Chapter 16: Financial Innovation - Refundable Annuities (Annuity Options)

16.1 The Timing of Annuity Purchases

In previous chapters (in particular chapters 8 and 10) we have seen that in the presence of a competitive annuity market, uncertainty with respect to the length of life can be perfectly insured by an optimum policy which invests all individual savings in long-term annuities. The implication of associating annuity purchases with savings is that the bulk of annuities are purchased throughout ones working life. This stands in stark contrast to empirical evidence that most private annuities are purchased at ages close to retirement (in the US the average age of annuity purchasers is 62).

A recent survey in the UK (Gardner and Wadsworth (2004)) reports that half of the individuals in the sample would, given the option, never annuitize. This attitude is independent of specific annuity terms and prices. By far, the dominant reason given for the reluctance to annuitize was a *"preference for flexibility"*. Among those willing to annuitize, the major factors that affected their decision were health (those in good health more likely to annuitize), education, household size (less likely to annuitize as household size increases) and income (higher earnings support annuitization).

Lack of flexibility in holding annuities was interpreted by the respondents as inability to short-sell (or borrow against) early purchased annuities when personal circumstances make such sale desirable. Preference to sell annuities arises typically upon the realization of negative information about longevity (disability) or income. In this survey, the reluctance to purchase annuities early in life was hardly affected by knowledge that annuities purchased later will be more expensive (due to adverse-selection).

Bodie (2003) also attributes the reluctance to annuitize to uncertain needs for long-term care:

> "Retired people do not voluntarily annuitize much of their wealth. One reason may be that they believe they need to hold on to assets in case they need nursing home care. Annuities, once bought, tend to be illiquid"...

Data about the timing of annuity purchases and surveys such as the above suggest a need to develop a model that incorporates uninsurable risks, such as income (or 'needs' such as long-term care) in addition to longevity risk. Further, to respond to individuals' desire for flexibility, the model should allow for short-sales of annuities purchased early or the purchase of additional short-term annuities when so desired. The first part of this chapter builds on a model developed by Brugiavini (1993) with this objective in mind.

With uncertainty extending to other variables beyond longevity, competitive annuity markets cannot attain a First-Best allocation (which requires income transfers accross states of nature). Sequential annuity market equilibrium is characterized by the purchase of long-term annuities, short-sale of some of these annuities later-on or purchase of additional short-term annuities.

Since the competitive equilibrium is Second-Best, it is natural to ask whether there are financial instruments which, if available, are welfare improving. We answer this question in the affirmative, proposing a new type of *refundable annuities.* These are annuities that can be refunded, if so desired, at a pre-determined price. Holding a portfolio of such refundable annuities with varying refund prices allows individuals more flexibility in adjusting their consumption path upon the arrival later in life of information about longevity and income.

We show that *refundable annuities* are equivalent to *annuity options*. These are options that entitle the holder to purchase annuities at a later date at a pre-determined price.

Interestingly, annuity options are available in the UK. It is worth quoting again from a textbook for actuaries (Fisher and Young (1965)):

> "*Guaranteed Annuity Options.* The option may not be exercised until a future date ranging perhaps from 5 to 50 years hence... The mortality and interest assumptions should be conservative... The estimates of future improvement implied by experience from which mortality tables were constructed suggest that there should be differences in rates according to the year in which the option is exercisable... A difference of about $\frac{1}{4}$% in the yield per £100 purchase price could arise between one option and another exercisable ten years later... (Such) differences in guaranteed annuity rates according to the future date on which they are exercisable do therefore seem to be justified in theory" (*Actuarial Practice of Life Assurance*, p. 421).

Behavioral economics, dealing with bounded rationality (see below) seems to provide additional support to the offering of annuity options which involve a small present cost and allow postponement of the decision to purchase annuities. It has been argued (e.g. Thaler and Benartzi (2004) and Laibson (1997)) that these features provide a positive inducement to purchase annuities for individuals with tendencies to procrastinate or heavily discount the short-run future.

16.2 Sequential Annuity Market Equilibrium Under Survival Uncertainty

Individuals live for two or three periods. Their longevity is unknown in period zero. They learn their period two survival probability, p $(0 \leq p \leq 1)$ at the beginning of period one. Survival probabilities have a continuous distribution function, $F(p)$, with support $[\underline{p},\ \bar{p}]$. In period zero, all individuals earn the same income, y_0, and do not consume. They purchase ('long-term') annuities, each of which pays \$1 in period two if the holder of the annuity is alive (all individuals survive to period one). Denote the amount of these annuities by a_0 and their price by q_0. Individuals can also save in non-annuitized assets who, for simplicity, are assumed to carry a zero rate of interest. The amount of savings in period zero is $y_0 - q_0 a_0$.

At the beginning of period one ('working years'), individuals earn an income, y_1, learn about their survival probability, p, $1 \geq p \geq 0$, and make decisions about their consumption in period one, c_1, and in period two, c_2 (if alive). They may purchase additional one-period ('short-term') annuities, a_1, $a_1 \geq 0$, or short-sell an amount b_1 of period-zero annuities, $b_1 \geq 0$. Since same consumption is invaluable, they will never sell *all* their long-term annuities, that is, $a_0 - b_1 > 0$. In period two, annuities' payout is $a_0 + a_1 - b_1$ if the holder of the annuities is alive and zero if the holder is dead.

(a) First-Best

Suppose that income in period one, y_1, is known with certainty so individuals are distinguished only by their realized survival probabilities in period one.

Expected lifetime utility, V, is

$$V = E\left[u(c_1) + pu(c_2)\right] \tag{16.1}$$

where $u'(c) > 0$, $u''(c) < 0$.

The economy's resource constraint is

$$E\left[c_1 + pc_2\right] = y_0 + y_1 \tag{16.2}$$

Optimum consumption, the solution to maximization of (16.1) s.t. (16.2), may depend on p, $(c_1(p),\ c_2(p))$. However, concavity of V and the linear constraint yield a First-Best allocation which is independent of p: $c_1^*(p) = c_2^*(p) = c^*$,

$$c^* = \frac{y_0 + y_1}{1 + E(p)} \tag{16.3}$$

where

$$E(p) = \int_{\underline{p}}^{\bar{p}} p dF(p) \tag{16.4}$$

is expected lifetime. We shall now show that a competitive long-term annuity market attains the First-Best allocation.

(b) Annuity Market Equilibrium: No Late Transactions

In period one, the issuers of annuities can distinguish between those who purchase additional annuities ('lenders') and those who short-sell period zero annuities ('borrowers'). Since borrowing and lending activities are distinguishable, their prices may be different. Denote the lending price by q_1^1 and the borrowing price by q_1^2.

The individual's maximization is solved backwards: Given a_0, p, q_1^1 and q_1^2, individuals in period one maximize utility

$$\underset{a_1 \geq 0,\ b_1 \geq 0}{Max} [u(c_1) + pu(c_2)] \tag{16.5}$$

where

$$\begin{aligned} c_1 &= y_0 + y_1 - q_0 a_0 - q_1^1 a_1 + q_1^2 b_1 \\ c_2 &= a_0 + a_1 - b_1 \end{aligned} \tag{16.6}$$

The F.O.C. are

$$-u'(c_1)q_1^1 + pu'(c_2) \leq 0 \tag{16.7}$$

and

$$u'(c_1)q_1^2 - pu'(c_2) \leq 0 \tag{16.8}$$

Denote the solutions to (16.6) - (16.8) by $\hat{a}_1(p)$, $\hat{b}_1(p)$, $\hat{c}_1(p)$ and $\hat{c}_2(p)$, where we suppress the dependence on q_0, q_1^1, q_1^2, y_0 and y_1. It can be shown (see Appendix) that when $\hat{a}_1(p) > 0$, so (16.7) holds with equality, $\frac{\partial \hat{a}_1}{\partial p} > 0$ and when $\hat{b}_1(p) > 0$, so (16.8) holds with equality, $\frac{\partial \hat{b}_1}{\partial p} < 0$. A higher survival probability increases the amount of lending and decreases the amount of borrowing whenever these are positive.

Assume that optimum consumption is strictly positive, $\hat{c}_i(p) > 0$, $i = 1, 2$, for all $\underline{p} \leq p \leq \bar{p}$ (a sufficient condition is that $u'(0) = \infty$).

It is assumed that $q_1^2 < q_1^1$. This condition ensures, by (16.7) - (16.8), that individuals are either lenders ($\hat{a}_1 > 0$) or borrowers ($\hat{b}_1 > 0$) and not both. It is shown below that this condition always holds in equilibrium.

In period zero, individuals choose an amount a_0 that maximizes expected utility, anticipating optimum behavior in period one:

$$\underset{a_0 \geq 0}{Max} E\left[u(\hat{c}_1) + pu(\hat{c}_2)\right] \tag{16.9}$$

s.t. (16.6). By the envelope theorem, the F.O.C. is

$$-E\left[u'(\hat{c}_1)\right]q_0 + E\left[pu'(\hat{c}_2)\right] = 0 \tag{16.10}$$

Denote the optimum amount of period zero annuities by $\hat{a}_0$. Since in period zero all individuals are alike and purchase the same amount of annuities, the equilibrium price, $\hat{q}_0$, is equal to expected lifetime, (16.4),

$$\hat{q}_0 = E(p) \tag{16.11}$$

The equilibrium prices of a_1 and of b_1 are determined as follows.

When (16.7) holds with equality at the 'kink', $\hat{a}_1 = \hat{b}_1 = 0$, this determines a survival probability, p_a, $p_a = \lambda q_1^1$, where

$$\lambda = \frac{u'(y_0 + y_1 - E(p)\hat{a}_0)}{u'(\hat{a}_0)} \tag{16.12}$$

with $\hat{a}_0$ determined by (16.10) and (16.11):

$$\begin{aligned} -E\left[u'(y_0 + y_1 - E(p)\hat{a}_0 - q_1^1\hat{a}_1(p) + q_1^2\hat{b}_1(p))\right]E(p) + \\ +E\left[pu'(\hat{a}_0 + \hat{a}_1(p) - \hat{b}_1(p))\right] = 0 \end{aligned} \tag{16.13}$$

When $\hat{a}_1(p) = \hat{b}_1(p) = 0$ for all p, $\bar{p} \geq p \geq \underline{p}$, then, from (16.13), $\lambda = 1$ (because marginal utilities are independent of p). When $p_a < \bar{p}$ then, by (16.7), $\hat{a}_1(p) > 0$ for $\bar{p} \geq p \geq p_a$ and $\hat{a}_1(p) = 0$ for $p_a \geq p \geq \underline{p}$.

Using a similar argument for short-sales, define $p_b = \lambda q_1^2$. The condition $q_1^2 < q_1^1$ implies that $p_b < p_a$. It can be seen from (16.8) that if $p_b > \underline{p}$, then

$\hat{b}_1 > 0$ for $\underline{p} \leq p < p_b$ and $\hat{b}_1 = 0$ for $\bar{p} \geq p \geq p_b$. Summarizing,

$$\begin{array}{ll} \hat{a}_1 > 0,\ \hat{b}_1 = 0 & p_a < p \leq \bar{p} \\ \hat{a}_1 = \hat{b}_1 = 0 & p_b \leq p \leq p_a \\ \hat{b}_1 > 0,\ \hat{a}_1 = 0 & \underline{p} \leq p < p_b \end{array} \tag{16.14}$$

The equilibrium prices $\hat{q}_1^1$ and $\hat{q}_1^2$ are determined by zero expected profits conditions:

$$\int_{p_a}^{\bar{p}} (\hat{q}_1^1 - p)\hat{a}_1(p)dF(p) = 0 \tag{16.15}$$

and

$$\int_{\bar{p}}^{p_b} (\hat{q}_1^2 - p)\hat{b}_1(p)dF(p) = 0 \tag{16.16}$$

Note that the bounds of integration, p_a and p_b, depend on the equilibrium values $\hat{q}_1^1$ and $\hat{q}_1^2$. As shown in chapter 8 and first stated by Brugiavini (1993), equilibrium prices that satisfy (16.15) and (16.16) are $q_1^1 = \bar{p}$ and $q_1^2 = \underline{p}$, which implies that $\hat{a}_1 = \hat{b}_1 = 0$ for all p. Under a certain condition, this solution is unique. Proof is provided in the Appendix to this chapter. This solution entails that $\hat{c}_1(p)$ and $\hat{c}_2(p)$ are independent of p and, by (16.13), equal to the First-Best allocation: $\hat{c}_i(p) = c^*$, $i = 1, 2$, given by (16.3).

Conclusion: *when uncertainty is confined to future survival probabilities, consumers purchase early in life an amount of annuities that generates zero demand for annuities in older ages, ensuring a consumption path that is independent of the state of nature ($\hat{c}_1$ and $\hat{c}_2$ independent of p). Consequently, there will be no annuity transactions late in life.*

This stark conclusion is in contrast to overwhelming empirical evidence which shows that private annuities are purchased by individuals in advanced ages[49]. Indeed, we shall now show that the above conclusion does not carry-over to more realistic cases with uncertainty about (uninsurable) future variables, such as income, in addition to survival probabilities.

16.3 Uncertain Survival and Future Incomes: Existence of a Separating Equilibrium

Suppose that in period zero, the probability of survival to period two *and* the level of income in period one, y_1, are both uncertain, their realizations occurring at the beginning of period one[50]. The realized levels of p and y_1 are assumed to be private information unknown to the issuers of annuities. For simplicity, assume that y_1 is distributed independently of p. Its distribution, denoted by $G(y_1)$, has a support $(\underline{y}_1,\ \bar{y}_1)$.

(a) First-Best

As before, the First-Best allocation maximizes expected utility, (16.1), subject to the resource constraint

$$E\ [c_1 + pc_2] = y_0 + E(y_1) \tag{16.17}$$

Again, the solution is independent of p: $c_1^* = c_2^* = \dfrac{y_0 + E(y_1)}{1 + E(p)}$. However, unlike the previous case where the early purchase of annuities could fully insure against survival uncertainty and, consequently, implement the First-Best

[49] See Brown *et-al* (2001).

[50] An alternative formulation is to make utility in period one depend on a parameter ('needs'), whose value is unknown in period zero and realized at the beginning of period one. This formulation yields the same results as below.

allocation, it is seen from (16.17) that the First-Best solution with income uncertainty requires income transfers, providing the expected level of income to everyone. Indeed, income insurance would enable such transfers. However, for obvious reasons, the level of realized income must be assumed to be private information and this precludes insurance contingent on the level of income. Consequently, the annuity market cannot, in general, attain the First-Best allocation.

(b) Sequential Annuity Market Equilibrium

As before, maximization is done backwards. In period one, utility maximization w.r.t. a_1 yields the F.O.C.

$$-u_1'(\hat{c}_1)q_1^1 + pu'(\hat{c}_2) \leq 0 \tag{16.18}$$

with equality when $\hat{a}_1 > 0$. Setting $\hat{a}_1 = \hat{b}_1 = 0$, (16.18), with equality

$$-u'(y_0 - q_0 a_0 + \tilde{y}_1^1(p))\ q_1^1 + pu'(a_0) = 0 \tag{16.19}$$

defines for each p a critical level of income, $\tilde{y}_1^1(p)$. Since $-u'(y_0 - q_0 a_0 + y_1 + q_1^2 b_1)\ q_1^1 + pu'(a_0 - b_1) > 0$ for all $y_1 > \tilde{y}_1^1(p)$ and $b_1 \geq 0$, it follows that $\hat{a}_1(p, y_1) > 0$ for all $\bar{y}_1 \geq y_1 > \tilde{y}_1^1(p)$ and $\hat{a}_1(p, y_1) = 0$ for all $\underline{y}_1 \leq y_1 < \tilde{y}_1^1(p)$ (see Figure 16.1).

Similarly, the F.O.C. w.r.t. b_1 is

$$u'(\hat{c}_1)q_1^2 - pu'(\hat{c}_2) \leq 0 \tag{16.20}$$

with equality when $\hat{b}_1 > 0$. Again, setting $\hat{a}_1 = \hat{b}_1 = 0$, (16.20) with equality defines for each p a critical level of income, $\tilde{y}_1^2(p)$. Since $u'(y_0 - q_0 a_0 + y_1)q_1^2 - pu'(a_0) > 0$ for all $\underline{y}_1 \leq y_1 < \tilde{y}_1^2(p)$ and $\hat{a}_1 \geq 0$, it follows that $\hat{b}_1(p, y_1) > 0$ for all $\underline{y}_1 \leq y_1 < \tilde{y}_1^2(p)$ and $\hat{b}_1(p, y_1) = 0$ for all $\bar{y}_1 \geq y_1 > \tilde{y}_1^2(p)$.

To make the pattern displayed in Figure (16.1) consistent, it is necessary that $\tilde{y}_1^2(p) < \tilde{y}_1^1(p)$ for all p, which is equivalent to the condition that $q_1^2 < q_1^1$. That is, the borrowing price is lower than the lending price[51]. We shall show that this condition is always satisfied in equilibrium

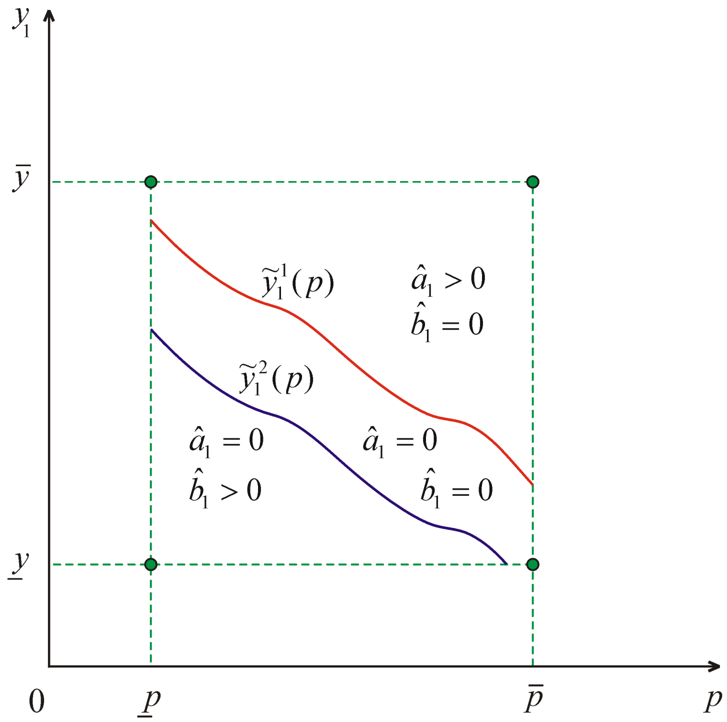


Figure 16.1
Pattern of Period One Annuity Purchases

Equilibrium prices, $(\hat{q}_1^1,\ \hat{q}_1^2)$, are defined by zero expected profits conditions:

$$\int_{\underline{p}}^{\bar{p}} (\hat{q}_1^1 - p)\hat{a}_1(p, \cdot)\ dF(p) = 0 \tag{16.21}$$

[51]For a 2×2 case, Brugiavini (1993) shows that the condition is that income variability be large relative to the variability of survival probabilities. This ensures that all individuals with a high income and with any survival probability purchase annuities, and *vice-versa*.

and

$$\int_{\underline{p}}^{\bar{p}} (\hat{q}_1^2 - p)\hat{b}_1(p,\cdot)\ dF(p) = 0 \tag{16.22}$$

where $\hat{a}_1(p,\cdot) = \int_{\tilde{y}_1^1(p)}^{\bar{y}_1} \hat{a}_1(p,y_1)dG(y_1)$ and $\hat{b}_1(p,\cdot) = \int_{\underline{y}_1}^{\tilde{y}_1^2(p)} \hat{b}_1(p,y_1)dG(y_1)$, are total demands for a_1 and b_1, respectively, by all relevant income recipients with a given p.

Recall that $\hat{a}_1$ and $\hat{b}_1$ depend implicitly on q_1^1 and q_1^2 and on $\tilde{y}_1^1(p)$ and $\tilde{y}_1^2(p)$, defined above. Thus, existence and uniqueness of $(\hat{q}_1^1,\ \hat{q}_1^2)$, defined by (16.20) and (16.21), requires certain conditions.

From (16.21) and (16.22),

$$\hat{q}_1^1 - \hat{q}_1^2 = \int_{\underline{p}}^{\bar{p}} p\varphi(p)dF(p) \tag{16.23}$$

where

$$\varphi(p) = \frac{\hat{a}_1(p,\cdot)}{\int_{\underline{p}}^{\bar{p}}\hat{a}_1(p,\cdot)\ dF(p)} - \frac{\hat{b}_1(p,\cdot)}{\int_{\underline{p}}^{\bar{p}}\hat{b}_1(p,\cdot)\ dF(p)} \tag{16.24}$$

Clearly, $\int_{\underline{p}}^{\bar{p}}\varphi(p)\ dF(p) = 0$. Hence, $\varphi(p)$ changes sign at least once over $[\underline{p},\ \bar{p}]$. Since $\hat{a}_1(p,\cdot)$ strictly increases and $\hat{b}_1(p,\cdot)$ strictly decreases in p, $\varphi(p)$ strictly increases in p. This implies that there exists a unique $\tilde{p}$, $0 < \tilde{p} < 1$, such that $\varphi(p) \lesseqgtr 0$ as $p \lesseqgtr \tilde{p}$. It follows that

$$\hat{q}_1^1 - \hat{q}_1^2 = \int_{\underline{p}}^{\bar{p}} p\varphi(p)dF(p) > \tilde{p}\int_{\underline{p}}^{\bar{p}} \varphi(p)dF(p) = 0 \tag{16.25}$$

Thus, the condition for an equilibrium with active lending and borrowing in period one is satisfied.

As before, the equilibrium price for period zero annuities is equal to life expectancy:

$$\hat{q}_0 = E(p) = \int_{\underline{p}}^{\bar{p}} p dF(p) \tag{16.26}$$

Of course, $0 < \hat{q}_0 < 1$. Notice that, since $\hat{a}_1(p, \cdot)$ strictly increases and $\hat{b}_1(p, \cdot)$ strictly decreases in p, $1 > \hat{q}_1^1 > \hat{q}_0$ while $\hat{q}_1^2 < \hat{q}_0$, reflecting adverse selection in period one.

16.4 Refundable Annuities

We have seen that when uncertainty early in life is confined to longevity then the optimum purchase of long-term annuities provides perfect protection against this uncertainty. Consequently, all annuity transactions occur early in life with no residual activities at later ages and hence no *adverse selection.* In contrast, when faced with uninsurable uncertainties in addition to longevity, individuals are induced to adjust their portfolios upon the arrival of new information. These adjustments are characterized by adverse selection, reflected in a higher price for (short-term) annuities purchased and a lower price for annuities sold. Recall that in the above discussion we have allowed the purchase of short-term annuities late in life as well as the short sale of long-term annuities purchased earlier. In spite of these "pro-market" assumptions, asymmetric information generates adverse selection.

In these circumstances, we pose the following question: are there financial instruments which, if available, may improve the market allocation in terms of expected utility[52]? We answer this question in the affirmative by proposing a

[52] We mean instruments which work via individual incentives, in contrast to fiscal means, such as taxes/subsidies, available to the government.

new financial instrument that may achieve this goal. The proposal is to have *a new class of annuities, each carrying a guaranteed commitment by the issuer to refund the annuity, when presented by the holder, at a (pre) specified price. Call these (guaranteed) refundable annuities.*

As shown below, the short-sale of annuities purchased in period zero is equivalent to the purchase in period zero of refundable annuities whose refund price is equal to $\hat{q}_1^2$. Therefore, in order to improve upon this allocation, it is proposed that individuals will hold a portfolio composed of *a variety of refundable annuities with different refund prices.* The purchase of refundable annuities with different refund prices will provide more flexibility in adjusting consumption to the arrival of information about longevity and income. With regular annuities, the revenue *per annuity* from short-sales in period one is independent of the quantity of annuities sold. With a variety of refundable annuities, this revenue may vary: depending on the realization of longevity and income, individuals will sell refundable annuities in *descending order,* from the highest guaranteed refund price and down.

A portfolio of refundable annuities with different refund prices will enable these adjustments to be more closely related to the realization of the level of income and longevity, and provide more flexibility to individuals' decisions about their optimum consumption path.

Formally, within the context of the previous three period model, the market for refundable annuities works as follows. Define a *refundable annuity of type* r as an annuity purchased in period zero with a guaranteed refund price of $r \geq 0$. This includes annuities with no refund price ($r = 0$). As before, individuals may borrow against these annuities at the market price for borrowing, as described in the previous section. Denote the amount of type r annuities by a_0^r, $a_0^r \geq 0$, and the amount refunded by b_1^r, $a_0^r \geq b_1^r \geq 0$.

Consider first only one type of refundable annuities. For any realization of y_1, consumption in periods one and two is:

$$\begin{aligned} c_1 &= y_0 + y_1 - q_0^r a_0^r + r b_1^r - q_1 a_1 \\ c_2 &= a_0^r - b_1^r + a_1 \end{aligned} \tag{16.27}$$

where $a_1 \geq 0$ are (short-term) annuities purchased in period one at a price of q_1 and q_0^r is the price of the refundable annuity[53].

In view of (16.11), maximization of (16.1) w.r.t. b_1^r and a_1 yields F.O.C.

$$u'(c_1)r - pu'(c_2) \leq 0 \tag{16.28}$$

and

$$-u'(c_1)q_1 + pu'(c_2) \leq 0 \tag{16.29}$$

Denote the solutions to these equations by $\hat{b}_1^r(p, y_1)$, and $\hat{a}_1(p, y_1)$. Again, these functions implicitly depend on $y_0 - q_0^r a_0^r$, r and q_1. The optimum level of period zero annuities is determined by maximization of expected utility, (16.3), assuming an optimum choice, $(\hat{c}_1,\ \hat{c}_2)$, in period one. The F.O.C. is

$$-E\ [u'(\hat{c}_1)]q_0^r + E\ [pu'(\hat{c}_2)] = 0 \tag{16.30}$$

[53] In period zero we allow annuities with no refund price ($r = 0$) and individuals may short-sell these annuities in period one (borrow) at a market determined price. For simplicity, we disregard this possibility here. See Appendix.

Extension of the model beyond three periods would allow to have refundable annuities which can be exercised at different dates.

Denote the solution to (16.16) by $\hat{a}_0^r$. The equilibrium price $\hat{q}_0^r$ satisfies a zero expected profits condition:

$$\hat{q}_0^r \hat{a}_0^r = \int_{\underline{p}}^{\bar{p}} p(\hat{a}_0^r - \hat{b}_1^r(p; \cdot) dF(p) + r \int_{\underline{p}}^{\bar{p}} \hat{b}_1^r(p; \cdot) dF(p)$$

or

$$\hat{q}_0^r = E(p) + \frac{1}{\hat{a}_0^r} \int_{\underline{p}}^{\bar{p}} (r - p) \hat{b}_1^r(p; \cdot) dF(p) \tag{16.31}$$

while $\hat{q}_1$ is determined by (16.8).

Two observations are in place. First, a condition for an active annuity market in period zero is that $r < \hat{q}_1$. This is equivalent to the requirement above (with no refundable annuities) that $\hat{q}_1^2 < \hat{q}_1^1$. When the refund price exceeds the price of period one annuities, $r > \hat{q}_1$, individuals will refund all the annuities purchased in period zero, $\hat{b}_1^r(p, y_1) = \hat{a}_0^r$, for all p and y_1. But then, by (16.15), $\hat{q}_0^r = r > \hat{q}_1$. However, when the price of annuities in period one is lower than their price in period zero, no annuities will be purchased in period zero, $\hat{a}_0^r = 0$.

Second, comparing (16.21) and (16.27), it is seen that *refundable annuities and short-sales of period zero annuities (borrowing) are equivalent when the refund and the borrowing price are equal:* $r = \hat{q}_1^2$. Thus, when short-sales are permitted, refundable annuities may be (ex-ante) welfare enhancing if they provide a refund price or a variety of refund prices different from the borrowing equilibrium price.

16.5 A Portfolio of Refundable Annuities

Now suppose that individuals can purchase in period zero a variety of refundable annuities. Type $r_i \geq 0$ annuities are annuities that each guarantees a refund of r_i when presented by the holder in period one. There are k types of such refundable annuities, ranked from the highest refund down, $r_1 > r_2$... $> r_k \geq 0$. Denote the price and the amount purchased of type r_i annuities by q_0^i and a_0^i, respectively. The amount refunded of type r_i annuities in period one is denoted b_1^i, $a_0^i \geq b_1^i \geq 0$.

Individuals' consumption is now given by

$$c_1 = y_0 + y_1 - \sum_{i=1}^{k} q_0^i a_0^i - q_1 a_1 + \sum_{i=1}^{k} r_i b_1^i \tag{16.32}$$

and

$$c_2 = \sum_{i=1}^{k} (a_0^i - b_1^i) + a_1 \tag{16.33}$$

Maximization of (16.5) w.r.t. a_1 and b_1^i, $i = 1, 2, .., k$, yields F.O.C.

$$-u'(c_1)q_1 + pu'(c_2) \leq 0 \tag{16.34}$$

and

$$u'(c_1)r_i - pu'(c_2) \leq 0, \qquad i = 1, 2, .., k \tag{16.35}$$

with equality when $a_1 > 0$ and $b_1^i > 0$, respectively. Denote the solutions to (16.34) and (16.35) by $\hat{a}_1$ and $\hat{b}_1^i$, $i = 1, 2, .., k$. These are functions of $\underline{r} = (r_1, r_2, ..r_k)$, $\bar{q}_0 = (q_0^1, q_0^2, ..q_0^k)$ and q_1.

It is seen from (16.35) that if $\hat{b}_1^i > 0$ then $\hat{b}_1^i = a_0^i$, for all $1 \geq i > j$. That is *all the higher ranked annuities (compared to marginally refunded annuity) are fully refunded.*

The amount of type r_i annuities purchased in period zero is determined by maximization of expected utility, (16.9), yielding F.O.C.

$$-E[u'(c_1)]q_0^i + E[pu'(c_2)], \qquad i = 1, 2, .., k \tag{16.36}$$

where the expectation is over p and y_1.

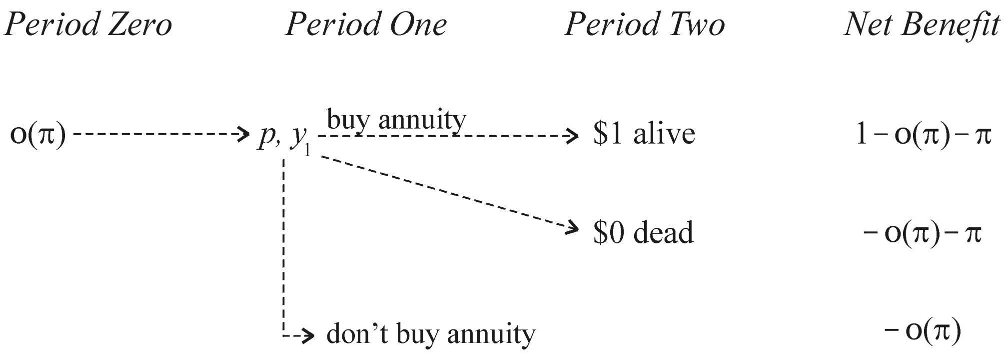
The value of holding a diversified optimum portfolio of refundable annuities clearly depends on specific assumptions about risk attitudes (utility function) and the joint distribution of longevity and income. To provide insight plan to do detailed calculations and report them in a separate paper[54].

16.6 Equivalence of Refundable Annuities and Annuity Options

We shall now demonstrate that refundable annuities are equivalent to *options to purchase annuities at a later date for a pre-determined price.* In terms of the above three period model, suppose that individuals can purchase in period zero options, each of which entitles the owner to purchase in period one an annuity at a given price. As before, the payout of each annuity is \$1 in period two if the owner is alive and nothing if dead. Denote by $o(\pi)$ the price of an option that, if exercised, entitles the holder to purchase in period one an annuity at a price of π. On a time scale, the scheme is as follows:

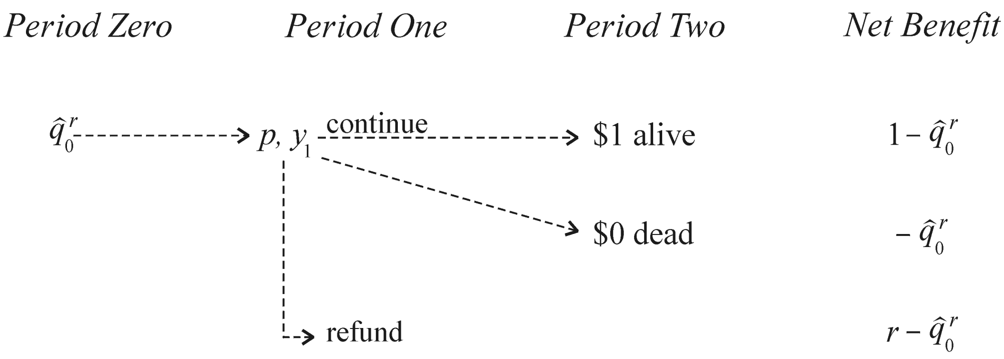
[54] This is joint research with Jerry Green of Harvard University, who has been instrumental in developing the ideas in this chapter.

Annuity Options



The comparable scheme for refundable annuities is:

Refundable Annuities



It is seen that when $\hat{q}_0^r = o(\pi) + \pi$ *and* $r = \pi$ (and hence, $\text{o}(\pi) = \hat{q}_0^r - r$), *these two schemes are equivalent.*

In addition to the above discussion about the advantages of the flexibility offered by holding a portfolio of options to annuitize, there may be additional 'behavioral' reasons in favor of such options. A vast economic literature reports experimental and empirical evidence of individuals' bounded rationality and shortsightedness (e.g.Rabin (1998) and (1999), Mitchell and Utkus (2004)). Of particular relevance to our case seems to be the plan designed by Thaler and Benartzi (2004), where individuals commit to save for pensions a certain fraction from future *increases in earnings*. The *'raison-detre'* for this plan is, presumably, cognitive shortcomings or self-control problems (procrastination, short-sightedness). Individuals are more willing to commit to the purchase of annuities from increases in earnings compared to the purchase by rational individuals. By deliberately delaying the implementation of the purchase of annuities, this plan may accommodate *hyperbolic discounters* (Laibson (1997)) who put a high discount rate on short-run saving. Thaler and Benartzi report that their plan has been successfully implemented by a number of firms. There seem to be parallels between the psychological insight that motivated this plan and the proposed options to annuitize at a later date.

Appendix to Chapter 16

We have seen in the text that $\hat{b}_1(p) = 0$ when $\hat{a}_1(p) > 0$ and (16.7) holds with equality. Differentiating w.r.t. p

$$\frac{\partial \hat{a}_1}{\partial p} = -\frac{1}{p}\left(\frac{1}{\frac{u''(\hat{c}_1)}{u'(\hat{c}_1)}q_1^1 + \frac{u''(\hat{c}_2)}{u'(\hat{c}_2)}}\right) > 0 \tag{A.1}$$

Similarly, when $\hat{b}_1(p) > 0$ then $\hat{a}_1(p) = 0$ and (16.8) holds with equality. Differentiating w.r.t. p

$$\frac{\partial \hat{b}_1}{\partial p} = \frac{1}{p}\left(\frac{1}{\frac{u''(\hat{c}_1)}{u'(\hat{c}_1)}q_1^2 + \frac{u''(\hat{c}_2)}{u'(\hat{c}_2)}}\right) < 0 \tag{A.2}$$

Consider the zero expected condition (16.5):

$$\int_{p_a}^{\bar{p}} (q_1^1 - p)\hat{a}_1(p)dF(p) = 0 \tag{A.3}$$

Where $p_a = \lambda q_1^1$, λ is given by (16.12):

$$\lambda = \frac{u'(y_0 + y_1 - E(p)\hat{a}_0)}{u'(\hat{a}_0)} \tag{A.4}$$

and $\hat{a}_0$ is determined by (16.13)

$$-E\left[u'(y_0 + y_1 - E(p)\hat{a}_0 - q_1^1\hat{a}_1(p) + q_1^2\hat{b}_1(p))E(p) + \right.$$

$$\left. + E\left[pu'(\hat{a}_0 + \hat{a}_1(p) - \hat{b}_1(p))\right] = 0 \right. \tag{A.5}$$

When $\hat{a}_1(p) = \hat{b}_1(p) = 0$ for $\bar{p} \geq p \geq \underline{p}$, then $\lambda = 1$ (because in (A.5), marginal utilities are independent of p). Whenever $\hat{a}_1(p) > 0$ and $\hat{b}_1(p) > 0$ for some ranges of p, this changes $\hat{a}_0$, and hence λ, compared to the previous case.

Denote by φ expected profits in the period one market for annuities,

$$\varphi(q_1^1) = \int_{p_a}^{\bar{p}} (q_1^1 - p)\hat{a}_1(p)dF(p) \tag{A.6}$$

An equilibrium price, $\hat{q}_1^1$, is defined by $\varphi(\hat{q}_1^1) = 0$. Since $p_a = \bar{p}$ when $q_1^1 = \bar{p}$ (because $\hat{a}_1(p) = 0$, and $\lambda = 1$), $\hat{q}_1^1 = \bar{p}$ is an equilibrium price, implying no purchase of annuities in period one. A similar argument applies to the market for b_1: here the equilibrium price is $\hat{q}_1^2 = \underline{p}$, implying $\hat{b}_1(p) = 0$ for all p.

Could there be another equilibrium with $p_a < \bar{p}$ (and $p_b > \underline{p}$)? Under a 'mild' condition the answer is negative.

Suppose that $q_1^1 = E(p)$. Then, by (16.7) - (16.8) and (A.5), $\hat{a}_0 = 0$ and $\hat{b}_1(p) = 0$ for all $\underline{p} \leq p \leq \bar{p}$. This is reasonable: when prices of annuities in period zero and in period one are equal, annuities are purchased only in period one. Then, by (A.4), $\lambda = 0$. It now follows from (A.1) and (A.6) that $\varphi(E(p)) < 0$. A sufficient condition that $\hat{q}_1^1 = \bar{p}$ be the only equilibrium price is that $\varphi(q_1^1)$ strictly increases for all q_1^1, $E(p) < q_1^1 < \bar{p}$. From (A.6), the condition is

$$\varphi'(q_1^1) = \int_{p_a}^{\bar{p}} [\hat{a}_1(p) + (\hat{q}_1^1 - p)\frac{d\hat{a}_1(p)}{dq_1^1}]\ dF(p) > 0 \tag{A.7}$$

Note that $\dfrac{d\hat{a}_1(p)}{dq_1^1}$ in (A.7) is the total derivative of $\hat{a}_1(p)$ w.r.t. q_1^1, taking into account the equilibrium change in $\hat{a}_0$ (from (A.5)). Condition (A.7) ensures that $\varphi(q_1^1) < 0$ for $E(p) \leq p < \bar{p}$.