \max(c_n^{\delta}, c_n^{-\delta}) \right\} \\ &\leqslant \sqrt{k} \left\{ c_n^{-\frac{\eta}{\gamma}+1} - \frac{G(c_n Q(1-k/n))}{G(Q(1-k/n))} \right\} \\ &\leqslant \sqrt{k} A_0(Q(1-k/n)) \left\{ c_n^{-\frac{\eta}{\gamma}+1} \frac{c_n^{\rho'} - 1}{\rho'} + \varepsilon c_n^{-\frac{\eta}{\gamma}+1+\rho'} \max(c_n^{\delta}, c_n^{-\delta}) \right\} \end{aligned} \tag{7.17}$$

It is clear that, for some c

$$c_n^{-\frac{\eta}{\gamma}+1} \frac{c_n^{\rho} - 1}{\rho} \to 0 \ and \ \varepsilon c_n^{-\frac{\eta}{\gamma}+1+\rho} \max(c_n^{\delta}, c_n^{-\delta}) \to c < \infty$$

since $c_n \to 1$ as $n \to \infty$. Moreover, from the condition of Theorem 4 we know $\sqrt{k} A_0(Q(1-k/n)) = o(1)$ for all large n. Consequently, both upper bound and lower bound in (7.17) converge to 0 if $n \to \infty$. This proves (7.16). Next, we consider (7.15). By Mean-Value Theorem, there exists ξ_n between $\hat{\gamma}_k$ and γ such that

$$c_n^{-\frac{\eta}{\hat{\gamma}_k}+1} - c_n^{-\frac{\eta}{\gamma}+1} = \frac{\eta}{\xi_n^2} c_n^{-\frac{\eta}{\xi_n}+1} \log(c_n) \{\hat{\gamma}_k - \gamma\}$$

Hence,

$$\sqrt{k} \left\{ c_n^{-\frac{\eta}{\hat{\gamma}_k}+1} - c_n^{-\frac{\eta}{\gamma}+1} \right\} = \frac{\eta}{\xi_n^2} c_n^{-\frac{\eta}{\xi_n}+1} \log(c_n) \sqrt{k} \{\hat{\gamma}_k - \gamma\} \xrightarrow{d} \frac{\eta}{\gamma^2} 1^{-\frac{\eta}{\gamma}+1} \cdot 0 \cdot \mathrm{N}(0, \gamma^2) = 0$$

by applying the consistency and asymptotic normality of Hill estimator in Theorem 1. ■

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