Pricing of equity-linked life insurance contracts, partial differential equations and neural networks

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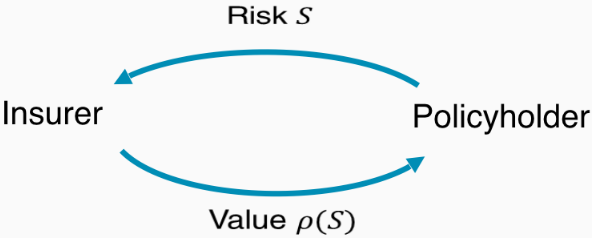
Based on joint works with Jan Dhaene (KU Leuven) and Karim Barigou (Universite Laval)

Introduction

Motivation

- The fundamental question in finance and insurance is how we should price contingent claims?
- We are interested in a portfolio of equity-linked life insurance policies.



- What is a value of an insurance liability S? What is a value of a portfolio of insurance liabilities?
  - Depends on financial risks (hedgeable and non-hedgeable): Interest rate, equity, volatility, rational lapses etc.
  - Depends on actuarial risks (non-hedgeable diversifiable and non-diversifiable): diversifiable and undiversifiable mortality risk, irrational lapses etc.

Actuarial and Financial valuation

- Actuarial valuations:

$$\boxed{\rho[S] = \mathbb{E}^{\mathbb{P}}[S] + \mathsf{RM}^{\mathbb{P}}[S]}$$

  - Based on the principle of diversification.
  - Risk margin (premium loading) RM to cover non-diversified risk.
  - The valuation is performed under the real-world measure $\mathbb{P}$.

- Financial valuation:

$$\boxed{\rho[S] = \mathbb{E}^{\mathbb{Q}}[S]}$$

  - Based on the principles of no-arbitrage and replication.
  - Risk-neutral measures $\mathbb{Q}$ and the implied prices follow from observed market prices of traded assets.
  - In complete markets: $\mathbb{Q}$ is unique, the price of S is the market price of the replicating portfolio.
  - In incomplete markets: Infinite choice of $\mathbb{Q}$ and we choose the pricing and the hedging objective, the price of S is the market price of the hedging portfolio.

Actuarial and Financial valuation

- The goal of the research is to combine actuarial and financial valuations.
- In accordance with Solvency II Directive, insurance liabilities should be priced in a so-called market-consistent way:

$$Price = \underbrace{Best\ Estimate}_{\text{Hedgeable and diversifiable risks}} + \underbrace{Risk\ Margin}_{\text{Non-hedgeable and non-diversifiable risks}}$$

- Can we give a formal (theoretical) derivation of this pricing principle in discrete and continuous time?

Two-step approach

- One-period two-step hedge-based approach (Dhaene et al. 2017):

$$\rho\,[S] = \underbrace{V^{\theta}(0)}_{\text{Hedgeable part}} + \underbrace{\pi[S - V^{\theta}(T)]}_{\text{Non-hedgeable part}}$$

  - Based on (non-unique) liability decomposition into hedgeable and non-hedgeable parts.
  - The price of the hedgeable part is determined by financial valuation and quadratic hedging, the price is related to the market cost of the hedging portfolio.
  - The price of the non-hedgeable part is determined by actuarial valuation, the price is related to the real-world cost of the residual claims left after the hedging portfolio is implemented (includes financial and actuarial non-hedgeable risks) .

Two-step approach

- The two-step hedge-based valuation of Dhaene et al. (2017) in a one-period setting was generalized in a discrete multi-period setting in Barigou et al. (2019) and in continuous-time in Delong et al. (2019*a*), Delong et al. (2019*b*) and Delong & Barigou (2021).
- The authors consider a fair valuation of insurance liabilities.
- We derive a system of partial differential equations for the continuous-time valuation operator.
- We can solve the system of PDEs with deep neural networks by applying Deep Backward Dynamic Programming Principle (Deep BDPP).

Two-step approach

- Two-step conditional approach by (Pelsser & Stadje 2014) and (Salahnejhad Ghalehjooghi & Pelsser 2021) :

$$\underbrace{\rho[S] = \mathbb{E}^{\mathbb{Q}}\left[\underbrace{\pi[S \mid \mathbf{Y}]}_{\text{Inner step}}\right]}_{\text{Outer step}}$$

  - Inner step: Actuarial valuation conditional on traded asset prices.
  - Outer step: Financial valuation.
- Three-step hedge-based and conditional approach by (Linders 2023).

Fair valuation of insurance liabilities and pricing PDEs

Financial market

- We consider a Black-Scholes type financial market:
- Risk-free asset:

$$\frac{dR(t)}{R(t)} = r dt, \quad 0 \le t \le T.$$

- Two risky assets:

$$\frac{dY(t)}{Y(t)} = \mu_Y dt + \sigma_Y dW_Y(t), \quad 0 \le t \le T,$$
$$\frac{dF(t)}{F(t)} = \mu_F dt + \sigma_F dW_F(t), \quad 0 \le t \le T,$$

where the processes (W_Y, W_F) are correlated Brownian motions defined by

$$W_Y(t) = W_1(t), \quad W_F(t) = \rho W_1(t) + \sqrt{1-\rho^2} W_2(t),$$

and (W_1, W_2) are independent Brownian motions.
- We assume that Y is traded in the financial market, and F is not traded.
- More generally, F can be interpreted as a background non-hedgeble and non-diversifiable noise which impacts the insurance payments.

Insurance portfolio

- We consider an equity-linked life insurance portfolio with n policies.
- The lifetimes of the policyholders $(\tau_k)_{k=1,\dots,n}$ are independent and identically distributed, conditional on $\mathcal{F}_T^{(Y,F)} = \sigma\big((Y(s), F(s), 0 \le s \le T\big)$:

$$\mathbb{P}\big(\tau_k > t | \mathcal{F}_T^{(Y,F)}\big) = e^{-\int_0^t \lambda(s,Y(s),F(s))ds}, \quad 0 \le t \le T.$$

- The counting process and the compensated counting process:

$$N(t) = \sum_{k=1}^{n} \mathbf{1}\{\tau_k \le t\},$$

$$\tilde{N}(t) = N(t) - \int_0^t (n - N(s-))\lambda(s, Y(s), F(s))ds, \quad 0 \le t \le T.$$

- The number of policies in force: $J(t) = n - N(t)$.
- The benefit process:

$$B(t) = \int_0^t (n - N(u-))A(u, Y(u), F(u))du + \int_0^t D(u, Y(u), F(u))dN(u) + (n - N(T))S(Y(T), F(T))\mathbf{1}_{t=T}, \quad 0 \le t \le T.$$

Hedging portfolio

- Since a part of the benefit process can be hedged, we introduce a hedging portfolio.
- Let $\theta = (\theta(t), 0 \leq t \leq T)$ denote a dynamic hedging strategy – the amount of money invested in Y,
- Let $V^\theta = (V^\theta(t), 0 \leq t \leq T)$ denote the self-financing hedging portfolio under the strategy θ given by the dynamics:

$$\begin{aligned} dV^\theta(t) = {} & \theta(t)(\mu_Y dt + \sigma_Y dW_Y(t)) + (V^\theta(t) - \theta(t))r dt \\ & - (n - N(t-))A(t, Y(t), F(t))dt - D(t, Y(t), F(t))dN(t), \end{aligned}$$

and the terminal payments $(n - N(T))S(Y(T), F(T))$ are subtracted from $V^\theta(T)$ at time T.

One-period valuation operator

- For the market price of the hedgeable liability, we minimize the mean-square hedging error at the terminal time under the unique equivalent martingale measure for Y:

$$\inf_{\theta} \mathbb{E}^{\hat{\mathbb{Q}}}\left[|(n - N(T))S(Y(T), F(T)) - V^{\theta}(T)|^2\right].$$

- We introduce the one-period valuation operator:

$$\varrho(B) = V_B^*(0) + \pi\left[\Big((n - N(T))S(Y(T), F(T)) - V_B^*(T)\Big)e^{-rT}\right],$$

with the actuarial valuation rule for the non-hedgeble liability:

$$\pi[\xi] = \mathbb{E}^{\mathbb{P}}[\xi] + \mathsf{RM}^{\mathbb{P}}[\xi],$$

where RM stands for a one-period actuarial risk margin.

- Consequently, we consider the one-period valuation operator:

$$\begin{aligned}\varrho(B) &= V_B^*(0) + \mathbb{E}^{\mathbb{P}}\left[\Big((n - N(T))S(Y(T), F(T)) - V_B^*(T)\Big)e^{-rT}\right] \\ &\quad + \mathsf{RM}^{\mathbb{P}}\left[\Big((n - N(T))S(Y(T), F(T)) - V_B^*(T)\Big)e^{-rT}\right] \\ &= \textit{Best Estimate of } B \; + \; \textit{Risk Margin for } B.\end{aligned}$$

Multi-period valuation operator

- Let us consider the time points $\mathcal{T} = 0, h, ..., T-h, T$.
- We iteratively and backward apply the one-period valuation operator with time step h:

$$\varphi_B(T) = (n - N(T))S(Y(T), F(T)),$$

$$\varphi_B(t) = \varrho_t\Big(\int_t^{t+h} d\tilde{B}(s)\Big), \quad t = 0, h, ..., T-h,$$

$$\tilde{B}(s) = \int_t^s (n - N(u-))A(u, Y(u), F(u))du$$

$$+ \int_t^s D(u, Y(u), F(u))dN(u)$$

$$+ \varphi_B(t+h)\mathbf{1}\{s = t+h\}, \quad t \le s \le t+h.$$

- We introduce the multi-period valuation operator:

$$\varphi_B(t) = V^*_{\tilde{B}}(t) + \mathbb{E}^{\mathbb{P}}\Big[\Big(\varphi_B(t+h) - V^*_{\tilde{B}}(t+h)\Big)e^{-rh}|\mathcal{F}_t\Big]$$

$$+ \mathrm{RM}^{\mathbb{P}}\Big[\Big(\varphi_B(t+h) - V^*_{\tilde{B}}(t+h)\Big)e^{-rh}|\mathcal{F}_t\Big], \quad t = 0, h, ..., T-h.$$

Continuous-time valuation operator

- We would like to extend the definition of the price $\varphi_B(t)$ from $t \in \mathcal{T}$ to all times $t \in [0, T]$.
- The continuous-time valuation operator φ_B is defined as an operator which satisfies the continuous-time limit of the discrete-time pricing equation.
- We are interested in finding φ which satisfies the limit:

$$\lim_{h\to 0}\left\{\frac{\mathbb{E}^{\mathbb{P}}_{t,y,f,k}\Big[\big(\varphi(t+h)-V^*_{\tilde{B}}(t+h)\big)e^{-rh}-\big(\varphi(t)-V^*_{\tilde{B}}(t)\big)\Big]}{h}\right.$$
$$\left.+\frac{\mathsf{RM}^{\mathbb{P}}_{t,y,f,k}\Big[\big(\varphi(t+h)-V^*_{\tilde{B}}(t+h)\big)e^{-rh}-\big(\varphi(t)-V^*_{\tilde{B}}(t)\big)\Big]}{h}\right\}=0,$$

  for any $(t, y, f, k) \in [0, T) \times (0, \infty) \times (0, \infty) \times \{0, ..., n\}$.
- We choose the actuarial risk margin:

$$\mathsf{RM}^{\mathbb{P}}[\xi] = \frac{1}{2}\gamma\sqrt{h}\sqrt{\mathsf{Var}^{\mathbb{P}}[\xi]}, \quad \text{on } [t, t+h].$$

The resulting system of non-linear PDEs

Theorem

Let us consider the system of non-linear PDEs:

$$\varphi_t^k(t,y,f) + \varphi_y^k(t,y,f)yr + \varphi_f^k(t,y,f)f\big(\mu_F - \frac{\mu_Y - r}{\sigma_Y}\sigma_F\rho\big)$$
$$+ \varphi_{yf}^k(t,y,f)yf\sigma_Y\sigma_F\rho + \frac{1}{2}\varphi_{yy}^k(t,y,f)y^2\sigma_Y^2 + \frac{1}{2}\varphi_{ff}^k(t,y,f)f^2\sigma_F^2$$
$$+ \big(\varphi^{k-1}(t,y,f) + D(t,y,f) - \varphi^k(t,y,f)\big)k\lambda(t,y,f) + kA(t,y,f) - \varphi^k(t,y,f)r$$
$$+ \Phi^k\Big(t, \varphi_f^k(t,y,f)f\sigma_F\sqrt{1-\rho^2}, \varphi^{k-1}(t,y,f) + D(t,y,f) - \varphi^k(t,y,f)\Big) = 0,$$
$$(t,y,f) \in [0,T) \times (0,\infty) \times (0,\infty),$$
$$\varphi^k(T,y,f) = kS(y,f), \quad (y,f) \in (0,\infty) \times (0,\infty),$$

for $k \in \{0,...,n\}$, *where* $\Phi^k(t,x_1,x_2) = \frac{1}{2}\gamma\sqrt{x_1^2 + x_2^2 k\lambda(t,y,f)}$ *for the standard deviation actuarial risk margin.*

We assume that there exist unique solutions $(\varphi^k)_{k=0,\ldots,n}$ *to the PDEs. The continuous-time valuation operator* φ *determined by the PDEs satisfies the continuous-time limit of the discrete-time pricing equation as* $h \to 0$.

The resulting system of non-linear PDEs

- We prove

$$\lim_{h\to 0}\frac{\mathrm{RM}_{t,y,f,k}\Big[\big(\varphi(t+h)-V^*_{\tilde{B}}(t+h)\big)e^{-rh}-\big(\varphi(t)-V^*_{\tilde{B}}(t)\big)\Big]}{h}$$
$$=\Phi^k\Big(t,\varphi^k_f(t,y,f)f\sigma_F\sqrt{1-\rho^2},\varphi^{k-1}(t,y,f)+D(t,y,f)-\varphi^k(t,y,f)\Big).$$

- We can call Φ an instantaneous actuarial risk margin,
- The instantaneous actuarial risk margin puts a price on two non-hedgeable components: the first is the delta-hedging perfect replication strategy for the independent component of the risky asset F, the second is the sum at risk in the event of the policyholder's death.
- The instantaneous actuarial risk margin gives an economic capital which should be hold for $[t,t+h]$ with $h\to 0$.
- The continuous-time valuation operator φ is market-consistent and actuarial, hence it is fair in the sense of Dhaene et al. (2017).

Theorem
The continuous-time valuation operator has the representation:

$$\varphi^k(t,y,f) = \mathbb{E}^{\hat{\mathbb{Q}}}_{t,y,f,k}\Big[\int_t^T e^{-r(s-t)}dB(s) + \int_t^T e^{-r(s-t)}\Phi(s)ds\Big],$$
$$(t,y,f) \in [0,T] \times (0,\infty) \times (0,\infty),\ k \in \{0,...,n\}.$$

- The valuation operator values liabilities as the best estimate of the liability plus the total actuarial risk margin for the liability.
- More interpretations and relation to Solvency II pricing principle can be found in Delong et al. (2019*a*) and Delong et al. (2019*b*).

The resulting system of non-linear PDEs

- Extensions to include more risk factors and more sophisticated pricing problems are possible (Delong & Barigou (2021)). We always end up with a system of non-linear PDEs.
- The link between pricing and solving PDEs is well-known in finance and insurance.
- Our system of PDEs includes Black-Scholes PDE and Thiele's DE as special cases.
- How to solve our system of PDEs numerically?

Solving the system of PDEs with neural networks

The system of PDEs

- Let $z = (y, f)$. We have to solve the system of non-linear PDEs:

$$\varphi_t^k(t,z) + \nabla_{y,f}\varphi^k(t,z)\cdot\mu(t,z) + \frac{1}{2}\operatorname{Tr}\left(\sigma\sigma^{\mathrm{T}}(t,z)\left(\operatorname{Hess}_{y,f}\varphi^k\right)(t,z)\right)$$
$$+\left(\varphi^{k-1}(t,z) - \varphi^k(t,z) + D(t,z)\right)k\lambda(t,z) + kA(t,z) - \varphi^k(t,z)r$$
$$+\Phi^k\left(t, \varphi_f^k(t,z) f\sigma_F\sqrt{1-\rho^2}, \varphi^{k-1}(t,z) + D(t,z) - \varphi^k(t,z)\right) = 0,$$
$$(t,z)\in[0,T)\times\mathbb{R}^2, \qquad \varphi^k(T,z) = g^k(z), \quad z\in\mathbb{R}^2,$$

  for $k \in \{0, ..., n\}$.
- If we apply finite difference methods, then we face the curse of dimensionality if the dimension of z is large.
- MC methods are only available for linear PDEs and they provide a solution for a single fixed initial point.
- LSMC methods require proper choice of the basis functions.
- Main motivation for the research: In order to get the price $\varphi^n(t,z)$ for a portfolio with n policyholders we have to solve $n-1$ non-linear PDEs to get $\left(\varphi^k(y,z)\right)_{k=1}^{n-1}$. We face large computational times if n is large.

Solving PDEs using deep learning

- In recent years there is a growing interest in applying machine learning methods to solve PDEs. In particular, deep neural networks have proved to be efficient in many learning (approximation) tasks (universal approximation theorems).
- Han et al. (2018) propose to represent a solution to a PDE as a solution to a BSDE and approximate the initial value function and the control process function of the forward version of the BSDE with deep neural networks in order to match the terminal condition of the BSDE – Deep BSDE.
- Huré et al. (2020) propose to split the global optimization problem from Han et al. (2018) into multiple local optimization problems constructed based on dynamic programming principle – Deep BDPP.
- Gnoatto et al. (2022) applies Deep BSDE to PIDEs.
- Castro (2022) applies Deep BDPP to PIDEs.
- In this presentation we apply Deep BDPP to our system of PDEs. In Delong & Barigou (2021) we applied Deep BSDE.

Representation as Forward Stochastic Differential Equation

- The price φ is also solution to the following FSDE:

$$
\begin{aligned}
\varphi^{J(t)}(t, Z(t)) = &\ \varphi^{n}(0, Z(0)) \\
& - \int_0^t \Upsilon^{J(s-)}\Big(s, Z(s), \varphi^{J(s-)}(s, Z(s)), \\
& \qquad \nabla_{y,f}\varphi^{J(s-)}(s, Z(s))\sigma(s, Z(s)), \\
& \qquad \Big(\varphi^{J(s-)-1}(s, Z(s)) - \varphi^{J(s-)}(s, Z(s))\Big) J(s-)\lambda(s, Z(s))\Big) ds \\
& + \int_0^t \nabla_{y,f}\varphi^{J(s-)}(s, Z(s))\sigma(s, Z(s)) dW(s) \\
& + \int_0^t \Big(\varphi^{J(s-)-1}(s, Z(s)) - \varphi^{J(s-)}(s, Z(s))\Big) d\tilde{N}(s), \\
\varphi^{J(T)}(T, Z(T)) = &\ g^{J(T)}(Z(T))
\end{aligned}
$$

- Starting from an initial price of the liability, the FSDE describes the price dynamics of the insurance liability controlled with two processes to match the terminal payment (a forward version of a BSDE).
- The intial price and the control processes are smooth functions of the risk factors and the time variable. We estimate these functions with deep neural networks by solving a control problem.

Discretization of the FSDE

- We consider time points $0 = t_0 < t_1 < ... < t_{n-1} < t_n = T$ and we discretize the FSDE with Euler scheme. On $[t_n, t_{n+1}]$ we consider:

$$
\begin{aligned}
&\widehat{\varphi}^{J(t_{n+1})}(t_{n+1}, Z(t_{n+1})) = \varphi^{J(t_n)}(t_n, Z(t_n)) \\
&- \Upsilon^{J(t_n)}\Big(t_n, Z(t_n), \varphi^{J(t_n)}(t_n, Z(t_n)), \\
&\qquad\qquad \nabla_{y,f}\varphi^{J(t_n)}(t_n, Z(t_n))\sigma(t_n, Z(t_n)), \\
&\qquad\qquad \Big(\varphi^{J(t_n)-1}(t_n, Z(t_n)) - \varphi^{J(t_n)}(t_n, Z(t_n))\Big)J(t_n)\lambda(t_n, Z(t_n))\Big)\Delta t_n \\
&+ \nabla\varphi^{J(t_n)}(t_n, Z(t_n))\sigma(t_n, Y(t_n), F(t_n))\Delta W(t_n) \\
&+ \Big(\varphi^{J(t_n)-1}(t_n, Z(t_n)) - \varphi^{J(t_n)}(t_n, Z(t_n))\Big)\Delta\tilde{N}(t_n)
\end{aligned}
$$

where

$$
\begin{aligned}
&\Delta t_n = t_{n+1} - t_n, \quad \Delta W(t_n) = W(t_{n+1}) - W(t_n), \\
&\Delta\tilde{N}(t_n) = N(t_{n+1}) - N(t_n) - J(t_n)\lambda(t_n, Y(t_n), F(t_n))\Delta t_n.
\end{aligned}
$$

with

$$
N(t_{n+1}) - N(t_n)\big|\mathcal{F}_{t_n} \sim Bin\Big(1, J(t_n)\lambda\big(t_n, Z(t_n)\big)\Delta t_n\Big).
$$

Deep BDPP for our system of PDEs

- Deep Backward Dynamic Programming Principle:
- We set $g^{J(T)}(T, Z(T)) = J(T)S(Y(T), F(T))$.
- We apply backward induction and go backward from $t = t_{n-1}$ to $t = t_0$:
  - At each point of time t, on $[t, t + \Delta t]$, we use two deep neural networks to approximate:

    N1: The price at time t:

    $$(k, z) \mapsto \varphi^k(t, z) \approx \mathcal{N}_t^{\phi}(k, z).$$

    N2. The price gradient at time t:

    $$(k, z) \mapsto \nabla\varphi^k(t, z)\sigma(t, z) \approx \mathcal{N}_t^{\chi}(k, z).$$

  - The parameters of the neural networks are estimated to minimize the quadratic loss function:

    $$\mathbb{E}\left[\left|\widehat{\varphi}^{J(t+\Delta t)}(t + \Delta t, Z(t + \Delta t)) - g^{J(t+\Delta t)}(t + \Delta t, Z(t + \Delta t))\right|^2\right].$$

  - We set $g^{J(t)}(t, Z(t)) = \widehat{\varphi}^{J(t)}(t, Z(t))$.

Deep BDPP for our system of PDEs

Final remarks:

- The number of policies in force k is used as a regressor in the price function, hence we compute the price for any k in one step.
- Compared to PIDEs and Castro (2022), we do not learn the jump component since it is related to the price function.
- The convergence should be proved with the arguments from Huré et al. (2020) and Castro (2022) – to be done.

Practical remarks:

- The minimization problems are solved via Stochastic Gradient Descent algorithm based on simulations of $\big(J(t), Y(t), F(t)\big)_{t=0}^{t_n}$.
- We have to specify the structure of the neural networks (number of hidden layers, number of nerons, activation functions) and the learning process (learning rate, batch size, number of epochs, early stopping).
- We use TensorFlow and Keras in R for implementation.

Numerical examples with Deep BDPP

Numerical examples

- We consider a portfolio of $n = 100$ equity-linked life insurance policies.
- By V we denote the policyholder's account value which results from investing policyholder's premium $V(0) = 1$ in the traded risky asset Y and collecting fee c from the account by the insurer.
- The death and survival guarantees:

$$D(t, Y(t)) = (1.05 - V(t))_+, \quad 0 \leq t \leq 1, \quad S(Y(1)) = (1.01 - V(1))_+.$$

- The insurer is also exposed to independent non-diversifiable mortality risk (stochastic intensity) modelled with F.
- We simulate $N = 100,000$ sample paths of the risk factors with time step $h = 0.01$.
- At each point of time t we use two deep neural networks to approximate:
  N1: The price at time t:

$$(k, v, \lambda, c) \mapsto \varphi^k(t, v, \lambda, c) \approx \mathcal{N}_t^{\phi}(k, v, \lambda, c).$$

  N2. The price gradient at time t:

$$(k, v, \lambda, c) \mapsto \nabla\varphi^k(t, v, \lambda, k, c)\sigma(v, \lambda) \approx \mathcal{N}_t^{\chi}(k, v, \lambda, c).$$

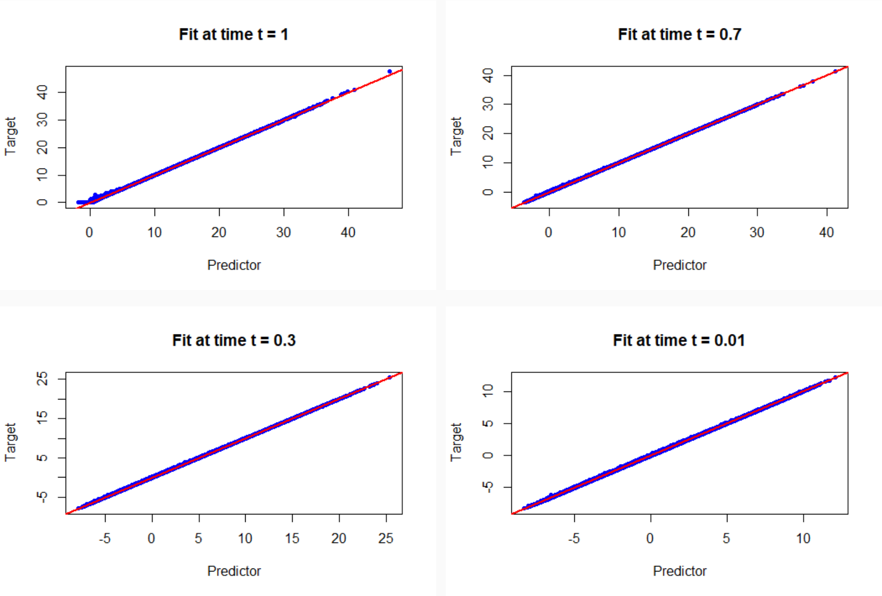


Figure 1: Validation of terminal replication errors for $\gamma = 0$ – one calibration.

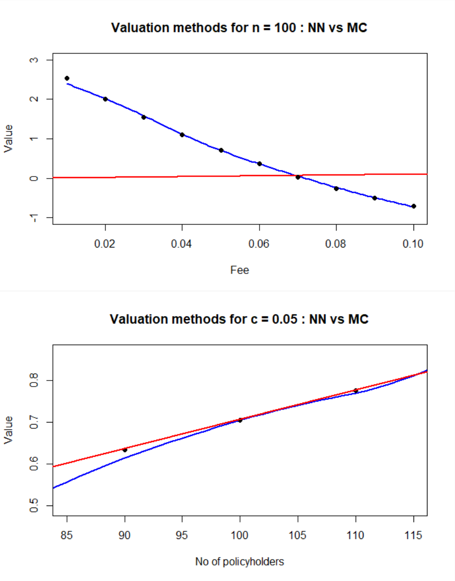


Figure 2: Estimation results for $\gamma = 0$ – one calibration.

Numerical examples

	c=0.03	c=0.04	c=0.05	c=0.06	c=0.07
NN (deep BDPP)	1.5579	1.1217	0.7182	0.3455	0.0089
MC (true)	1.5370	1.0994	0.7043	0.3641	0.0221
95% MC_upper	1.5006	1.0612	0.6641	0.3221	-0.0215
95% MC_lower	1.5734	1.1377	0.7445	0.4061	0.0657

	n=90	n=100	n=110
NN (deep BDPP)	0.6410	0.7182	0.7966
MC (true)	0.6329	0.7043	0.7755
95% MC_upper	0.5967	0.6641	0.7313
95% MC_lower	0.6691	0.7445	0.8196

Table 1: Estimation results from 10 calibrations with NN and MC estimates (the true value based on 1,000,000 obs. and the confidence intervals for the mean value in a sample with 100,000 obs.).

Numerical examples

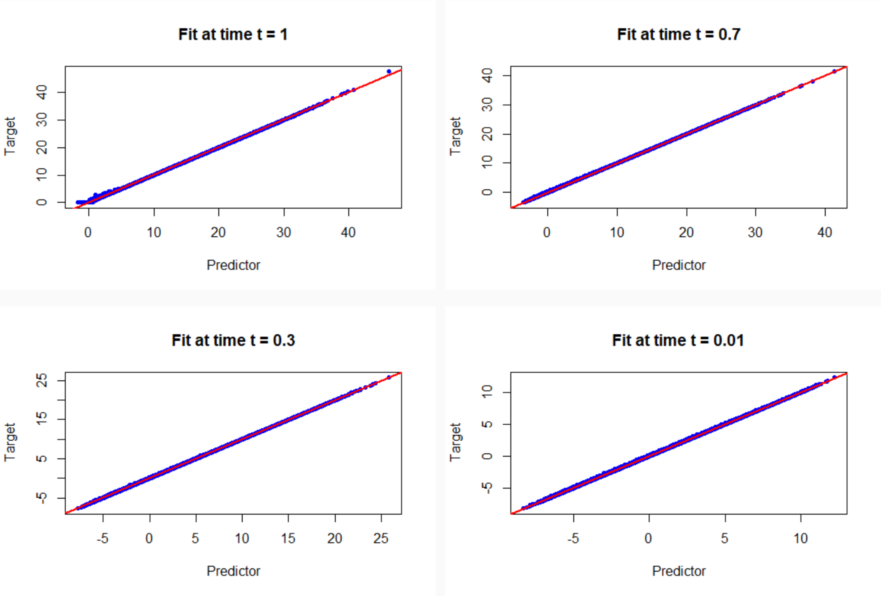


Figure 3: Validation of terminal replication errors for $\gamma = 5$ – one calibration.

Numerical examples

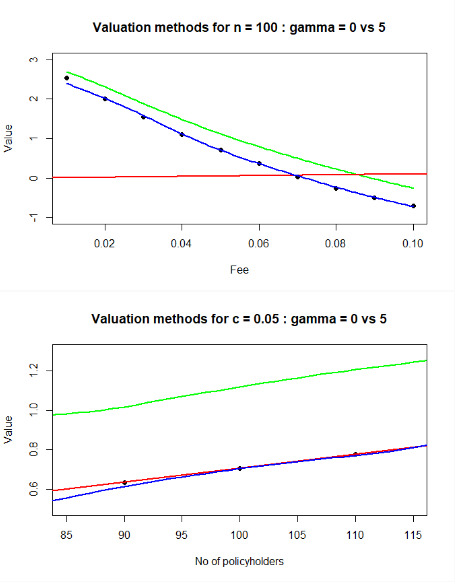


Figure 4: Estimation results for $\gamma = 0$ and $\gamma = 5$ – one calibration.

Conclusion

Conclusion

- We derived a system of non-linear PDEs for the fair pricing of a portfolio of equity-linked life insurance contracts in a general stochastic framework with various types of financial and insurace risks.
- We used the connection with BSDEs with jumps and proposed an efficient neural network architecture to solve our system of PDEs with multiple non-linear PDEs.

Thank you very much.

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